Systems control theory applied to natural and synthetic musical sounds[‡]

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Context. This research report sums up the research carried out under the title: *Systems control theory applied to natural and synthetic musical sounds*, which was supported by the CONSONNES (CONtrôle de SONs instrumentaux Naturels Et Synthétiques) project, supported by the French National Research Agency (ANR), under grant ANR-05-BLAN-0097-01.

A structured (special) session *MU07 : Control of natural and synthetic musical sounds* was organized by J. Kergomard and M. Wanderley for Acoustics'08 Paris international conference, in July 2008; this conference was co-organized by the Acoustical Society of America (ASA), the European Acoustics Association (EAA) and the French Acoustical Society (SFA). The opening talk, by both authors, aimed at giving an overview presentation of the links between systems control theory and simulation of natural or synthetic musical sounds. The present research report is an extended version of the talk.

Abstract. Systems control theory is a far developped field which helps to study stability, estimation and control of dynamical systems. The physical behaviour of musical instruments, once described by dynamical systems, can then be controlled and numerically simulated for many purposes. The aim of this paper is twofold: first, to provide the theoretical background on linear system theory, both in continuous and discrete time, mainly in the case of a finite number of degrees of freedom; second, to give illustrative examples on wind instruments, such as the vocal tract represented as a waveguide, and a sliding flute.

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***Abstract*— Systems control theory is a far developped field which helps to study stability, estimation and control of dynamical systems. The physical behaviour of musical instruments, once described by dynamical systems, can then be controlled and numerically simulated for many purposes. The aim of this paper is twofold: first, to provide the theoretical background on linear system theory, both in continuous and discrete time, mainly in the case of a finite number of degrees of freedom; second, to give illustrative examples on wind instruments, such as the vocal tract represented as a waveguide, and a sliding flute.**

Keywords— state space, stability, control, observation, inverse problem, oscillators, damping.

I. Introduction

Musical instruments can be modelled as dynamical systems, which can be decomposeed into a linear resonator, excitated through a non-linear oscillator. Hence, the development of theoretical **models** with a high degree of refinement with respect to both the asymptotic regime as well as transient behavior is of great musical importance. Moreover, related numerical simulation methods, as well as joint experimental work are fundamental, and imply a broad interaction among researchers coming from a wide variety of disciplines such as acoustics, control theory, signal processing and numerical analysis.

Mastering the playing technique of an acoustic instrument (i.e. its control) is a difficult and lifelong pursuit. In particular, it is important to characterize the **stability property** of an operating mode in order to prevent false notes. The quality and reproducibility of a musical performance depend on the level of technique that has been reached. This technique could well be described in terms of various system parameters: physical parameters of the instrument and extra-parameters applied by the player to the instrument, which we call **control inputs**. The ability to modify the behavior of the instrument relies on what is called the **controllability property** of the system.

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Numerical simulation is an important tool for the validation and investigation of theoretical models, provided that the fidelity to the continuous time model or to physical measurements is respected. It is natural that such numerical models could be used to estimate parameters based on **measured output** signals. This problem is linked to the **observability property** of the system. Indeed, this will be the first step to address **inverse problems**: recovering the full oscillating internal state of the physical model from external measurements only, considering the control as locally constant; the second step would then consist in computing this control parameter from the recovered state, using e.g. adpative filtering techniques, see e.g. [35].

A robot musician, independently of how controllable it might be, is generally deaf, and thus unable to adapt its behavior in response to the sound it produces. That is why the introduction of a **feedback loop** based on the measured output signals will be necessary to obtain better sounds.

The present paper is devoted to the application of modeling and control theory to a wide class of dynamical systems: *musical instruments*, which exhibit a rich variety of behaviors: they can be modeled either by Ordinary Differential Equations (ODEs) or by Partial Differential Equations (PDEs), linear or non-linear, with or without time delays, using constant or time-varying coefficients, etc.

The outline of the paper is the following:

- in § II, we consider the finite dimensional setting (a finite number degrees of freedom). In § II-A and § II-B, we give stability, controllability and observability properties and define the synthesis of observer-based controllers (OBC) for both state estimation and control purposes. In § II-C, we illustrate these on the example of Kelly-Lochbaum structure for digital waveguides. In § II-D, realization theory and feedback structure for viscoelastic damping models are then presented on the example of a standard 1 d.o.f. oscillator.
- In § III, we consider the infinite dimensional case (infinitely many d.o.f.) for which we also introduce notions from control theory in §III-B, with an emphasis on boundary control for stability using Lyapunov tech-

niques (i.e. energy based methods). The elaboration of the control is made easier once the hyperbolic PDEs have been diagonalized thanks to Riemann invariants, in § III-A. This technique is fully illustrated in § III-C on the example of a sliding flute. Finally in § III-D, some damping due to viscothermal losses is introduced in the flute model, and it is shown how the realization technique with memory variables helps to reformulate the model into a dynamical system, and study its stability thanks to an appropriate Lyapunov function.

II. Finite dimensional case

We give definitions, and select most important properties that are standard for *linear* finite-dimensional systems, which can be found in more details in e.g. [31]. Two theoretical frameworks are presented: the continuous-time setting for ordinary differential equations in § II-A, and the discrete-time setting for ordinary difference equations in § II-B. Two examples are then developped, as applications to acoustics in § II-C, or to mechanics in § II-D.

A. Control, Observation and Stability of ODEs

A scalar ODE of nth order in time can be transformed into a first order ODE with vector values, it is enough to set $X = [x\,\dot{x}\,\cdots\,x^{(n-1)}]$ as state vector.

1) Mathematical setting: Consider the following continuous-time dynamical system

$$\frac{d}{dt}X(t) = A\,X(t) + B\,v(t) \quad \forall t > 0,\ X(0) = X_0, \tag{1}$$

$$y(t) = C\,X(t) + D\,v(t)\,, \tag{2}$$

with vector-valued functions of time:

- input, or control v: a vector of dimension m,
- state vector X of dimension n,
- output, or observation or measurement y: a vector of dimension p.

The matrices in (1)-(2) are :

- input matrix B, of dimension $n \times m$,
- matrix of dynamics A, of dimension $n \times n$,
- output matrix C, of dimension $p \times n$,
- feedthrough matrix D, of dimension $p \times m$.

2) Solution: This system is affine, so when there is no control ($v = 0$), the free solution reads $X(t) = e^{tA}\,X_0$; on the contrary when the initial condition is zero ($X_0 = 0$), the forced solution of the system can be written in the time domain $y(t) = D\,v(t) + \int_0^t C\,e^{(t-s)A}\,B\,v(s)\,\mathrm{d}s$, or $y = h \star v$ with impulse response $h(t) = D\,\delta_0(t) + C\,e^{tA}\,B$ for $t \geq 0$ and in the frequency domain, thanks to causal or one-sided *Laplace transform* (LT), $Y(s) = H(s)\,V(s)$, with $p \times m$ *transfer matrix*:

$$H(s) = D + C(s\,I_n - A)^{-1}B\,, \quad \text{for} \quad \Re e(s) > \alpha\,.$$

3) Stability: The system is said to be *externally stable* when any bounded input gives rise to a bounded ouput, that is when $\int_0^\infty \|h(t)\|\,\mathrm{d}t < \infty$ or, equivalently, when the transfer matrix $s \mapsto H(s)$ has **no poles in the closed right half plane** $\Re e(s) \geq 0$, that is if and only if $\alpha < 0$. Only in this case are we allowed to compute the *Fourier transform* (FT) directly from the LT:

$$\hat{h}(f) := H(s = 2i\pi\,f), \quad \forall f \in \mathbb{R},$$

where f is the analog frequency expressed in Hz.

The system is said to be *asymptotically stable* when the free trajectories converge to 0 in the phase space $\mathbb{R}^n$: this happens if and only if $\Re e(\mathrm{spec}(A)) < 0$, i.e. matrix A has **no eigenvalues in the closed right half plane**.

4) Controllability and Observability: System (1) is said to be **controllable** if for any given initial condition X_0, any time T and any final condition X_T, there exists a control v, say of finite energy on $[0, T]$, which allows to drive the initial state to the final one. To this end, let us define the $n \times (nm)$ controllability matrix:

$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\,. \tag{3}$$

The fundamental property is that the system is controllable on $[0, T]$ if and only if $\mathcal{C}$ has rank n. Hence, the controllability of the system is that of the pair of matrices (A, B).

System (1)-(2) is said to be **observable** if the state of the free system ($v = 0$) can be recovered or reconstructed from output measurements only, that is the possibility to reconstruct X_0 from the only knowledge of the output $t \mapsto y(t)$ on $[0, T]$. To this end, let us define the $(np) \times n$ observability matrix:

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\,. \tag{4}$$

The fundamental property is that the system is observable on $[0, T]$ if and only if $\mathcal{O}$ has rank n. Hence, the observability of the system is that of the pair of matrices (A, C).

Hence, both controllability and observability can be checked thanks to simple *algebraic criteria*. Another notion is important and linked to the previous ones: the notion of **minimality**.

Given a state-space representation (A, B, C, D), and an invertible square matrix $P \in GL_n(\mathbb{R})$, it can be easily seen that $(P^{-1}\,A\,P, P^{-1}\,B, C\,P, D)$ gives rise to the *same* transfer matrix H, hence the same input-ouput behaviour; a useful application is the celebrated **modal decomposition**, which consists in choosing P such that $P^{-1}\,A\,P$ be diagonal. Now, a state-space representation (A, B, C, D) of some given input-ouput system is said to be *minimal* if the size n of the state space is the lowest possible, which gives the same transfer matrix H. A nice theorem, due to Kalman, states that:

$$(A, B, C, D) \text{ is minimal} \Leftrightarrow \begin{cases} (A, B) \text{ is controllable} \\ \quad\text{and} \\ (A, C) \text{ is observable.} \end{cases}$$

In this latter case, both notions of stability do coincide: in other words, all the eigenvalues of the $n \times n$ matrix A are poles of H.

5) Observer-Based Controllers: The stabilization of an unstable system is possible when it is controllable, thanks to a feedback control law of the form $v = K X + w$, where w is an external input; the $m \times n$ feedback matrix K must be chosen so that the matrix $A + BK$ be stable, i.e. it has no poles in the closed right-half plane.

In this stabilization process, one has to use the full state X of the system, which is never measured in practise. That is the reason why it will be estimated. This latter estimation of the full state X from the only knowledge of external measurements y is possible, under the above observability hypothesis. Let us define the estimated state $\hat{X}$ as follows:

$$\frac{d}{dt}\hat{X}(t) = A X(t) + L e(t) + B v(t), \quad \hat{X}(0) = 0, \tag{5}$$

$$\hat{y}(t) = C \hat{X}(t) + D v(t), \tag{6}$$

$$e(t) = \hat{y}(t) - y(t). \tag{7}$$

The dynamics of the estimated state $\hat{X}$ is now driven by the output error measurement e, through the $n \times p$ matrix L, which must be chosen so that the matrix $A + LC$ be stable, i.e. it has no poles in the closed right-half plane. Then, the coupled system can be easily studied in the $[X, \hat{X} - X]$ coordinates, and the *separation principle*, due to Kalman, proves that the estimated state $\hat{X}$ converges asymptotically to the full state X; moreover, the dynamics of the feedback system with control law $v = K \hat{X} + w$ is now stable. See e.g. [8] for an application of this to inverse problems.

B. Control, Observation and Stability of discrete-time systems

A scalar difference equation of nth order in discrete time can be transformed into a first order difference equation with vector values, it is enough to set as state vector e.g. $X_k = [x_k \, x_{k-1} \, \cdots \, x_{k-(n-1)}]$.

1) Mathematical setting: Consider the following discrete-time dynamical system

$$X_{k+1} = A X_k + B v_k, \quad \text{for } k \in \mathbb{N}, \quad \text{and } X_0, \tag{8}$$

$$y_k = C X_k + D v_k \tag{9}$$

with the same size of matrices as before, see § II-A.1. It must be taken care that, even if the letters being used are the same, the meaning is different in continuous or discrete time; for sampled-data systems, there is a link such as $A_d = e^{T_s A}$ for the discrete and continuous matrices of dynamics, where T_s is the sampling period.

2) Solution: This system is affine, so when there is no control ($v = 0$), the free solution reads $X_k = A^k X_0$; on the contrary when the initial condition is zero ($X_0 = 0$), the forced solution of the system can be written in the time domain $y_k = D v_k + \sum_{l=0}^{k-1} C A^{k-l-1} B v_l$, or $y = h \overset{\mathrm{d}}{\star} v$ with impulse response $h_0 = D$ and $h_k = C A^{k-1} B$ when $k \geq 1$; and in the frequency domain, thanks to causal or one-sided *z-transform* (zT), $Y(z) = H(z) V(z)$, with $p \times m$ *transfer matrix*:

$$H(z) = D + C(z I_n - A)^{-1} B, \quad \text{for} \quad |z| > \rho.$$

3) Stability: The system is said to be *externally stable* when any bounded input gives rise to a bounded ouput, that is when $\sum_{k=0}^{\infty} \|h_k\| < \infty$ or, equivalently, when the transfer matrix $z \mapsto H(z)$ has **no poles outside the open unit disc** $|z| \geq 1$, that is if and only if $\rho < 1$. Only in this case are we allowed to compute the discrete Fourier transform (DFT) directly from the zT:

$$\hat{h}(\nu) := H(z = e^{2i\pi\nu}), \quad \forall \nu \in (-\frac{1}{2}, \frac{1}{2}),$$

where ν is the dimensionless discrete frequency, such that the corresponding analog frequency reads $f = \nu / T_s$.

The system is said to be *asymptotically stable* when the free trajectories converge to 0 in the phase space $\mathbb{R}^n$: this happens if and only if $|\operatorname{spec}(A)| < 1$, i.e. matrix A has **no eigenvalues outside the open unit disc**.

4) Controllability and Observability: The definitions and notions of controllability on $[0, N]$ are the same as in continuous time, as summarized in § II-A.4, the only difference is that the number N of time steps for control or observation needs to be greater than or equal to the state dimension: $N \geq n$, otherwise the properties are not equivalent.

Here again, both controllability and observability notions can be checked thanks to simple algebraic criteria, rank n for $\mathcal{C}$ in (3) or $\mathcal{O}$ in (4).

The discussion on *minimal representation* also applies to the discrete time setting, with the interesting consequences listed above.

5) Observer-Based Controllers: The stabilization of an unstable system is possible when it is controllable; and the estimation of the full state from the measurements is possible when the system is observable. Thus, as in continuous time, the separation principle applies, and efficient observer-based controller can be built, provided the system is minimal, and both $A + BK$, $A + LC$ are stable, i.e. their eigenvalues are located inside the unit circle. A specificity of discrete time relies on the existence of exact observer in a finite number of steps: it is enough to choose L such that all the eigenvalues of $A + LC$ are 0, so that the observer will be exact in at most n steps: these are called deadbeat observers.

C. Application to a pipe modelled by a connection of cylinders

1) Physical model: Following [18], [32], or [33] and references therein, we uniformly discretize the profile of an acoustic tube in n elementary cylinders of length L_s and radii R_i, and write the conservation laws for pressure and volume velocity at the interfaces: decomposing the solution into *ingoing* and *outgoing* waves, we obtain a classical Kelly-Lochbaum network with reflection coefficients r_i related to the impedances $Z_i = \frac{\rho c}{\pi R_i^2}$:

$$r_i := \frac{Z_i - Z_{i+1}}{Z_i + Z_{i+1}} = \frac{{R_{i+1}}^2 - {R_i}^2}{{R_{i+1}}^2 + {R_i}^2} \tag{10}$$

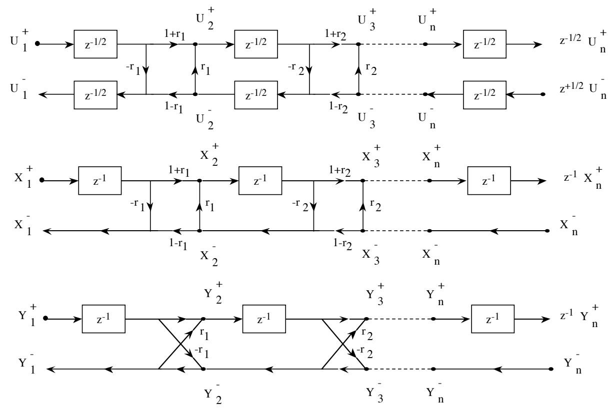

Fig. 1. Three equivalent representations of the discretized vocal tract: (top): original volume velocity variables, (middle): delayed variables, (bottom): normalized and delayed variables.

and delays on each branch of the network; sampling the signals, delaying and normalising them gives rise to a classical *lattice filter* with delays on the upper branch of the network only, see bottom of Fig. 1.

2) Mathematical model: Taking the n delayed and normalized quantities as state variables, the $m = 2$ ingoing waves (at the opposite ends) as inputs, and the $p = 2$ outgoing waves (at the opposite ends) as outputs, following [19], [20], we obtain the following state space representation of the network in discrete-time, of the form (8)-(9), with the special choice of variables:

- input vector $v = [Y_1^+ Y_n^-]$ of dimension $m = 2$,
- state vector $X = [z^{-1}\, Y_1^+ \cdots z^{-1}\, Y_n^+]$ of dimension n,
- output vector $y = [Y_n^+ Y_1^-]$ of dimension $p = 2$.

and corresponding matrices:

- the matrix of dynamics

$$A_n = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1-{r_1}^2 & -r_1 r_2 & \cdots & -r_1 r_{n-1} & \vdots \\ 0 & 1-{r_2}^2 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & -r_{n-2} r_{n-1} & 0 \\ 0 & \cdots & 0 & 1-{r_{n-1}}^2 & 0 \end{pmatrix} \tag{11}$$

- the two columns of the control matrix B_n

$$b_n^+ = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad b_n^- = \begin{pmatrix} 0 \\ r_1 \\ \vdots \\ r_{n-1} \end{pmatrix} \tag{12}$$

- the two rows of the observation matrix C_n

$$\begin{aligned} c_n^- &= \begin{pmatrix} -r_1 & \cdots & -r_{n-1} & 0 \end{pmatrix} \\ c_n^+ &= \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \end{aligned} \tag{13}$$

- the direct link matrix

$$D_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{14}$$

3) Analysis of the model: Following [19], we can conclude to:

- stability, since $|r_i| < 1$ and the characteristic polynomial of A_n is a Levinson polynomial,
- controllability, thanks to computations of $\mathcal{C}$,
- observability, thanks to computations of $\mathcal{O}$;

so far for the bi-port system, i.e. 2 input and 2 outputs, also called a MIMO system (multi input, multi output).

4) Boundary conditions: Taking into account static *boundary conditions* at both ends of the pipe amounts to applying a **feedback loop** on the system, and makes it a dipole or SISO system (single input, single output).

In the case of *frequency independent* reflexion coefficient R_L at the *lips* and R_G at the *glottis*, the two ends of the transmission line, we obtain a closed-loop system with multipliers in the feedback loop. These static boundary conditions applied to a lossless transmission line lead to the following matrix of dynamics:

$$A_{\text{dipole}} = \begin{pmatrix} -R_G r_1 & -R_G r_2 & \cdots & \cdots & -R_G R_L \\ 1-{r_1}^2 & -r_1 r_2 & \cdots & -r_1 r_{n-1} & -r_1 R_L \\ 0 & 1-{r_2}^2 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & -r_{n-2} r_{n-1} & -r_{n-2} R_L \\ 0 & \cdots & 0 & 1-{r_{n-1}}^2 & -r_{n-1} R_L \end{pmatrix}$$

It is nicely structured as follows:

$$A_{\text{dipole}}(R_G, R_L) = A_n - R_G\, b_n^+ c_n^- - R_L b_n^- c_n^+ - R_G R_L b_n^+ c_n^- \,.$$

The control column matrix is $b_{\text{dipole}} = \left(\frac{1+R_G}{2}\right) b_n^+$, and the observation row matrix $c_{\text{dipole}} = (1 + R_L)\, c_n^+$, with no feedthrough constant ($d_{\text{dipole}} = 0$).

In a more general case however, the boundary conditions are not static but *dynamical*: when the reflection coefficients at the boundaries *depend upon frequency* (i.e. $\omega \mapsto R_G(i\,\omega)$, $\omega \mapsto R_L(i\,\omega)$) – which is the case as soon as the physical description at the boundaries is slightly refined –, the closed-loops at the two ends of the transmission line are in fact loops of *feedback systems*. We first proceed to a causal and minimal realizations of these two transfer functions into discrete-time dynamical systems, namely (A_G, b_G, c_G, d_G) with state vector x_G for R_G, and (A_L, b_L, c_L, d_L) with state vector x_L for R_L. Then, thanks to the augmented state $X_a = [x'_G\, X'\, x'_L]'$ of dimension $n_a = n_G + n + n_L$, it is not difficult to build the global system, for which the matrix of dynamics reads:

$$A_d = \begin{pmatrix} A_G & b_G\, (c_n^- - d_L\, c_n^+) & -b_G\, c_L \\ b_n^+\, c_G & A_{\text{dipole}}(d_G,\, d_L) & -\left(b_n^- + d_L\, b_n^+\right)\, c_L \\ 0 & b_L\, c_n^+ & A_L \end{pmatrix}.$$

Autonomous dynamics can be seen on the block diagonal, while coupling is represented by off-diagonal terms; R_G, R_L

in the static case have now been replaced by d_G, d_L respectively.

These ideas stemming for automatic control will now be presented in details on another example: an augmented state space is built, that is devoted to the refined description of damping, thus introducing extra state variables (so-called memory variables).

D. *Application to damping models of classical oscillators*

In this subsection we show how realization theory together with an energy (or Lyapounov) analysis is helpful for both the simulation in time domain and the stability analysis of oscillating systems damped by a collection of memory variables. This toy model is detailed here so as to prepare a nice (but somewhat more involved) extension to fractional derivatives coupled to a wave equation in §III-D below.

1) The model: We want to analyze a 1-d.o.f oscillator, modelled by:

$$\ddot{x} + z + \dot{x} + y + \omega^2 x = 0\,, \qquad (15)$$

with 3 different types of damping:

- $\dot{x} = v$, instantaneous w.r.t v,
- $y(v)$, with memory, low-pass behaviour,
- $z(v)$, with memory, high-pass behaviour,

We now detail the last two *filters*, and provide passive minimal realizations for them.

a) Low-pass filters: A number K of RC circuits with input v and output y can be realized by the dynamical system:

$$\dot{\phi_k}(t) = -\xi_k\,\phi_k(t) + v(t), \quad \phi_k(0) = 0 \qquad (16)$$

$$y(t) = \sum_{k=1}^{K} \mu_k\,\phi_k(t) \qquad (17)$$

with energy ($\mu_k > 0$) $E_\phi(t) = \frac{1}{2}\sum_{k=1}^{K} \mu_k\,{\phi_k}^2(t)$ and energy balance:

$$\dot{E}_\phi = -\sum_{k=1}^{K} \mu_k\,\xi_k\,\phi_k^2 + v\,y \qquad (18)$$

Note that a *positive* aggregation of RC-circuits is passive and low-pass (-6 dB/oct):

$$H_{RC}^K(s) = \sum_{k=1}^{K} \mu_k\,\frac{1}{s+\xi_k} \quad \text{with } \mu_k > 0. \qquad (19)$$

since 1 RC-circuit is passive and low-pass (-6 dB/oct): indeed $\Re e(H_{RC}(s)) = \frac{\Re e(s)+\xi}{|s+\xi|^2} \geq 0$ for $\Re e(s) \geq 0$; and for any causal input v of finite energy, $\int_0^\infty y(t)v(t)\,\mathrm{d}t = \int_{\mathbb{R}} \hat{y}(f)\,\hat{v}(f)^*\,\mathrm{d}f = \int_{\mathbb{R}} H(2i\pi f)\,|\hat{v}(f)|^2\,\mathrm{d}f = 2\int_0^\infty \Re e(H(2i\pi f))\,|\hat{v}(f)|^2\,\mathrm{d}f \geq 0$.

b) High-pass filters: A number L of RL circuits with input v and output z can be realized by the dynamical system:

$$\dot{\psi_l}(t) = -\xi_l\,\psi_l(t) + v(t), \quad \psi_l(0) = 0 \qquad (20)$$

$$z(t) = \sum_{l=1}^{L} \nu_l\,(v(t) - \xi_l\,\psi_l)(t) \qquad (21)$$

with energy ($\nu_l > 0$) $E_\psi(t) = \frac{1}{2}\sum_{l=1}^{L} \nu_l\,\xi_l\,{\psi_l}^2(t)$ and energy balance:

$$\dot{E}_\psi = -\sum_{l=1}^{L} \nu_l\,(v - \xi_l\,\psi_l)^2 + v\,z \qquad (22)$$

Note that a *positive* aggregation of RL-circuits is passive and high-pass ($+6$ dB/oct):

$$H_{RL}^L(s) = \sum_{l=1}^{L} \nu_l\,\frac{s}{s+\xi_l} \quad \text{with } \nu_l > 0.$$

since one RL-circuit is passive and high-pass ($+6$ dB/oct): indeed, $\Re e(H_{RL}(s)) = \frac{|s|^2+\Re e(s)\,\xi}{|s+\xi|^2} \geq 0$, for $\Re e(s) \geq 0$.

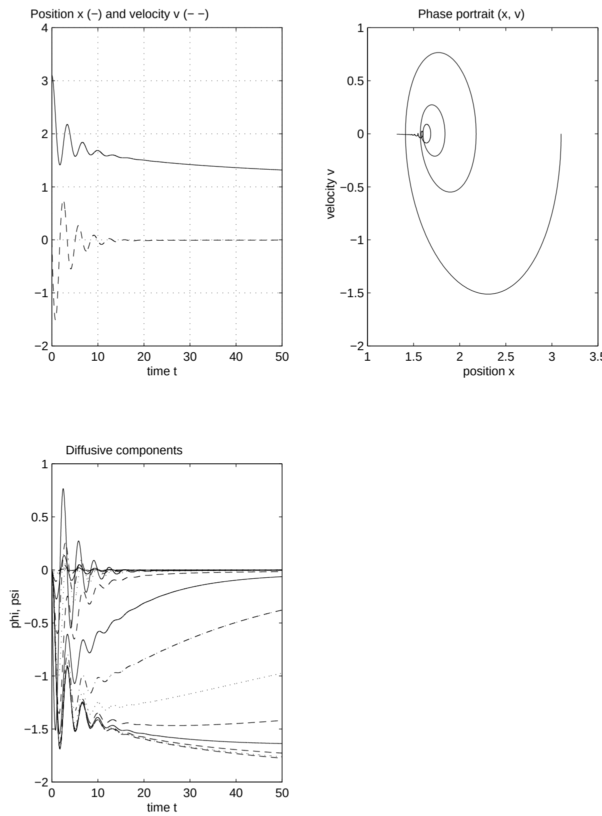

Fig. 2. Oscillator with memory damping. (top left): position x and velocity v versus time, (top right): cut in the phase space $(x,\,v) \in \mathbb{R}^2$, (bottom left): diffusive variables $\phi_k(t)$ and $\psi_l(t)$.

2) Analysis of the coupled system: The analysis of the 1-d.o.f oscillator (15), with associated mechanical energy $E_m(t) := \frac{1}{2}v^2(t) + \frac{1}{2}\omega^2\,x^2(t)$, can be easily performed thanks to an augmented energy, or *Lyapunov functional*:

$$\mathcal{E} := E_m + E_\phi + E_\psi$$

of the global system, with internal variables $[x, v, \phi, \psi]$. Indeed, using (18) and (22), we get:

$$\dot{\mathcal{E}} = -v^2 - \sum_{k=1}^{K} \mu_k\,\xi_k\,\phi_k^2 - \sum_{l=1}^{L} \nu_l\,(v - \xi_l\,\psi_l)^2 \leq 0\,.$$

Hence, we can conclude to asymptotic stability, i.e. $\mathcal{E}(t) \to 0$ as $t \to +\infty$, thanks to LaSalle's invariance principle, which is easy to apply in finite dimension. Note that there is no reason for the mechanical energy $E_m(t)$ to be a strictly decreasing function of time.

This behaviour is clearly illustrated in Fig. 2; especially the behaviour of the diffusive components (on bottom left) highly depends on the values of ξ_k (short-time memory for great ξ_k, or long-time memory for small ξ_k).

III. Infinite dimensional case

In this part, we consider the case of wind instruments which can be seen as time delay systems. In fact, a wind instrument is usually made of a linear acoustic resonator (the pipe) coupled with a nonlinear oscillator (the mouth of the instrument) (see e.g. [16], [6]). The resonator can be modeled through hyperbolic wave equations. We will first recall the d'Alembert equation in § III-A, its controllability property in the case of ideal boundary conditions of Cauchy type in § III-B, together with the notion of exponential stability of a periodic orbit corresponding to a desired note.

Then, in § III-C, we will consider an example of a **slide flute**, that is a kind of recorder without finger holes but which is ended by a piston mechanism to modify the length of the resonator (see Fig. 3). The fact that we can control the piston produces a moving boundary system. But it will be possible to control this virtual instrument and stabilize a periodic orbit through a suitable control of the pipe length. The proof of stability will be handled using a Lyapunov function, e.g. a kind of energy function which can decay w.r.t. time using a suitable boundary feedback to control the pipe length. This result will be obtained assuming an ideal boundary condition at the mouth of the instrument, saying that the pressure at the entrance of the pipe is zero.

Note that in another paper [5], the authors consider a more realistic model of the mouth, taking into account the coupling effects between the acoustic field of the resonator and the air jet obtained by blowing through a flue channel and formed by flow separation at the flue exit. The resulting boundary conditions are much more complicated and have been linearized to perform a modal analysis and a control algorithm. But in that case, the stability question and the elaboration of an associated Lyapunov function remain open (see also [7] for details). This is the reason why we have chosen to consider a simplified boundary condition at the entrance of the pipe to be able to perform the stability analysis of the periodic orbit through Lyapunov analysis.

Finally, in § III-D, we examine the introduction of damping in the flute model, and make use of the toy model, as developped in § II-D, to reformulate the system in view of his control.

A. Physical model of the pipe

We will denote ρ_0 the fluid (here the air) density at rest, S_p the constant section of the pipe which is supposed to be cylindrical. We assume that the flow rate $u(x,t)$ at time t and point x in the pipe and the relative pressure $p(x,t) = P - P_{atm}$ (P_{atm} denoting the atmospheric pressure) are uniform on a section. Therefore the Euler equation, giving the fluid dynamical properties (23) neglecting the viscous and thermal effects near the walls:

$$\frac{\partial u}{\partial t} = -\frac{S_p}{\rho_0}\frac{\partial p}{\partial x} \tag{23}$$

and the mass conservation law (24)

$$\frac{\partial p}{\partial t} = -\frac{\rho_0 c^2}{S_p}\frac{\partial u}{\partial x} \tag{24}$$

lead to the wave equation also known as d'Alembert equation:

$$\frac{\partial^2 p}{\partial t^2} - c^2\frac{\partial^2 p}{\partial x^2} = 0. \tag{25}$$

Equations (23) and (24) allow to write the system dynamics in the following state-space form with $X = \begin{pmatrix} u \\ p \end{pmatrix}$:

$$\frac{\partial X}{\partial t} + A\frac{\partial X}{\partial x} = 0\,,\ \text{with } A = \begin{pmatrix} 0 & S_p/\rho_0 \\ \rho_0 c^2/S_p & 0 \end{pmatrix}. \tag{26}$$

This representation can be diagonalized :

$$\partial_t Z + \Lambda \partial_x Z = 0\,,\ \text{with } \Lambda = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \tag{27}$$

where the change of coordinates is given by :

$$Z = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} u + \dfrac{S_p}{\rho_0 c}p \\ u - \dfrac{S_p}{\rho_0 c}p \end{pmatrix} \tag{28}$$

and

$$X = \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha+\beta}{2} \\ \dfrac{\rho_0 c(\alpha-\beta)}{2S_p} \end{pmatrix}. \tag{29}$$

The eigenvalues $c > 0$ and $-c < 0$ being respectively the velocity of the ingoing wave $\alpha(x,t)$ and of the outgoing wave $\beta(x,t)$. $\alpha(x,t)$ and $\beta(x,t)$ satisfy two classical wave equations:

$$\frac{\partial \alpha}{\partial t} + c\frac{\partial \alpha}{\partial x} = 0 \text{ and} \tag{30}$$

$$\frac{\partial \beta}{\partial t} - c\frac{\partial \beta}{\partial x} = 0. \tag{31}$$

Since $\alpha(x,t)$ and $\beta(x,t)$ are constant along the "characteristic curves", α and β are called the **Riemann invariants** (see e.g. [30, Tome II, Chap. 12]).

Let us introduce the following notations:

$$\alpha_0(t) = \alpha(x=0,t) \text{ and } \beta_0(t) = \beta(x=0,t). \tag{32}$$

Then, the behavior of $\alpha(x,t)$ is a time delay system from α_0:

$$\alpha(x,t) = \alpha_0(t - \frac{x}{c}) \tag{33}$$

and conversely we have:

$$\beta(x,t) = \beta_0(t + \frac{x}{c}). \tag{34}$$

We will denote in the sequel τ_p the delay due to the length of the pipe:

$$\tau_p = \frac{L}{c}. \tag{35}$$

B. Controllability and Stability of a simplified Cauchy problem

Using (25), we consider in this section a simple Cauchy problem of the following form:

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{\partial^2 p}{\partial x^2} = 0 \tag{36}$$

for $t \in [0,T]$ and $x \in [0,L]$ with boundary conditions:

$$p(L,t) = 0 \ , \ p_x(0,t) = v(t). \tag{37}$$

Equations (36) and (37) can model a wind instrument with a lossless resonator of length L, an open end corresponding to $p(L,t) = 0$ and a controlled flow at the entrance corresponding to $p_x(0,t) = v(t)$, $v(t)$ being a *control function*. Let us also consider initial conditions:

$$p(x,0) = p^0(x) \ , \ p_t(x,0) = q^0(x) \ , \ x \in [0,L]. \tag{38}$$

Let us denote $H^1_{(0)}(0,L)$ the Sobolev space:

$$\{p \in L^2(0,L), \ p_x \in L^2(0,L), \ p(0) = 0\}.$$

Assuming that p^0 belongs to $H^1_{(0)}(0,L)$ and q^0 belongs to $L^2(0,L)$ we have the following controllability result (see e.g. [36], [9, Sec. 2.4.2]):

Theorem 1. *Let $T > 2L$. The control system (36)-(37) is controllable in time T, that is: for every $(p^0,q^0) \in H^1_{(0)}(0,L) \times L^2(0,L)$ and every $(p^1,q^1) \in H^1_{(0)}(0,L) \times L^2(0,L)$, there exists $v(t) \in L^2(0,T)$ such that the solution p of the Cauchy problem (36)-(37)-(38) satisfies $(p(.,T), p_t(.,T)) = (p^1,q^1)$.*

Let us now recall the so-called exponential stability property of an equilibrium point or equilibrium orbit $\bar{p}(x,t)$ of (36)-(37). If e denotes the error $e(x,t) = p(x,t) - \bar{p}(x,t)$, we define:

Definition 1. *The equilibrium solution $\bar{p}(x,t)$ of (36)-(37) is exponentially stable if there exist constants $\lambda > 0$, $C > 0$ such that for every $E(0) > 0$ the following inequality holds:*

$$| \, E(t) \, | \leq Ce^{-\lambda t} E(0) \tag{39}$$

where $E(t)$ is the norm:

$$E(t) = \frac{1}{2} \int_0^L (p_t^2 + p_x^2) dx. \tag{40}$$

The objective of the control is usually to render exponentially stable an equilibrium point or orbit which is not naturally stable. We will now examine the case of a particular wind instrument which is called a slide flute.

C. Boundary control of a slide flute

a) The control model: Since we modify the length of the resonator using the piston, the boundary is moving. Therefore, as it has been done in [3] in the case of an overhead crane with a **variable length flexible cable**, it is interesting to apply the following change of variable

$$x = L\sigma \tag{41}$$

to transform the system in a one with a fixed spatial domain for σ, i.e. $\sigma \in [0,1]$.

According to (41), if we denote:

$$\begin{cases} \tilde{\alpha}(\sigma,t) = \alpha(x,t) = \alpha(L(t)\sigma,t) \\ \tilde{\beta}(\sigma,t) = \beta(x,t) = \beta(L(t)\sigma,t) \end{cases} \tag{42}$$

equations (30) and (31) become:

$$\begin{cases} \dfrac{\partial \tilde{\alpha}}{\partial t}(\sigma,t) + \left(\dfrac{c - \dot{L}\sigma}{L}\right) \dfrac{\partial \tilde{\alpha}}{\partial \sigma}(\sigma,t) = 0 \\ \dfrac{\partial \tilde{\beta}}{\partial t}(\sigma,t) - \left(\dfrac{c + \dot{L}\sigma}{L}\right) \dfrac{\partial \tilde{\beta}}{\partial \sigma}(\sigma,t) = 0. \end{cases} \tag{43}$$

We still have two wave equations, but with time variable velocities depending on the pipe length and on the control variable $\dot{L}$.

b) The boundary conditions: The ideal case of a mouth aperture and a rigid closed end due to the piston mechanism is such that the slide flute can be viewed as a closed-open pipe for which the fundamental frequency f of the note is related to the pipe length as follows (see e.g. [27], [12], [15]):

$$f = \frac{c}{4L}. \tag{44}$$

We have considered a lossless model without friction. Moreover, as we have already said in introduction, we do not consider any mouth model excited by a blowing pressure, but only a simplified model with an ideal mouth aperture leading to a zero pressure at the entrance of the flute. Consequently, with suitable initial conditions for the pressure and the flow, close to the desired equilibrium orbit, it is not necessary to introduce an extra input air flow to compensate losses. We can then write at $x = 0$ the ideal boundary condition:

$$p(0,t) = \frac{\rho_0 c}{2S_p}(\alpha(0,t) - \beta(0,t)) = 0. \tag{45}$$

Similarly, $u(L,t) = 0$, means that there is no loss at the pipe end which can be viewed as a ridig wall (see e.g. [12]). But, in our case, the piston mechanism leads to the following mechanical dynamics at $x = L$:

$$p(L,t) + F = m\ddot{L} \tag{46}$$

where F is the force exerted on the piston and m its mass (see Fig. 3).

Remark 1. *Notice that $u(L,t) = S_p\dot{L}$. Then from (46) we have $p(L,t) + F = \dfrac{m}{S_p} u_t(L,t)$. But from Euler equation,*

$u_t(L,t) = -\frac{S_p}{\rho_0} p_x(L,t)$. *Therefore, if we introduce the control function* $v(t) = -\frac{\rho_0}{S_p}(p(L,t)+F)$, F *being the external force and* $p(L,t)$ *the known pressure usually measured at the end of the instrument, the boundary conditions can be interpreted as Cauchy type conditions analogous to (37) (bounds* $x=0$ *and* $x=L$ *have just simply been inverted):*

$$p(0,t) = 0\ ,\ p_x(L,t) = v(t).$$

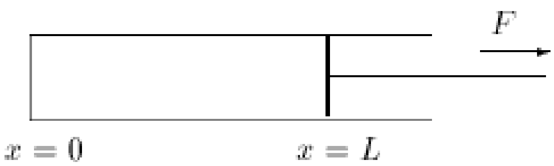

Fig. 3. The slide flute

In a first step, we can consider that the control input is the piston velocity $\dot{L}$, related to the physical control F (homogeneous to $\ddot{L}$) by an integrator or "cascaded system" (46). Therefore, using "backstepping" technics, we can compute the physical control input F which should be applied to the system if we know $\dot{L}$ (see for example e.g. [2]). Without loss of generality we can then consider the boundary condition at $x=L$:

$$u(L,t) = S\dot{L}. \tag{47}$$

Using (29), (42) and (45)-(47), the boundary conditions can be rewritten in the $\tilde{\alpha}$ and $\tilde{\beta}$ variables:

$$\begin{cases} \tilde{\alpha}(0,t) - \tilde{\beta}(0,t) = 0 \\ \tilde{\alpha}(1,t) + \tilde{\beta}(1,t) = 2S\dot{L} \end{cases} \tag{48}$$

Finally, the control model which is considered is given by (43) with the associated boundary conditions (48). The equilibrium orbit we want to stabilize is for example of the form:

$$\bar{p}(x,t) = \cos(\frac{2\pi f_0 x}{c} + \frac{\pi}{2})\cos(2\pi f_0 t) \tag{49}$$

where f_0 is the frequency of the expected note (see 44)). Using Euler equation, we deduce that the corresponding flow has the following form:

$$\bar{u}(x,t) = \frac{S_p}{\rho_0 c}\sin(\frac{2\pi f_0 x}{c} + \frac{\pi}{2})\sin(2\pi f_0 t). \tag{50}$$

We have seen in Remark 1 that the boundary conditions are of Cauchy type. But, if we would like to rewrite d'Alembert equation (25) in (σ,t) to work in the fixed domain $[0,1]$ for σ, the resulting equation would be also hyperbolic but with more complicated terms as in [3], and the controllability property has not yet been precisely established for that system. It is nevertheless possible to analyze the stability of the natural periodic orbit and to exponentially stabilize it for the slide flute through a suitable Lyapunov function, analogous to the energy function given by (40). This is the object of the following sections.

c) The control problem: We want the solution $(\tilde{\alpha}(\sigma,t), \tilde{\beta}(\sigma,t))$ to converge asymptotically towards the periodic orbit $(\bar{\alpha}(\sigma,t), \bar{\beta}(\sigma,t))$ given by (49) and (50), which can be written from (28):

$$\begin{cases} \bar{\alpha}(\sigma,t) = \frac{S}{\rho_0 c}\cos(2\pi f_0(t - \frac{L_0\sigma}{c}) - \pi/2) \\ \bar{\beta}(\sigma,t) = -\frac{S}{\rho_0 c}\cos(2\pi f_0(t + \frac{L_0\sigma}{c}) + \pi/2) \end{cases} \tag{51}$$

f_0 being the frequency corresponding to the desired note, related to the set value of the pipe L_0 by (44).

d) Study of the equilibrium orbit stability: In the case of an equilibrium orbit with frequency f_0, we know that for an open-closed pipe the modes are odd multiples of f_0 related to L_0 by (44) (see [27], [12], [15]). Therefore, the solution is purely oscillatory and it is important to elaborate a boundary control to obtain local exponential stability.

e) Elaboration of the stabilizing boundary control: As in [10], [13], [11] or [1], [2], [3], we will consider a Lyapunov function candidate to elaborate the stabilizing control law. Let us define V be as follows:

$$\begin{aligned} V &= A\int_0^1 (\tilde{\alpha}(\sigma,t) - \bar{\alpha}(\sigma,t))^2\, d\sigma \\ &+ A\int_0^1 (\tilde{\beta}(\sigma,t) - \bar{\beta}(\sigma,t))^2\, d\sigma + \frac{1}{2}(L - L_0)^2 \end{aligned} \tag{52}$$

where A is an arbitrary strictly positive constant.

If we denote to simplify:

$$\begin{aligned} &\tilde{\alpha}_0 = \tilde{\alpha}(0,t),\ \bar{\alpha}_0 = \bar{\alpha}(0,t),\ \tilde{\alpha}_1 = \tilde{\alpha}(1,t),\ \bar{\alpha}_1 = \bar{\alpha}(1,t) \\ &\tilde{\beta}_0 = \tilde{\beta}(0,t),\ \bar{\beta}_0 = \bar{\beta}(0,t),\ \tilde{\beta}_1 = \tilde{\beta}(1,t),\ \bar{\beta}_1 = \bar{\beta}(1,t) \end{aligned} \tag{53}$$

differentiating V with respect to time and using (43)-(48), we obtain:

$$\begin{aligned} &\dot{V} = -\frac{\dot{L}}{L}V + \dot{L}(L - L_0) + \\ &\frac{Ac}{L}\left((\tilde{\alpha}_0 - \bar{\alpha}_0)^2 - (\tilde{\alpha}_1 - \bar{\alpha}_1)^2 + (\tilde{\beta}_1 - \bar{\beta}_1)^2 - (\tilde{\beta}_0 - \bar{\beta}_0)^2\right) \\ &+ \frac{A\dot{L}}{L}\left((\tilde{\alpha}_1 - \bar{\alpha}_1)^2 + (\tilde{\beta}_1 - \bar{\beta}_1)^2\right) \end{aligned} \tag{54}$$

But from (48) we have:

$$\tilde{\alpha}(0,t) - \tilde{\beta}(0,t) = 0$$

and from (51) we can also write:

$$\bar{\alpha}(0,t) - \bar{\beta}(0,t) = 0,$$

which leads to:

$$(\tilde{\beta}_0 - \bar{\beta}_0)^2 = (\tilde{\alpha}_0 - \bar{\alpha}_0)^2$$

then the term $\left[(\tilde{\alpha}_0 - \bar{\alpha}_0)^2 - (\tilde{\beta}_0 - \bar{\beta}_0)^2\right]$ is zero in the expression (54) of $\dot{V}$.

It remains to study the following term:

$$\mathcal{Z} = (\tilde{\beta}_1 - \bar{\beta}_1)^2 - (\tilde{\alpha}_1 - \bar{\alpha}_1)^2. \tag{55}$$

From (48), (51) and (44) we have:

$$\begin{cases} \tilde{\alpha}(1,t) + \tilde{\beta}(1,t) = 2S\dot{L} \text{ and} \\ \bar{\alpha}_1 + \bar{\beta}_1 = \dfrac{S}{\rho_0 c}(\cos(2\pi f_0 t - \pi) - \cos(2\pi f_0 t + \pi)) = 0 \end{cases}$$

denoting

$$\delta = \tilde{\alpha}_1 - \bar{\alpha}_1 \tag{56}$$

$\mathcal{Z}$ can be rewritten:

$$\mathcal{Z} = 4S^2\dot{L}^2 - 4S\dot{L}\delta, \tag{57}$$

which yields finally:

$$\begin{cases} \dot{V} = \mathcal{A}\dot{L}^2 + \dot{L}\left(\mathcal{B} + \dfrac{A(2\delta^2 + \mathcal{Z})}{L}\right) \text{ with } \mathcal{A} = \dfrac{4AcS^2}{L} \\ \text{and } \mathcal{B} = L - L_0 - \dfrac{V}{L} - \dfrac{4AcS\delta}{L} \end{cases} \tag{58}$$

This expression of $\dot{V}$ as a second order polynomial in $\dot{L}$ implies the following boundary control $\dot{L}$ to make the expected periodic solution (51) asymptotically stable, at least locally:

$$\dot{L} = -\frac{1}{2\mathcal{A}}\left(\mathcal{B} + \frac{A(2\delta^2 + \mathcal{Z})}{L}\right). \tag{59}$$

¿From (55), (59) is a second order equation in $\dot{L}$ of the form:

$$a(L)\dot{L}^2 + b(L,\delta)\dot{L} + c(L,V,\delta) = 0,\ a(L) > 0. \tag{60}$$

Near the equilibrium orbit, the discriminant Δ of (60) is close to $(8AcS^2)^2$. Therefore, Δ is positive and the solution $\dot{L}$ corresponding to the suitable root is:

$$r_1 = \frac{-b(L,\delta) + \sqrt{\Delta}}{2a(L)}. \tag{61}$$

Let us now state the main theorem of the section.

Theorem 2. *For every constant $\epsilon > 0$, there exists a constant $\nu > 0$ such that each solution of system (43) with boundary conditions (48) in closed loop with the control law (59) is such that:*

$$\begin{aligned} &\mid L(0) - L_0 \mid + \mid \tilde{\alpha}(.,0) - \bar{\alpha}(.,0) \mid_{L^2(0,1)} + \\ &\mid \tilde{\beta}(.,0) - \bar{\beta}(.,0) \mid_{L^2(0,1)} < \nu \end{aligned} \tag{62}$$

is defined for all $t \geq 0$ and satisfies:

$$\begin{aligned} &\mid L(t) - L_0 \mid + \mid \tilde{\alpha}(.,t) - \bar{\alpha}(.,t) \mid_{L^2(0,1)} + \\ &\mid \tilde{\beta}(.,t) - \bar{\beta}(.,t) \mid_{L^2(0,1)} < \epsilon. \end{aligned} \tag{63}$$

Therefore, there exists $\eta > 0$ such that if:

$$\begin{aligned} &\mid L(0) - L_0 \mid + \mid \tilde{\alpha}(.,0) - \bar{\alpha}(.,0) \mid_{L^2(0,1)} + \\ &\mid \tilde{\beta}(.,0) - \bar{\beta}(.,0) \mid_{L^2(0,1)} < \eta \end{aligned} \tag{64}$$

then for all t tending to $+\infty$:

$$\begin{aligned} &\mid L(t) - L_0 \mid + \mid \tilde{\alpha}(.,t) - \bar{\alpha}(.,t) \mid_{L^2(0,1)} + \\ &\mid \tilde{\beta}(.,t) - \bar{\beta}(.,t) \mid_{L^2(0,1)} \longrightarrow 0. \end{aligned} \tag{65}$$

For sake of clarity, the proof is given in Appendix.

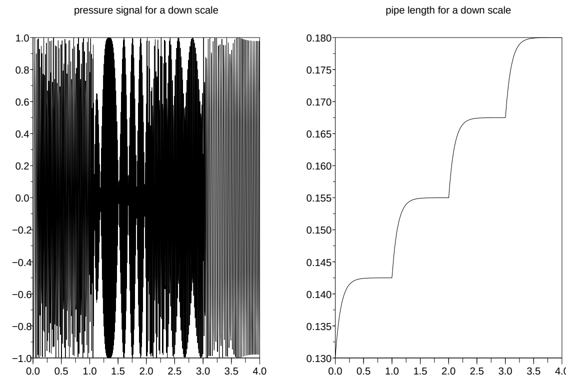

Fig. 4. Acoustic pressure and pipe length for a down scale

f) Simulation results: We have applied a simplified version of the control law (59), neglecting the values of δ and $\mathcal{Z}$. Fig. 4 displays the results we obtain when the player wants to produce a down scale by sliding the piston.

We can see that the length L of the pipe tends to the successive set points in about $1\,s$ (the sampling period being equal to $22050\,Hz$) and the piston velocity is smaller than $20\,cm/s$, which corresponds to realistic player gestures. The system has been numerically simulated using the software package "Scilab" (see for example [4]) from equations (30), (31) and (29) which allow to compute the physical values of the pressure and the flow. The Scilab function `playsnd` (see [4]) allows to appreciate the transient effect of the dynamics of the piston. The different notes sound a little bit "metallic", which is not surprising since we have considered an ideal boundary condition at the entrance of the pipe. Of course, quite realistic sounds have been synthesized when we take into account physical models for the excitation mechanism and the dynamics of the air jet as in [7]. But, in that case, the entrance boundary conditions are much more complicated so that the problem of finding a Lyapunov function candidate to prove the stability remains still open.

D. *Introducing damping in the flute model*

Realistic pipes are lossy, due to friction of the air near the wall which induces viscous and thermal effects; a refined description of those gives rise to the Lokshin model with damping of fractional order in time, see e.g. [28]: for $z \in (0,\,1)$, with $r(z) > 0$, $\eta(z), \varepsilon(z) \geq 0$, $w(t,z)$ satisfies:

$$\partial_t^2 w + \eta(z)\,\partial_t^{3/2} w + \varepsilon(z)\,\partial_t^{1/2} w - \frac{1}{r^2}\partial_z(r^2\,\partial_z w) = 0\,; \tag{66}$$

with static boundary conditions in $z = 0$ and $z = 1$.

The model is non standard, since:

- there is no simple energy property, due to fractional derivatives in time,
- the coefficients are variable with space: $\eta \mapsto \eta(z)$, and $\varepsilon \mapsto \varepsilon(z)$: no closed-form solution as in [23].

But still, existence, uniqueness and asymptotic stability can be proved, using diffusive realization in a somewhat more involved form as in § II-D, but based on the same principles. The underlying idea is to go from the finite number of d.o.f. to an infinite (and even continous) number of d.o.f., while preserving the fundamental passivity property. It relies on:

$$\int_0^\infty \mu_\beta(\xi)\,\frac{1}{s+\xi}\,\mathrm{d}\xi = \frac{1}{s^\beta}\,, \quad \text{with} \quad \mu_\beta(\xi) \propto \frac{1}{\xi^\beta}\,,$$

for $\Re e(s) > 0$, and also:

$$\int_0^\infty \nu_\alpha(\xi)\,\frac{s}{s+\xi}\,\mathrm{d}\xi = s^\alpha\,, \quad \text{with} \quad \nu_\alpha(\xi) \propto \frac{1}{\xi^{1-\alpha}}\,.$$

Hence, fractional integrals of order $0 < \beta < 1$ and fractional derivatives of order $0 < \alpha < 1$ are continuous positive aggregations of low-pass (RC) & high-pass (RL) circuits, respectively.

The quantities of interest are the classical wave energy

$$E_m(t) = \frac{1}{2}\int_0^1 [(\partial_t w)^2(z,t) + (\partial_z w)^2(z,t)]\, r^2(z)\,\mathrm{d}z\,,$$

and the diffusive energies for fractional integrals

$$E_\phi(t) = \frac{1}{2}\int_0^1 \int_0^\infty \mu_{1/2}(\xi)\,\phi(\xi,z,t)^2\,\mathrm{d}\xi\,\epsilon(z)\,\mathrm{d}z\,,$$

and for fractional derivatives

$$E_\psi(t) = \frac{1}{2}\int_0^1 \int_0^\infty \nu_{1/2}(\xi)\,\xi\,\psi(\xi,z,t)^2\,\mathrm{d}\xi\,\eta(z)\,\mathrm{d}z\,.$$

Defining an augmented first-order dynamical system with an augmented energy $\mathcal{E} = E_m + E_\phi + E_\psi$, the methodology of § II-D can be applied, at least formally, to study the Webster-Lokshin equation as a coupled system of the form (15). Once again, this reformulation of (66) as a first order system in both the wave variables $(\partial_z w, \partial_t w)$ and the continuous collection of memory (or diffusive) variables $(\phi(\xi,.), \psi(\xi,.))$ proves useful for analysis and numerical simulations (especially the design of ad hoc numerical schemes which preserve the energy balance at a discrete level). But, since the mathematical tools involved are very technical, it has been chosen not to present them in this overview paper. The interested reader is referred to [22], [21], [25], [26].

IV. Conclusion and perspectives

In this paper, we have shown both the interests of realization theory and automatic control when applied to modelling, simulation and control of musical instruments, at least for simplified ones.

In the finite dimensional case, we have presented a first example in discrete time of a lossless pipe, such as the vocal tract or a flared acoustic pipe; we have studied a second example of a mechanical oscillator damped by memory mechanism, which is being used in viscoelasticity.

In the infinite dimensional case, we have proved exponential stability of a slide flute using a suitable boundary control for ideal boundary conditions of Cauchy type. For more realistic models of the mouth taking into account the coupling effects between the acoustic field of the resonator and the air jet (see e.g. [14], [5], [7]), the resulting boundary conditions at the mouth are more complicated and have been linearized to perform a modal analysis and a control algorithm. But in that case, the stability question and the elaboration of an associated Lyapunov function remain open (see also [7] for details). Moreover, we have given hints to show how some non standard damping models for the wave equation can also be recast into the framework of linear control systems, introducing extra memory variables and checking energy balances in order to get a well-posed model.

The questions of optimal control have not been addressed so far in the present paper, they mix control systems with optimization techniques: for short, the dynamical system is viewed as a constraint, and so-called Lagrange variables are defined so as to respect this family of constraints; the nice result is that, upon minimisation of a cost functional, the Lagrange multipliers do follow an adjoint dynamical system (which involves the adjoint of the matrix of dynamics in the linear case, for example). A very interesting and original application of these techniques to the design of wind instruments is presented in [17].

Now, it is of utmost importance to emphasize that even if sounds are usually decomposed into harmonics thanks to Fourier analysis (even short-time Fourier analysis), it is not sufficient for realistic descriptions especially in the transient regimes. It thus proves necessary to go beyond the linear and time-invariant framework. But the problems of control, when non-linear, time-dependent and even more infinite-dimensional are really hard to tackle, and not suited for a tutorial paper; nevertheless, the interested reader is referred to e.g. [9] for up to date mathematical issues, and also to [29], [34] for very interesting and promising developments of nonlinear tools in the field of simulation and control of musical instruments.

V. Acknowledgements

The authors are indebted to J. Kergomard for his numerous advices, that helped greatly improve the original version of this research report.

Appendix

To prove the local exponential stability result of Theorem 2, we first notice that from (58) and (59) $\dot{V}$ can be rewritten:

$$\dot{V} = -\frac{\left(\mathcal{B} + \dfrac{A(2\delta^2 + \mathcal{Z})}{L}\right)^2}{4\mathcal{A}} \leq 0. \qquad (67)$$

The Lyapunov function is clearly decreasing. To conclude, we have to study the convergence of the solutions. For that purpose, we will use a LaSalle invariance principle, in the infinite dimension case. Let us then study the invariant solutions satisfying $\dot{V} = 0$, which is equivalent from (67) and (59) to the condition $\dot{L} = 0$. In that case, V and L are constant, $\mathcal{Z}$ is zero and from (60), $c(L, V, \delta) = 0$, which implies that δ given by (56) is itself constant.

Moreover, if $\dot{L} = 0$, using (43), $\tilde{\alpha}(\sigma, t)$ est de la forme:

$$\tilde{\alpha}(\sigma, t) = \phi(t - \frac{L\sigma}{c}). \qquad (68)$$

Then using (51), we can write:

$$\phi(t) = \delta + \frac{S}{\rho_0 c}\sin(2\pi f_0(t + \frac{L - L_0}{c})). \quad (69)$$

Similarly, using (43), $\tilde{\beta}(\sigma, t)$ is of the form:

$$\tilde{\beta}(\sigma, t) = \psi(t + \frac{L\sigma}{c}). \quad (70)$$

From the boundary condition at $\sigma = 0$ in (48), we deduce:

$$\psi(t) = \phi(t) = \delta\frac{S}{\rho_0 c}\sin(2\pi f_0(t + \frac{L - L_0}{c})). \quad (71)$$

Using now the boundary condition at $\sigma = 1$ in (48) with (71) and the relation $\dot{L} = 0$, we obtain:

$$\phi(t - \frac{L}{c}) = -\phi(t + \frac{L}{c}). \quad (72)$$

Replacing ϕ by its expression (69), we deduce from (72) and (44) the relation:

$$\delta + \frac{S}{\rho_0 c}\sin(\frac{\pi c t}{2L_0} - \pi/2) = -\delta - \frac{S}{\rho_0 c}\sin(\frac{\pi c t}{2L_0} + \pi/2). \quad (73)$$

This leads locally for L close to L_0 to the solution:

$$L = L_0 \text{ and } \delta = 0. \quad (74)$$

Therefore, using (74), the expression (51) of $(\bar{\alpha}(\sigma, t), \bar{\beta}(\sigma, t))$ as well as (68), (70) and (71), the expression of V given by (52) becomes:

$$V = 2A\delta^2 \quad (75)$$

which implies from (74) that V is also zero. So, the application of LaSalle invariance principle leads to:

$$\begin{cases} L = L_0 \\ \delta = V = 0 \end{cases} \quad (76)$$

which implies due to (52), that the solution $(\tilde{\alpha}(\sigma, t), \tilde{\beta}(\sigma, t))$ asymptotically converges to the equilibrium orbit $(\bar{\alpha}(\sigma, t), \bar{\beta}(\sigma, t))$ given by (51).

From a mathematical point of view, it then suffices to check the pre-compactness of the solutions to conclude as in the finite dimensional case. To do that, we can procced like in [3, Sec. 4]. $\diamondsuit$

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