Open Archive TOULOUSE Archive Ouverte (OATAO)

OATAO is an open access repository that collects the work of Toulouse researchers and makes it freely available over the web where possible.

This is a publisher version published in : [http://oatao.univ-toulouse.fr/](http://oatao.univ-toulouse.fr/)
Eprints ID : 3109

**To link to this article**
URL : [http://dx.doi.org/10.1109/TAP.2005.844434](http://dx.doi.org/10.1109/TAP.2005.844434)

**To cite this version :**

Any correspondance concerning this service should be sent to the repository administrator: staff-oatao@inp-toulouse.fr.
Lacunarity of Fractal Superlattices: A Remote Estimation Using Wavelets

Y. Laksari, H. Aubert, Senior Member, IEEE, D. L. Jaggard, Fellow, IEEE, and J. Y. Tourneret

Abstract—The lacunarity provides a useful parameter for describing the distribution of gap sizes in discrete self-similar (fractal) superlattices and is used in addition to the similarity dimension to describe fractals. We show here that lacunarity, as well as the similarity dimension, can be remotely estimated from the wavelet analysis of superlattices impulse response. As a matter of fact, the skeleton—the set of wavelet-transform modulus-maxima—of the reflected signal overlaps two hierarchical structures in the time-scale domain: such that one allows the direct remote extraction of the similarity dimension, while the other may provide an accurate estimation of the lacunarity of the interrogated superlattice. Criteria for the choice of the mother wavelet are established for impulse response corrupted by additive Gaussian white noise.

Index Terms—Inverse problem, lacunarity, noise, wavelet analysis.

I. INTRODUCTION

THE remote analysis of scaling properties of multiscale (fractal) objects may be achieved from interrogation by an incident electromagnetic wave. From the observed reflection data in the spectral (see [1] and, the references therein) or in the time [2]–[6] domain the problem consists of determining the primary fractal descriptors of interrogated objects. Recently some of us have reported [7], [8] the remote determination of fractal descriptors of discrete self-similar laminated structures denoted fractal superlattices. We have shown that the time-scale analysis is a powerful tool for detecting singularities (or abrupt changes) in the impulse response that are due to direct reflections and consequently, that such analysis may be used advantageously for the resolution of inverse problems involving discrete self-similar objects [9], [10]. We explore here the efficiency of the wavelet-based analysis when the impulse responses of fractal superlattices are corrupted by additive Gaussian white noise. In addition to the admissibility conditions, criteria for the choice of the mother wavelet are established for improving the efficiency of wavelet-based analysis. Next the inverse problem involving laminated structures constituted by two different superlattices is analyzed with success.

The present article is organized as follows: the theoretical developments of Section II deal with the determination of the impulse response of fractal superlattices corrupted by an additive Gaussian white noise. Section III is devoted to the choice of the mother wavelet and to the remote estimation of the similarity dimension and lacunarity from the wavelet analysis of this impulse response. In Section IV the proposed approach is applied to the cascade of two different superlattices. Finally, conclusions are drawn in the last section.

II. IMPULSE RESPONSE OF FRAC TAL SUPERLATTICES CORRUPTED BY AN ADDITIVE GAUSSIAN WHITE NOISE

A. Similarity Dimension and Lacunarity of Fractal Superlattices

Discrete self-similar (fractal) superlattices are multilayered structures that are designed by alternating dielectric layers of refractive index \( n_1 \) and \( n_0 (\leq n_1) \) according to an iterative process. As displayed in Fig. 1(a) for the first stages of growth, the stage of growth \( S \) consists in \( \eta \) replica of one obtained at the previous stage \( S = 1 \), each replica being reduced by a factor \( \rho \). The similarity dimension \( D_S \) associated to the resulting superlattice is defined by \( \ln(\eta)/\ln(1/\rho) \).

Two superlattices, with the same similarity dimension, may differ from the size \( \varepsilon \) of the outermost gap (normalized to the total length \( L \)) at the first stage of growth (here, a “gap” is a layer of refractive index \( n_0 \)). Such quantity characterizes the “gappiness” or lacunarity of the superlattices [11]: when all gaps have the same lengths at the first stage of growth, the structure presents a low lacunarity; when some gaps are large while the others are very small (e.g., for small values of \( \varepsilon \)) the structure is said to be highly lacunar. In addition to the similarity dimension \( D_S \), the lacunarity-related parameter \( \varepsilon \) provides a second useful descriptor for characterizing fractal superlattices.

B. Impulse Response Corrupted by Noise

The noiseless impulse response \( r_\sigma(t) \) of a fractal superlattice is given by the following inverse Fourier transform:

\[
\begin{align*}
  r_\sigma(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{S,L}(\omega)e^{-j\omega t} e^{-(\sigma \omega)^2} d\omega \\
  &\quad \text{where } \sigma \text{ denotes the radius (or root mean square duration) of the normally incident (electromagnetic or optical) Gaussian pulse and } R_{S,L} \text{ designates the reflection coefficient of the superlattice of length } L \text{ at the stage of growth } S. \text{ This coefficient is computed by using an efficient double recursive computational technique.}
\end{align*}
\]
LAKSARI et al.: LACUNARITY OF FRACTAL SUPERLATTICES: REMOTE ESTIMATION

Fig. 1. (a) Generation, for the first two stages of growth, of a lacunar (polyadic) Cantor distribution of dielectric layers of refractive index \( n_0 \) and \( n_1 \) immersed in a medium with refractive index \( n_0 \). Here, each stage consists in \( \eta = 4 \) replica of the distribution at previous stage, each being reduced by a factor \( \rho = 1/7 \). The similarity dimension of this fractal structure is then \( D_s = \ln(\eta) / \ln(1/\rho) = \ln 4 / \ln 7 \). (b) Pulse interrogation of the lacunar Cantor superlattice. As illustrated in the insert, due to multiple reflections the reflected signal is wildly irregular. The spectral radius of the pulse is chosen close to the highest spatial frequency of the multilayered structure at \( S = 5 \). The constitutive parameters of the multigap superlattice are: \( S = 5, \xi = 1/14, \eta = 4, \rho = 1/7, \delta = 2/7, n_0 = 1, \) and \( n_1 = 1.5 \).

(see [11] for details). An impulse response \( r(t) \) corrupted by an additive Gaussian white noise \( n(t) \) of zero mean and variance \( \sigma_n^2 \) can then be written as follows:

\[
r(t) = r_\sigma(t) + n(t).
\]

As illustrated in Fig. 1(b), due to multiple reflections inside the superlattice, the noisy or noiseless impulse response exhibits a succession of abrupt changes or singularities in time. For noiseless impulse responses it has recently been reported that the wavelet analysis allows the remote estimation of the similarity dimension [7] and lacunarity [8] of fractal superlattices. We show here that this wavelet-based approach holds even if an additive Gaussian white noise corrupts the impulse responses.

III. REMOTE ESTIMATION OF THE SIMILARITY DIMENSION AND LACUNARITY FROM WAVELETS

A. Wavelet Analysis of Highly Irregular Signals Corrupted by Noise

The continuous wavelet transform (CWT) of the impulse response \( r(t) \) consists of expanding the response in terms of wavelets constructed from a single function, named the mother
wavelet $\Psi(t)$, by means of dilations and translations. The wavelet coefficient of $r(t)$ at scale $a$ and time $b$ is then given by

$$W_\Psi[r](a,b) = \frac{1}{a} \int_{-\infty}^{+\infty} r(t) \Psi \left( \frac{t - b}{a} \right) dt. \quad (3)$$

The (real valued) mother wavelet $\Psi(t)$ is required to be of zero mean and well localized in both time and scale [13]. Admissibility conditions are now well-known and established [14]. Additional criteria may be proposed for the choice of this wavelet in order to maximize the wavelet-based signal-to-noise ratio ($\text{SNR}_\Psi$) given by [15]

$$\text{SNR}_\Psi(a,b) = \frac{||W_\Psi[r_n](a,b)||^2}{E[||W_\Psi[n](a,b)||^2]} \quad (4)$$

where $E[x]$ designates the mathematical expectation of $x$, $r_n(t)$ is the noiseless impulse response and $n(t)$ denotes an additive Gaussian white noise of zero mean and variance $\sigma_n^2$ (see Section II). As observed previously, the noiseless impulse response consists of a succession of abrupt changes. Assume that these singularities are analog to a succession of Dirac delta functions. For a Dirac delta function singularity occurring at time $t_0$, i.e., for $r_\sigma(t) = A\delta(t - t_0)$ where $A = r_\sigma(t_0)$, the SNR$_\Psi$ defined by (4) becomes

$$\text{SNR}_\Psi(a,b) = \frac{A^2}{a \sigma_n^2} \frac{\psi^2(t_0/b)}{R_\psi(0)} \quad (5)$$

where $R_\psi(t) = \int \psi(u) \cdot \psi(u - t) du$ is the auto-correlation function of the (real valued) mother wavelet $\psi(t)$. From (5) it follows that the SNR$_\Psi(a,b)$ is inversely proportional to the scale parameter. Consequently, the wavelet analysis is more efficient at fine scales than at coarse scales. Moreover, (5) provides a direct relationship between the SNR$_\Psi$ in the time-domain $A^2/\sigma_n^2$, and the SNR$_\Psi$ in the time-scale domain SNR$_\Psi(a,b)$. Finally, for maximizing the SNR$_\Psi(a,b)$ at time $b = t_0$, we conclude that a mother wavelet $\psi$ which renders maximum the ratio $\psi^2(0)/R_\psi(0)$ should be chosen. For a mother wavelet belonging to the class of unit energy—that is for $\psi(t)$ satisfying the condition $R_\psi(t) = 1$—this criteria reverts to ensure a high value to $\psi^2(0)$. We adopt here the second derivative of the Gaussian function $\psi(t) = (1 - t^2)e^{-t^2/2}$ (also referred to as the Mexican hat wavelet).

From the maxima of the CWT modulus at a given scale, named the wavelet-transform modulus-maxima (WTMM), one can extract the skeleton of the impulse response, that is, the set
of all WTMM in the time-scale domain. This skeleton allows us to capture the scaling properties of the temporal distribution of singularities in the analyzed signal [16]. Here, we use this property of wavelet analysis to explore the locations of singularities in the impulse response \( r(t) \). In order to illustrate our method, Fig. 2(a) displays the skeleton of the impulse response (with high SNR) of a lacunar superlattice at stage \( S = 5 \), with \( \eta = 4, \rho = 1/7 \), and \( \varepsilon = 1/14 \). We observe that this skeleton overlaps two hierarchical structures that are reported in Fig. 2(b) and 2(c). Thus, fractal superlattices, when interrogated by an electromagnetic or optical pulse, imprint their discrete self-similarity properties on the reflected signal by generating discrete self-similar hierarchical structures in time-scale domain. We now show that these two hierarchical structures allow the remote estimation of the similarity dimension and lacunarity.

**B. Wavelet-Based Estimation of the Similarity Dimension**

Following the process developed in [7] for nonlacunar superlattices, the hierarchical structure displayed in Fig. 2(b) allows the estimation of the similarity dimension of the superlattice interrogated by the pulse from the following relationship:

\[
D_W = \lim_{a \to 0^+} \left\{ -\frac{\ln[Z(a)]}{\ln(a)} \right\}
\]

where \( Z(a) \) designates the number of maxima at scale \( a \). As a matter of fact, from the illustrative example of Fig. 2(b) \( \eta = 4, \rho = 1/7, \) and \( \varepsilon = 1/14 \) we note that the number of maxima \( Z \) increases by \( 4^m \) when the scale decreases from \( k/\alpha^{m-1} \) to \( k/\alpha^m \) (\( k > 0 \) and \( m \) is an integer), where \( \alpha \geq 7 \) denotes the constant scale factor between successive bifurcations; thus, from (6), we deduce that \( D_W = \ln(4)/\ln(7) \). This wavelet-based dimension provides an excellent approximation of the similarity dimension of the interrogated superlattice. For fractal superlattice with arbitrary \( \eta \) and \( \rho \), we have observed that the number \( Z \) of maxima increases by \( 2(N - 1)N^m \) when the scale decreases from \( k/\alpha^{m-1} \) to \( k/\alpha^m \) (\( \alpha > 1 \)), where \( N \) designates the number of arch-like structures replicated in the time-scale and \( \alpha \) represents the constant scale factor between successive bifurcations. We remark that the values of \( N \) and \( \alpha \) provide a very good approximation of \( \eta \) and \( 1/\rho \), respectively. Moreover, the number of maxima at a given scale \( a = k/\alpha^m \) is found to be \( Z(a) = \sum_{m=0}^{m} 2(N - 1)N^m = 2(N^{m+1} - 1) \). Consequently, from (6) we deduce \( D_W = \ln(N)/\ln(\alpha) \), that is, a remote estimation of the similarity dimension \( D_S = \ln(\eta)/\ln(1/\rho) \).

As shown in Fig. 3 where a large range of similarity dimensions \( D_S \) for various lacunarities and signal-to-noise ratios is investigated, the computed wavelet-based dimension \( D_W \) still provides a good estimation of the similarity dimension for many fractal superlattices. As mentioned in our previous research involving non lacunar superlattices and noiseless impulse responses [7], the wavelet-based dimension provides a good estimation of the similarity dimension if refractive indices \( n_1 < 3(\rho_01 = 1) \) and if the radius \( \sigma \) of pulses is close to the inverse of the highest spatial frequency of the interrogated multilayered media. For SNR_\phi greater than 10 dB we show here that the error on the estimation of the similarity dimension does not exceed 6% (see Fig. 3).

**C. Wavelet-Based Estimation of the Lacunarity**

Consider the lacunarity-related parameter \( \varepsilon \) of a fractal superlattice with arbitrary \( \eta \) and \( \rho \) (see Section II). At the first stage of growth and for even values of \( \eta \), we deduce that \( \varepsilon = (1 - \eta \rho)/[\Delta + (\eta - 2)] \) where \( \Delta = \delta/\varepsilon \) denotes the ratio between the length of the center gap and the length the outermost gap. Consequently, since \( \eta \) and \( \rho \) are estimated by \( N \) and \( 1/\alpha \), respectively, a wavelet-based approximation \( \varepsilon_W \) of the lacunarity-parameter can be obtained by the following relationship:

\[
\varepsilon_W = \frac{1 - N/\alpha}{\Delta_W + (N - 2)} \quad \text{if } N \text{ is even} \tag{7a}
\]

where \( \Delta_W \) designates a scale ratio between the first bifurcations at large scales. For odd values of \( \eta (\text{or } N) \) the wavelet-based lacunarity parameter is found to be

\[
\varepsilon_W = \frac{1 - N/\alpha}{2\Delta_W + (N - 3)} \quad \text{if } N \text{ is odd}, \tag{7b}
\]

Fig. 4 display the wavelet-based lacunarity \( \varepsilon_W \) as a function of the lacunarity-parameter \( \varepsilon \) for various SNRs. In Fig. 4(a) the lacunarity-parameter \( \varepsilon \) is estimated by substituting \( \alpha \) by \( 1/\rho \) in (7a), that is, by using the exact value of \( \alpha \). The error does not exceed 17% for SNRs greater than 10 dB. From (7) we note that the determination of the wavelet-based lacunarity suffers from two sources of errors: one on the evaluation of the constant scale factor \( \alpha \) between successive bifurcations and one on the estimation of \( \Delta_W \). As shown in Fig. 4(b) where these two inaccuracies are taken into account the wavelet-based lacunarity-parameter \( \varepsilon_W \) given by (7a) is still close to the lacunarity-related parameter \( \varepsilon \) (the error does not exceed 8%).
Consider a multilayered dielectric structure composed of two Cantor superlattices (separated by a distance \( d \)) with similarity dimensions \( D_{S1} \) and \( D_{S2} \), respectively. For each superlattice the transmission and reflection coefficients are deduced from the above-mentioned recursive algorithm (see [11]). By cascading three chain-matrices the reflection coefficient of the overall laminated structure is then deduced and finally, the impulse response is computed. As shown in Fig. 5 the skeleton associated with the cascade of two superlattices exhibits two hierarchical structures in time-scale domain. The hierarchy associated to the second superlattice emerges clearly in the late time response of the first one. Each hierarchy allows the accurate remote extraction of the similarity dimensions \( D_{S1} \) and \( D_{S2} \) of each constitutive Cantor superlattice, that is: \( D_{S1} = \ln(4)/\ln(7) \) and \( D_{S2} = \ln(2)/\ln(5) \).

IV. CASCADE OF SUPERLATTICES

Fig. 4. Wavelet-based lacunarity \( \varepsilon_{W} \) as a function of the lacunarity parameter \( \varepsilon \) for a lacunar Cantor superlattice: (a) from the exact value of \( \alpha \), that is, by substituting \( \alpha \) by \( 1/\rho \) in (7a), and for various signal-to-noise ratios \( S/B : (\bigcirc) S/B = 10 \text{ dB}, (+) S/B = 20 \text{ dB}, \) and \((*) S/B = 30 \text{ dB}\); and (b) for \( S/B = 30 \text{ dB} \) and \((*) \) from the exact value of \( \alpha \) and \((---) \) from the value of \( \alpha \) estimated from the skeleton of Fig. 3(b). Here, the similarity dimension \( D_{S} = \ln(4)/\ln(7) \).

V. CONCLUSION

An accurate remote estimation of the similarity dimension and lacunarity of discrete self-similar superlattices has been achieved from the wavelet analysis of impulse response in presence of an additive Gaussian white noise. Criteria for the choice of the mother wavelet are established in order to maximize a wavelet-based \( \text{SNR}_{W} \) and consequently to improve the remote estimation of fractal descriptor from reflection data. Illustrative examples are given for the case of one-dimensional Cantor superlattices. However, the analysis is also applicable to multidimensional discrete self-similar objects.

REFERENCES

Yahia Laksari was born in Sidi Bel Abbes, Algeria, in September 1975. He received the Engineer Diploma in electronics engineering from the Electronic Institute of Sidi Bel Abbes, in 1998, and the Master’s degree in microwave and optical telecommunication and the Ph.D. degree (with high-honors) from the Ecole Nationale Supérieure d’Électronique, d’Électrotechnique, d’Informatique, d’Hydraulique et des Télécommunications (ENSEEIHT), Institut National Polytechnique, Toulouse, France, in 1999 and 2003, respectively. He has performed research work on remote analysis of discrete self-similar object from a wavelet-based partition function.


In January 1993, he joined the faculty of the Institut National Polytechnique de Toulouse, as an Assistant Professor, where he became an Associate Professor in February 1997 and a Professor in February 2001. From April 1997 to March 1998, he was a Visiting Associate Professor at the School of Engineering and Applied Science, University of Pennsylvania, Philadelphia. Since July 2001, he has been the Associate Chairman of the Electronics Laboratory at INPT. He was a contributor to the book Fractals: Theory and Applications in Engineering (Berlin, Germany: Springer-Verlag, 1999). He has performed research work on integral-equation and variational methods applied to electromagnetic wave propagation and scattering. Currently his research activities involve the electromagnetic modeling of multiscale structures and fractal objects.

Dr. Aubert is a Member of the International Scientific Radio Union (URSI) Commission B. In 1994, he was awarded with the Leopold Escande Prize for his Ph.D. dissertation. He is the secretary of the IEEE Antennas and Propagation French Chapter and is the Founder and Counselor of the IEEE Student Branch at ENSEEIHT.

Dwight L. Jaggard (S’68–M’77–SM’86–F’91) was born in Oceanside, New York. He received the B.S.E.E. and M.S.E.E. degrees from the University of Wisconsin-Madison, in 1971 and 1972, respectively, and the Ph.D. degree in electrical engineering and applied physics from the California Institute of Technology (Caltech), Pasadena, in 1976.

From 1976 to 1978, he was a Postdoctoral Research Fellow at Caltech and a Consultant to the NASA Jet Propulsion Laboratory, Pasadena, CA. In 1978, he joined the faculty at the University of Utah, Salt Lake City, as Assistant Professor of electrical engineering. Since 1980, he has performed research in complex (chiral and fractal) media, inverse scattering, and high-resolution imaging at the Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, where in 1989, he was a Co-Founder of the Complex Media Laboratory. Additionally, he was Associate Dean for Graduate Education and Research in the School of Engineering and Applied Science from 1992 through 1999 and is presently Professor of Systems and Electrical Engineering and Co-Director of the Technology Management Program at the same university. He holds several patents in the field and serves as a Consultant to government and industry. He has served as an Editor of the Journal of Electromagnetic Wave Applications and of several special sections of the Journal of the Optical Society regarding fractals and their applications. He is the co-editor and contributor to Recent Advances in Electromagnetic Theory (Berlin, Germany: Springer-Verlag, 1990) and is a contributor to Symmetry in Electromagnetics (New York: Taylor and Francis, 1995), Fractals in Engineering (Berlin, Germany: Springer-Verlag, 1997), Fractals: Theory and Applications in Engineering (Berlin, Germany: Springer-Verlag, 1999), and Frontiers of Electromagnetics (Piscataway, NJ: IEEE Press, 2000). His research is currently involved with the characteristics and applications of electromagnetic chirality; fractal electrodynamics; high-resolution imaging and inverse scattering; and the effects of knot symmetry and topology on waves.

Prof. Jaggard is a Fellow of the Optical Society of America and a Member of the Electromagnetics Academy. He received the S. Reid Warren Award for Distinguished Teaching, the Christian F. and Mary R. Lindback Award for Distinguished Teaching, and the Executive Master’s in Technology Management Award for Excellence in Teaching. He has served as an Editor and was a Member of the editorial board of the PROCEEDINGS OF THE IEEE regarding fractals and their applications.

Jean-Yves Tournéret received the ingénieur degree in electrical engineering from the Ecole Nationale Supérieure d’Électronique, d’Électrotechnique, d’Informatique, d’Hydraulique et des Télécommunications (ENSEEIHT), Toulouse, France, in 1989 and the Ph.D. degree from the Institut National Polytechnique, Toulouse, in 1992. He is currently a Professor at ENSEEIHT. He is a member of the IRIT Laboratory (UMR 5505 of the CNRS), where his research activity is centered around estimation, detection and classification of non-Gaussian and nonStationnary processes.

Prof. Tournéret was the Program Chair of the European Conference on Signal Processing (EUSIPCO), which was held in Toulouse, France, in 2002. He is also a member of different technical committees including the Signal Processing Theory and Methods (SPTM) Committee of the IEEE Signal Processing Society.