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K-set agreement bounds in round-based models through combinatorial topology

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ABSTRACT
Round-based models are very common message-passing models; combinatorial topology applied to distributed computing provides sweeping results like general lower bounds. We combine both to study the computability of \textit{k}-set agreement.

Among all the possible round-based models, we consider oblivious ones, where the constraints are given only round per round by a set of allowed graphs. And among oblivious models, we focus on closed-above ones, that is models where the set of possible graphs contains all graphs with more edges than some starting graphs. These capture intuitively the underlying structure required by some communication model, like containing a ring.

We then derive lower bounds and upper bounds in one round for \textit{k}-set agreement, such that these bounds are proved using combinatorial topology but stated only in terms of graph properties. These bounds extend to multiple rounds when limiting our algorithms to be oblivious – recalling only pairs of processes and initial value, not who send what and when.

1 INTRODUCTION
1.1 Motivation
Rounds structure many models of distributed computing: they simplify algorithms, capture the distributed equivalent of time complexity [13], and underly many fault-tolerant algorithms, like Paxos [23].

KEYWORDS
Distributed Computability, Set-agreement, Round-based models, Combinatorial Topology, Lower bounds, Upper Bounds

A recent trend, with parallel results by Charron-Bost and Schiper [9] on one hand, and Afek and Gafni [1] and Raynal and Stainer [26] on the other hand, is using this concept of round for formalizing many different models within a common framework. But the techniques used for proving results in these models tend to be ad-hoc, very specific to some model or setting. What is required going forward is a general approach to proving impossibility results and bounds on round-based models.

Actually, there is at least one example of a general mathematical technique used in this context: the characterization of consensus solvability through point-set topology by Nowak et al. [24]. We propose what might be seen as an extension to higher dimension of this intuition, by applying combinatorial topology (instead of point-set topology) to bear on \textit{k}-set agreement (instead of just consensus).

Combinatorial topology abstracts the reasoning around knowledge and indistinguishability behind many impossibility results in distributed computing. It thus provide generic mathematical tools and methods for deriving such results [15]. Moreover, this approach is the only one that managed to prove impossibility results and characterization of solvability of the \textit{k}-set agreement [10], our focus problem.

Concretely, we look at closed-above round-models, that is models where constraints happens round per round, and the set of communication graphs allowed is the closure-above of a set of graphs. These models capture some safety properties, where we require some underlying structure in communication, like having an underlying star, ring or tree. This is a strict generalization of the models with a fixed communication graph considered by Castañeda et al. [6].

For our models, we derive upper bound and lower bounds on the \textit{k} for which \textit{k}-set agreement is solvable. And although the proofs of the bounds use combinatorial topology, they are stated in terms of variants of the domination number, a well-known and used combinatorial number on graphs.

1.2 Overview
- We start by defining closed-above models in Section 2.
- Then we give various upper bounds for \textit{k}-set agreement in one round on those models in Section 3. These have the advantage of not requiring any combinatorial topology.
- Next, we introduce in Section 4 the combinatorial topology necessary for our lower bounds, both the basic definitions and our main technical lemma.
- We then go to lower bounds on round-based models for \textit{k}-set agreement in one round in Section 5. Recall that these
bounds use combinatorial topology, but are stated in terms of graph properties.

- Finally, Section 6 generalize both upper and lower bounds to the case of multiple rounds.

Due to size constraints, most of the proof can be found in the full version [28]

1.3 Related Works

Round-based models. The idea of using rounds for abstracting many different models is classical in message-passing. This includes the synchronous adversary models of Afek and Gafni [1] and Raymond and Stainer [26]; the Heard-Of model of Charron-Bost and Schiper [9]; and the dynamic networks of Kuhn et al [21].

Rounds are also used for building a distributed theory of time complexity [13] and for structuring fault-tolerant algorithms like Paxos [23].

Previous work on the solvability of consensus and $k$-set agreement include the characterization of consensus solvability for oblivious round-based models of Coulouma et al [11], the failure-detector-based approach of Jeanneau et al. [19], and the focus on graceful degradation in algorithms for $k$-set agreement of Bietyl et al. [3].

Combinatorial Topology. Combinatorial Topology was first applied to the problem of $k$-set agreement in wait-free shared memory by Herlihy and Shavit [18], Saki and Zarafshon [27] and Borowsky and Gafni [4].

Beyond these first forays, many other results got proved through combinatorial topology. Among others, we can cite the lower bounds for renaming by Castañeda and Rajsbaum [7] and the derivation of lower-bounds for message-passing by Herlihy and Rajsbaum [16].

There is even a result by Alistarh et al. [2] showing that traditional proof techniques (dubbed extension-based proofs) cannot prove the impossibility of $k$-set agreement in specific shared-memory models, whereas techniques from combinatorial topology can.

For a full treatment of combinatorial topology applied to distributed computing, see Herlihy et al. [15].

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Definition 2.1 (Communication model). Let $\text{Graph}_{\mathbb{N}}$ be the set of graphs. Then $\text{Com} \subseteq (\text{Graph}_{\mathbb{N}})^{\omega}$ is a communication model.

Any set of infinite sequences of graphs defines a model. In order to make models more manageable, we focus on a restricted form, where the graph for each round is decided independently of the others. The model is thus entirely characterized by the set of allowed graphs. We call these communication models oblivious, following Coulouma et al [11].

Definition 2.2 (Oblivious communication models). Let $\text{Com}$ be a communication model. Then $\text{Com}$ is oblivious $\equiv \exists S \subseteq \text{Graph}_{\mathbb{N}} : \text{Com} = S^{\omega}$.

Intuitively, oblivious models capture safety properties: bad things that must not happen. Or equivalently, good things that must happen at every round. Usually, these good properties are related to connectivity, like containing a cycle or a spanning tree. Since such a property tends to be invariant when more messages are sent, we can look at oblivious models defined by a set of subgraphs.

Definition 2.3 (Closed-above communication models). Let $\text{Com}$ be an oblivious communication model. Then $\text{Com}$ is closed-above $\equiv \exists S \subseteq \text{Graph}_{\mathbb{N}} : \text{Com} = ( \bigcup_{G \in S} \uparrow G )^{\omega}$, where $\uparrow G \equiv \{ H \mid V(H) = V(G) \land E(H) \supseteq E(G) \}$.

We call the graphs in $S$ the generators of $\text{Com}$.

If $S$ is a singleton, then $\text{Com}$ is simple closed-above.

Classical examples of closed-above models are the non-empty kernel predicate (only graphs where at least one process broadcasts) and the non-split predicate (only graphs where each pair of processes hears from a common process), used notably by Charaton-Bost et al. [8] for characterizing the solvability of approximate consensus (the variant of consensus where the decided value should be less than $\varepsilon$ apart, where $\varepsilon > 0$ is fixed beforehand).

Another closed-above model is the one satisfying the tournament property of Afek and Gafni [1], which they show is equivalent to wait-free read-write shared memory.

One example of an oblivious model which is not closed-above is the one generated by all graphs containing a cycle, except the clique. More generally, the closure-above forces us to have all graphs with more edges than our generators.

Nonetheless, closed-above models capture a fundamental intuition behind distributed computing models: specifying what should not happen. They also have a good tradeoff between expressivity and simplicity, since the “combinatorial data” used to build them is contained in a small number of graphs. Finally, the patterns
expected by safety properties tend to be independent of which processes play which roles – what matters is the existence of a ring or spanning tree, not who is where on it.

We call such closed-above models symmetric.

Definition 2.4 (Symmetric models). Let \( \text{Com} \) be a closed-above model, and \( S \) be the set of graphs generating it. Then \( \text{Com} \) is symmetric if \( S = \text{Sym}(S) \), where \( \text{Sym}(S) = \{ \pi(G) \mid G \in S \land \pi : \Pi \rightarrow \Pi \text{ a permutation on } \Pi \} \).

In the rest of the paper, we will limit ourselves to closed-above models, both symmetric and not.

2.2 Oblivious algorithms

Because most applications of combinatorial topology to distributed computing aim towards impossibility results, the traditional algorithms considered err on the side of power: full information protocols, which exchange at each round the view of everything ever heard by the process. For example, after a couple of rounds, views will contain nested sets of views, themselves containing views, recursively until the initial values.

In contrast, we focus on oblivious algorithms. That is, we limit each process to remember only the initial values it knows, not who sent them or when. This amounts to a function from \( \Pi \) to the set of initial values (with a \( \perp \) when the value is not known). In turn, these algorithms lose the ability to trace the path of the value.

We can view oblivious algorithms as full-information protocol whose decision map (the function from final view to decision value) depends only on the set of known pairs (process,initial value). The full-information protocol might still be used for deciding when to apply the decision map, but this map loses everything except the known pairs. That is, the decision map is constrained to decide similarly in situations where it received the same values, even when they were from different processes.

Definition 2.5 (Oblivious algorithm). Let \( \mathcal{A} \) be a full-information protocol, with decision map \( \delta \). Then \( \mathcal{A} \) is an oblivious algorithm if \( \forall v \text{ a view } : \delta(v) = \delta(\text{flat}(v)) \), where \( \text{flat}(v) = \bigcup_{(p,v_p) \in \Pi} \text{flat}(\{p,v_p\}) \) and \( \text{flat}((p,v_p)) = \{ \{p,v_p\} \text{ if } v_p \text{ is a singleton from } V_{in} \} \bigcup \text{flat}(v_p) \text{ otherwise} \).

3 ONE ROUND UPPER BOUNDS: A START WITHOUT TOPOLOGY

Although lower bounds are our targets, they require upper bounds to gauge their strength. We thus start with upper bounds on \( k \)-set agreement [10] for closed-above models. Another advantage of starting with our upper bounds is that they rely on concrete algorithms, and allow us to introduce generalizations of the classical domination number that will be used for our lower bounds.

Lastly, we also start with bounds for the one round case in this section and the next one. Bounds for multiple rounds depend on these one round bounds.

These bounds follow from a very simple algorithm for solving \( k \)-set agreement. We assume the set of initial values is totally ordered. Then everybody sends its initial value for one round, and decide the minimum it received.

3.1 Simple closed-above models: almost too easy

Recall that the domination number of a graph is the size of its smallest dominating set, that is the size of the smallest set of nodes whose set of outgoing neighbors is \( \Pi \). Note that the outgoing neighbors of a set \( S \subseteq \Pi \) contains \( S \) – that is, we assume self-loop.

Definition 3.1 (Domination number). Let \( G \) be a graph. Then its domination number \( \gamma(G) \triangleq \min \{i \in [1,n] \mid \exists P \subseteq \Pi : |P| = i \land \bigcup_{p \in P} \text{Out}_G(p) = \Pi \} \).

Because the simple closed-above model generated by \( G \) only allows graphs containing \( G \), their domination number is at most \( \gamma(G) \). This entails a very simple upper bound on \( k \)-set agreement.

Theorem 3.2 (Upper bound on \( k \)-set agreement by \( \gamma(G) \)). Let \( G \) be a graph. Then \( \gamma(G) \)-set agreement is solvable in one round on the simple closed-above model generated by \( G \).

Proof. The algorithm is just slightly different from the one stated at the start of the section: after one round, each process decides the minimum value of the ones of a fixed minimum dominating set of \( G \). Since \( G \) is known, this minimum dominating set can be computed beforehand. And because it is a dominating set, every process receives at least one value from it, so every process can decide.

Finally, since the minimum dominating set has at most \( \gamma(G) \) distinct values, at most \( \gamma(G) \) values are decided, and thus our algorithm solves \( \gamma(G) \)-set agreement.

From Castañeda et al. [6, Thm 5.1], we know this bound is tight: the oblivious model with a single graph \( G \) cannot solve \( k \)-set agreement in one round for \( k < \gamma(G) \). Hence the weaker simple closed-above model generated by \( G \) cannot solve \( k \)-set agreement in one round for \( k < \gamma(G) \).

Still, simple closed-above models are somewhat artificial, as can be seen in the proof: we know exactly the subgraph that must be contained in the actual communication graph. A more realistic take requires to spread the uncertainty to the underlying subgraph; we thus look next at general closed-above models.

3.2 General closed-above models: tweaking of upper bounds

For general closed-above models, we must deal with a set of possible underlying subgraphs. This makes our previous approach inapplicable: we cannot hardcode a dominating set because we don’t know the underlying subgraph for sure.

This new issue motivates the definition of a weakening of the domination number: the equal-domination number of a set of graphs. Intuitively, any set of that much process is a dominating set in all the graphs considered.

Definition 3.3 (Equal Domination number of a set of graphs). Let \( S \) be a set of graphs. Then its equal domination number \( \gamma^e(S) \triangleq \max_{G \in S} \gamma^e(G) \), where \( \gamma^e(G) \triangleq \min \{i \in [1,n] \mid \forall P \subseteq \Pi : |P| = i \Rightarrow \bigcup_{p \in P} \text{Out}_G(p) = \Pi \} \).
Theorem 3.4 (Upper bound on k-set agreement by $y^Q(S)$ for general closed-above models). Let $S$ be a set of graphs. Then $y^Q(S)$-set agreement is solvable on the closed-above model generated by $S$.

Proof. Let $P$ be a set of $y^Q(S)$ processes with the smallest initial values. They have thus at most $y^Q(S)$ distinct initial values. By definition of $y^Q(S)$, $P$ dominates every graph in $S$, and thus every graph in the closed-above model generated by $S$.

Thus taking the minimum after one round will result in deciding one of those initial values, and thus one of at most $y^Q(S)$ values. We conclude that our algorithm solves $y^Q(S)$-set agreement after one round on the closed-above model generated by $S$.

Since the equal-domination number is independent of which process does what, it is the same for any permutation of the graph. This entails an upper bound on symmetric models as a corollary.

Corollary 3.5. Let $S$ be a set of graphs. Then $y^Q(S)$-set agreement is solvable on the closed-above model generated by $\text{Sym}(S)$.

Now, the natural question to ask is whether we can improve this bound. Or equivalently, is it tight?

The answer depends on the graphs. To see it, let us look at another combinatorial number: covering numbers. Given fewer processes than the equal-domination number of the graph, they do not always form a dominating set. Nonetheless, they might still get heard by some minimum number of processes. We call such minimums the covering numbers of the graph: the $i$-th covering number of $G$ is, given any set of $i$ processes, the minimum number of processes hearing this set in $G$.

Definition 3.6 (Covering numbers of a set of graphs). Let $S$ be a set of graphs. Then $\forall i < y^Q(S)$, its $i$-th covering number $\text{cov}_i(S) \triangleq \min_{G \in S} \text{cov}_i(G)$, where $\text{cov}_i(G) \triangleq \min \{ |\bigcup_{p \in P} \text{Out}_G(p)| \}$.

These numbers capture the ability of a set of processes to disseminate their values in the graph. If we take the $i$ processes with the smallest initial values, we can be sure that at least $\text{cov}_i(S)$ processes will hear, and thus choose one of these. This then gives a solution to $(i + (n - \text{cov}_i(S))$-set agreement in one round.

Theorem 3.7 (Upper bounds on $k$-set agreement by covering numbers for general closed-above models). Let $S$ be a set of graphs. Then $\forall i \in [1, y^Q(S)]$, $(i + (n - \text{cov}_i(S)))$-set agreement is solvable on the oblivious closed-above model generated by $S$.

Proof. For a set of $i$ processes with the $i$ smallest initial values, they will reach at least $\text{cov}_i(S)$ processes after the first round. Thus these processes will decide one of the $i$ values when taking the smallest value they received.

As for the rest of the processes, we can’t say anything about what they will receive, and thus we consider the worst case, where they all decide differently, and not one of the $i$ smallest values. Then the number of decided values is at most $i + (n - \text{cov}_i(S))$, and the theorem follows.

The covering numbers are also independent of processes names; we thus get a similar upper bound on symmetric models as a corollary.

Corollary 3.8. Let $S$ be a set of graphs. Then $\forall i \in [1, y^Q(S)]$, $(i + (n - \text{cov}_i(S)))$-set agreement is solvable on the oblivious closed-above model generated by $\text{Sym}(S)$.

When is this new bound better than the one using the equal-domination number? When there is some $i$ such that $n - \text{cov}_i(S) < y^Q(S) - i$. Let us take the symmetric models generated by the two graphs in Figure 1.

In the first model, $n - \text{cov}_2(S) < y^Q(S) - i$ never happens, because every covering number of a star equals 1 (the biggest set of outgoing neighbors different from $\Pi$ contains only one process), and its equal-domination number equals $n$ (because when taking only $n - 1$ processes, the center of the star might not be in there). Thus $n - \text{cov}_2(S) = n - 1 \geq y^Q(S) - i = n - i$.

On the other hand, this is the case in the second model, because $\text{cov}_2(S) = 3$ and $y^Q(S) = 4$. We thus have $n - \text{cov}_2(S) = 4 - 3 = 1 < y^Q(S) - i = 4 - 2 = 2$. Hence the upper bound with covering numbers ensure 3-set agreement solvability while the upper bound with the equal-domination number only ensures 4-set agreement solvability.

3.3 Intuitions on upper and lower bounds

Why do our upper bounds hold? Because we can extract from the underlying graphs some minimal connectivity of sets of processes. Hence, we know from these combinatorial numbers how much the minimal values will spread in the worst case, and thus we bound the maximum number of values decided.

On the other hand, our lower bounds will follow from studying how much values can spread in the best case. Why? Because the more values can spread, the more processes can distinguish between initial configurations, and the more they have a chance to decide correctly. Ensuring enough indistinguishability thus entails an impossibility at solving $k$-set agreement.

This indistinguishability is linked to higher-dimension connectivity in combinatorial topology [15, Thm. 10.3.1]; we thus turn to the topological approach to distributed computing for our lower bounds.

4 ELEMENTS OF COMBINATORIAL TOPOLOGY

4.1 Preliminary definitions

First, we need to introduce the mathematical objects that this approach uses. These are simplexes and complexes. A simplex is simply a set of values, and can be represented as a generalization of a triangle in higher dimensions. Simplexes capture configurations in
Then the uninterpreted complex of a colored simplicial complex captures all considered configurations.

**Definition 4.1 (Simplex).** Let Cols and Views be sets. Then \( \sigma \subseteq \text{Cols} \times \text{Views} \) is a simplex on Cols and Views (or colored simplex) if \( \forall p \in \text{Cols} : |\{ v \in \text{Views} | (p, v) \in \sigma \} | \leq 1 \).

We have \( \text{col}(\sigma) \) or \( \text{names}(\sigma) \triangleq \{ p \in \text{Cols} | \exists v \in \text{Views} : (p, v) \in \sigma \} \). And we have \( \text{views}(\sigma) = \{ v \in \text{Views} | \exists p \in \text{Cols} : (p, v) \in \sigma \} \). We also write \( \text{views}(\sigma)(p) \) for the \( v \in \text{Views} \) such that \( (p, v) \in \sigma \).

The dimension of \( \sigma \) is \( |\text{dim}(\sigma) - 1| \).

Although we define Views to be any set for readability, the traditional view is of sets of pairs, the first element being a process name, and the second being either another view or an initial value. For more details, refer to [15].

Then a complex is a set of simplexes that is closed under inclusion. It captures all considered configurations.

**Definition 4.2 (Complex).** Let Cols and Views be sets. Then \( C \in \mathcal{P}(\text{Cols} \times \text{Views}) \) is a simplicial complex on Cols and Views (or colored simplicial complex) if

- \( \forall (p, v) \in \text{Cols} \times \text{Views} : \{ (p, v) \} \in C \).
- \( \forall \sigma, \tau \text{ simplexes on } \text{Cols} \times \text{Views} : \sigma \cap \tau \subseteq \sigma \Rightarrow \tau \in C \).

The facets of \( C \) are \( \{ \sigma \in C | \forall \tau \in C : \sigma \subseteq \tau \Rightarrow \tau = \sigma \} \).

The dimension of \( C \) is the maximum dimension of its facets. \( C \) is called pure if all its facets have the same dimension.

How can we go from our round-based models, which are generated by graphs, to simplexes and complexes?

Starting with a single graph, we define the uninterpreted simplex induced by this graph. This simplex captures the configuration after a round using graph \( G \), simply in terms of who hears from whom. It disregards input values, which makes it uninterpreted.

**Definition 4.3 (Uninterpreted simplex of a graph).** Let \( G \) be a graph. Then the uninterpreted simplex of \( G \) is \( \sigma_G \triangleq \) the colored simplex \( \{ (p, \text{inc}_G(p)) | p \in \Pi \} \).

Given a set of graphs \( A \) representing the possible graphs, we generalize the previous definition to give the uninterpreted complex of \( A \).

**Definition 4.4 (Uninterpreted complex of an oblivious model).** Let \( A \) be an oblivious model defined by a set of graphs \( S \). Then the uninterpreted complex of \( A \) is \( C_A \triangleq \) the complex whose facets are exactly the \( \{ \sigma_G | G \in S \} \).

4.2 Uninterpreted complexes of closed-above models

It so happens that closed-above models give rise to uninterpreted complex that are easy to define and study. Indeed, they are unions of pseudospheres, where pseudospheres are colored complexes topologically equivalent to \( n \)-spheres. These pseudospheres have already been used in the literature to study multiple models of computation [15, Chap. 13].

**Definition 4.5 (Pseudospheres [15, Def 13.3.1]).** Let \( V_1, V_2, \ldots, V_n \) be sets. Then the pseudosphere complex \( \phi(\Pi; V_i | i \in [1, n]) \triangleq \)

- \( \forall i, \forall v \in V_i : (P_i, v) \) is a vertex of \( C \).
- \( \forall J \subseteq [1, n] : \{ (P_i, v_i) | j \in J, v_j \in V_j \} \) is a simplex of \( C \) iff all \( P_j \) are distinct.

We can think of these complexes as a generalization of complete bipartite graphs in \( n \) dimensions. Recall that a complete bipartite graph is a graph that can be split into two sets of nodes, the nodes of each set not linked to each other and each node of one set linked to all nodes of the other set. For example, Figure 3a is a bipartite graph.

Now a pseudosphere is the same, except that nodes can be partitioned into \( n \) sets, so no simplex contains more than one element of each set as a vertex, and all the simplexes built from one element of each set are in the complex. Figure 3b is an example of a pseudosphere built from processes \( P_1, P_2, P_3 \), and the three sets \( V_1 = \{ v_1, v_2 \}, V_2 = \{ v_3, v_4 \} \) and \( V_3 = \{ v \} \).

Among other things, pseudospheres are closed under intersection, and are \( (n - 2) \)-connected.

**Lemma 4.6 (Intersection of pseudospheres [15, Fact 13.3.4]).** \( \phi(\Pi; U_i | i \in [1, n]) \cap \phi(\Pi; V_i | i \in [1, n]) = \phi(\Pi; U_i \cap V_i | i \in [1, n]) \).

One advantage of pseudosphere is that they have high connectivity [15, Def. 3.5.6]. Intuitively, connectivity concerns the (non-)existence of high-dimensional generalisation of holes in the complexes. Since pseudospheres are topologically equivalent to spheres [15, Sect. 13.3], they only have these holes in the highest dimensions.

**Lemma 4.7 (Connectivity of pseudospheres [15, Cor. 13.3.7]).** \( \phi(\Pi; V_i | i \in [1, n]) \) is \( (n - 2) \)-connected, where \( n \triangleq |\{ i \in [1, n] | V_i \neq \emptyset \}| \).
The connectivity of the uninterpreted complex for a simple closed-above model follows, because such a complex is a pseudosphere. Intuitively, for any process $p$, its possible views are exactly the upward closure of its view in the defining graph $G$. Then the $n$-simplexes of the uninterpreted complex are exactly the simplex can be deduced from the way that the cover elements $\mathcal{A}$ for general closed-above models.

**Lemma 4.8 (Uninterpreted complex of a simple closed-above model is a pseudosphere).** Let $A$ be a simple closed-above model, and $G$ be the graph from which it is built. Then $C_A = \{ \pi \mid \Pi \subseteq I \}$. The nerve complex can be deduced from the way that the cover elements of this cover. First, by Theorem 4.9, $C_A$ is $(n - 2)$-connected.

As for the intersection of any set $I$ of $C_A$, we have two properties. First, it cannot be empty, since all $C_A$ contains the uninterpreted simplex of the complete graph on $\Pi$, by definition of $G$. This gives us that the nerve complex is a simplex, and thus $\omega$-connected.

And second, the intersection is also a pseudosphere, by application of Lemma 4.6. Indeed, these are intersections of pseudospheres with the same processes which have an non-empty intersection for each color : the view of this process in the complete graph.

We can thus conclude by application of the nerve lemma and Theorem 4.9.

**4.3 Interpretation of uninterpreted complexes**

We can only go so far with uninterpreted complexes; at some point, we need to consider initial values.

**Definition 4.13 (Interpretation of uninterpreted simplex).** Let $\pi$ be an uninterpreted simplex on $\Pi$ and $r$ be a $(n - 1)$-simplex colored by $\Pi$. Then the **interpretation of $\pi$ on $r$**, $\pi(r) \overset{\text{def}}{=} \{(p, V) \mid p \in \Pi \land (v \in V \implies (3q \in \text{view}_v(p) : v = \text{view}_r(q)))\}$.

Then the same intuition can be applied to a full uninterpreted complex.

**Definition 4.14 (Interpretation of uninterpreted complex).** Let $\mathcal{A}$ be an uninterpreted complex on $\Pi$ and $I$ be a pure $(n - 1)$ complex colored by $\Pi$. Then the **interpretation of $\mathcal{A}$ on $I$**, $\mathcal{A}(I) \overset{\text{def}}{=} \bigcup_{r \in \text{facet of } I} \pi(r)$.

These interpretations give us protocol complexes, on which known results on computability are applicable.

**4.4 A Powerful Tool**

On the combinatorial topology front, our results leverage two main tools: the impossibility result on $k$-set agreement based on connectivity [15, Thm. 10.3.1], and a way to compute the connectivity of a complex from the way it is built. This section develops the second idea.

Let $\mathcal{A}$ be a pure complex of dimension $d$. We say that $\mathcal{A}$ is **shellable** if there is an ordering $\phi_1, \ldots, \phi_t$ of its facets such that for every $1 \leq t \leq r - 1$, $\bigcup_{i=1}^{t} \phi_i \cap \phi_{t+1}$ is a pure subcomplex of dimension $d - 1$ of the boundary complex of $\phi_{t+1}$, i.e., of $\text{skel}^{d-1} \phi_{t+1}$.
Let $13.2.2$]).

that if both the output complexes and the mapping are “nice”, the
of complexes we need here. While Castañeda et al. studied the
result from Castañeda et al. $[6]$ and adapts it to the interpretation
of $\mathcal{A}$’s facets $\sigma_1, \ldots, \sigma_t$,
with each $\sigma_i$ and $\varphi’$ sharing a $(d-1)$-face, it holds that
$(\bigcup_{i=1}^{t} \alpha(\varphi_i)) \cap \alpha(\varphi’) = \bigcup_{i=1}^{t} (\alpha(\varphi’) \cap \alpha(\sigma_i))$.

(2) For every $t \geq 0$ and every collection $\varphi_0, \varphi_1, \ldots, \varphi_t$ of $t + 1$
facets of $\mathcal{A}$ with each $\varphi_i$ and $\varphi_0$ sharing a $(d-1)$-face, it holds that
$\bigcap_{i=0}^{t} \alpha(\varphi_i)$ is least $(t - t)$-connected.

Then, $\mathcal{B}$ is $t$-connected.

5 ONE ROUND LOWER BOUNDS FOR ONE ROUND: A TOUCH OF TOPOLOGY

As before, we start with the simple closed-above case, where the
model is the closure of a single graph. In this case the tight lower
bound follows from Castañeda et al. $[6, Thm. 5.1]$, as mentioned
above.

Theorem 5.1 (Lower bound on k-set agreement for simple
closed-above models). Let $A$ a simple closed-above model
generated by the graph $G$. Let $k \leq \gamma(G)$. Then $k$-set agreement is not
solvable on $A$ in a single round.

We thus focus on general closed-above models. Here we have to
leverage the underlying structure of the protocol complex. We do
so through two tools: the main theorem from Section 4.4, as well
as two graph parameters: the equal-domination number over a set
of graphs, and the max-covering numbers of a set of graphs.

Definition 5.2 (Distributed domination number of a set of graphs).
Let $S$ be a set of graphs. Then the distributed domination num-
ber $S, \gamma_{\text{dist}}(S) \triangleq \min \left\{ i > 0 \mid \text{exists } G_i \subseteq S : \forall G \subseteq S : G \cap S_i \subseteq \bigcup_{G \in S_i} \text{Out}_G(P) = \Pi \right\}$.

The difference between $\gamma^{eq}(S)$ and $\gamma_{\text{dist}}(S)$ is that a set of $\gamma^{eq}(S)$
processes dominates each graph of $S$ separately, whereas a set
of $\gamma_{\text{dist}}(S)$ processes might not dominate any graph of $S$, but it
dominates every subset of $i$ graphs of $S$ together. Thus $\gamma_{\text{dist}}(S) \leq \gamma^{eq}(S)$. Fitting, considering the former is used in lower bounds and
the latter in upper bounds.

Next, the max-covering numbers are quite subtle. For $i < \gamma_{\text{dist}}(S)$,
the $i$-th max-covering number of $S$ is the maximum number of processes
hearing a set of $i$ procs, summed over $i$ graphs in $S$.

That is, the max-covering numbers capture how much values
can be disseminated in the best case. They serve in lower bounds
by giving a best case scenario on which we can focus to prove
impossibility.

Definition 5.3 (Max-covering numbers of a set of graphs). Let $S$
be a set of graphs and $i < \gamma_{\text{dist}}(S)$. Then the $i$-th max-covering number of $S, \text{max-cov}_i(S) \triangleq \max_{\text{max-cov}_i(S) \leq i} \min_{\text{max-cov}_i(S) \leq i} \left\{ \frac{\text{max-cov}_i(S) - 1}{n - 1} \right\}$.

We also define the $i$-th max-covering coefficients on $S, M_i(S) \triangleq \left\{ \frac{\text{max-cov}_i(S) - 1}{n - 1} \right\}$.
Theorem 5.4 (Lower bound on k-set agreement for general closed-above models). Let A be a closed-above model generated by the set of graphs S. Let l = \min\{\gamma_{\text{dist}}(S) - 2, \min\{t + M_t(S) - 2 \mid t \in \{1, \gamma_{\text{dist}}(S) - 1\}\}\). Then (l + 1)-set agreement is not solvable on A in a single round.

The term depending on \(\gamma_{\text{dist}}(S)\) in the lower bound serves when the max-covering numbers are not sufficient to distinguish adversaries with different properties. Consider for example the symmetric models of all unions of s stars, with s \(\leq n\). Then for those graphs, for \(t < \gamma_{\text{dist}}(S)\), we have \(\max_{\text{coy}}(S) = t\), and thus \(M_t(S) = n - t\). Hence the minimum over the \(t + M_t(S) - 2\) is \(n - 2\).

But this would mean that \((n - 1)\)-set agreement is impossible for \(s < n\), whereas we can clearly solve 2-set agreement for \(s = n - 1\), for example. What depends on \(s\) is \(\gamma_{\text{dist}}(S)\) itself. More precisely, \(\gamma_{\text{dist}}(S) = n - s + 1\), because given P, we can consider only the graph where the s centers of stars are in \(P\), up until the point where \(|P| > n - s\).

Hence our lower bound shows that for the symmetric union of s stars, \((n - s)\)-set agreement is impossible in one round. Given that our upper bounds above tell that \((n - s + 1)\)-set agreement is possible in one round for this model, the bound is tight.

Finally, the bound can be specialized for symmetric models.

Corollary 5.5 (Lower bounds for symmetric closed-above model). Let G be a graph. Let l = \(\min\{\gamma_{\text{dist}}(\text{Sym}(G)) - 2, \min_{t \in \{1, \gamma_{\text{dist}}(\text{Sym}(G)) - 1\}} \left\{ t + \frac{n - t - 1}{(\max_{\text{coy}}(\{G\}) - t)} - 2 \right\} \right)\). Then (l + 1)-set agreement is not solvable on \(\text{Sym}(G)\) in a single round.

Note that all these lower bounds are valid for general algorithms, not only oblivious ones. The reason is that a one round full information protocol is an oblivious algorithm.

6 MULTIPLE ROUNDS

Given that we focus on oblivious algorithms, a natural approach to extending our lower bounds to the multiple rounds case is to look at the product of our graphs. By product, we mean the graph of the paths with one edge per graph. Thus the products of \(t\) graphs capture who will hear who after \(t\) corresponding communication rounds.

Definition 6.1 (Graph path product). Let G and H be graphs with auto-loops (\(\forall v \in \Pi : (v, v) \in E(G) \land (v, v) \in E(H)\)). Then their graph path product \(G \times H \equiv (\Pi, E)\) such that \(\forall u, v \in \Pi : (u, v) \in \Pi \iff \exists w \in \Pi : (u, w) \in E(G) \land (w, v) \in E(H)\).

Since we have a graph as the result, we can apply our lower bounds for one round. At least, if the resulting graph still satisfy the hypotheses of our lower bounds. It does, although product doesn’t maintain closure-above. This subtlety is explained in the next subsection.

6.1 Closure-above is not invariant by product, but its still works

What is the pitfall mentioned above? Quite simply, that the product of two closed-above models does not necessarily gives a closed-above model. This follows from the fact that the closure-above of a product of graphs doesn’t always equal the product of the closure-above of the graphs.

Let’s take an example: the product of a cycle with itself.

Then we cannot build the following graph by extending the cycles and taking the product:

Why? Simply put, adding the new edge to either of the two cycles necessarily creates other edges in the product. Adding an edge from \(p_2\) to any other node than \(p_3\) and \(p_1\) also creates new edges: so does adding an edge to \(p_1\) and then an edge from \(p_4\) to \(p_6\), or an edge from \(p_3\) to \(p_6\) in the second graph.

Hence the product of the closure above of this cycle with itself is not the closure-above of the squared cycle. To put it differently, closure-above is not invariant by the product operation.

Nonetheless, the bell does not toll for our hopes of extending our properties. What is used in the lower bound proofs above is not closure-above itself, but its consequences: being a union of pseudospheres containing the full simplex, such that for each pseudosphere, all graphs contain the smallest graph.

All three properties are present in a specific subset of the product of two simple closed-above models: all products where edges might be added to the last graph in the product but not to the other. Each added edge only alters the view of its destination, since it is in the second graph, and multiple added edges don’t interfere because they are all added to the same graph. Hence we can change the views of processes one at a time, and thus we get a pseudosphere. Since adding no edge gives the original product and adding all missing edges gives the clique, we get the other two properties. Then taking this subset of the product of two general closed-above models result in a union of pseudospheres, one for each product of the underlying graphs.

Therefore we can extract relevant subcomplexes from the product of closed-above models, and then the lower bounds only depends on the properties of the underlying product of graphs.

6.2 Upper bounds for multiple rounds

Even if we just explained how to deal with lower bounds for multiple bounds, we still start by giving upper bounds for multiple rounds.

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This is for the same reason as in the one round case: the upper-bounds require no combinatorial topology, and they allow us to introduce concepts needed for the lower bounds.

First, we need to prove a little result that is enough for our upper bounds: that the product of closed-above models is included in the closure-above of the product.

**Lemma 6.2 (Product and inclusion for closed-above).** Let $G$ and $H$ be two graphs. Then $\uparrow G \bigotimes \uparrow H \subseteq \uparrow (G \bigotimes H)$.

**Proof.** Let $K \subseteq \uparrow G \bigotimes \uparrow H$. Thus $\exists G' \subseteq \uparrow G, \exists H' \subseteq \uparrow H : K = G' \bigotimes H'$. Let $u, v \in \Pi$ such that $(u, v) \in G \bigotimes H$. We show that $(u, v) \in K$; this will entail that $K \subseteq \uparrow (G \bigotimes H)$.

Because $(u, v) \in G \bigotimes H$, $\exists w \in \Pi : (u, w) \in E(G) \wedge (w, v) \in E(H)$. But $G' \subseteq \uparrow G$ and $H' \subseteq \uparrow H$, therefore $(u, w) \in E(G') \wedge (w, v) \in E(H')$. We conclude that $(u, v) \in G' \bigotimes H' = K$.

What this means is that taking the closure-above of the products of our graphs over-approximate the actual model after $r$ rounds. And thus, algorithms working on these approximations work on the actual model.

Now, let us start with simple closed-above models. Just like for the one round case, they are completely characterized by the domination number of their underlying graph.

**Theorem 6.3 (Upper bound (multiple rounds) for simple closed-above models).** Let $A$ be a simple closed-above model defined by the graph $G$. Let $r > 0$. Then $\gamma(G')$-set agreement is solvable in $r$ rounds in $A$.

**Proof.** We have that $\gamma(G')$-set agreement is solvable on $G'$ by Theorem 3.2. This then implies by Lemma 6.2 that it is solvable on $\uparrow (G')$, that is on $A$.

But for general closed-above models, one cannot use the domination number itself, because one cannot know which of the underying graphs will be there. As in the one round case, we use the equal-domination number and covering numbers.

**Theorem 6.4 (Upper bound (multiple rounds) on $k$-set agreement by $\gamma_k(S)$ for general closed-above models).** Let $A$ be a general closed-above model generated by the set of graphs $S$. Let $r > 0$. Then $\gamma_k(S'^{\uparrow})$-set agreement is solvable in $r$ rounds in $A$.

**Proof.** We have that $\gamma_k(S'^{\uparrow})$-set agreement is solvable on $\bigcup_{G_1, \ldots, G_r \in S} \uparrow \bigotimes G_i$ by Theorem 3.4. This then implies by Lemma 6.2 that it is solvable on $\bigcup_{G_1, \ldots, G_r \in S} \uparrow \bigotimes G_i$, that is on $A$.

**Theorem 6.5 (Upper bounds (multiple rounds) on $k$-set agreement by covering numbers for general closed-above models).** Let $A$ be a general closed-above model generated by the set of graphs $S$. Let $r > 0$. Then $\forall i \in [1, \gamma_k(S')] : (i + (n - \gamma_k(S')))\text{-set agreement is solvable on the oblivious closed-above model generated by $S$ in $r$ rounds.}$

**Proof.** We have that $\forall i \in [1, \gamma_k(S')] : (i + (n - \gamma_k(S')))\text{-set agreement is solvable on} \bigcup_{G_1, \ldots, G_r \in S} \uparrow \bigotimes G_i$ by Theorem 3.7. This then implies by Lemma 6.2 that it is solvable on $\bigcup_{G_1, \ldots, G_r \in S} \uparrow \bigotimes G_i$, that is on $A$.

One issue with these bounds is that they require the computation of possibly many products, as well as the computation of the combinatorial numbers for a lot of graphs. One alternative is to forsake the best bound we can get for one that can be computed using only the numbers for the initial graphs.

This hinges on covering number sequences. Recall that the $i$-th covering number of a graph is the minimum number of processes hearing a set of $i$ processes that do not broadcast. In a sense, it gives the guaranty of propagation of information by a set of $i$ processes.

That’s the whole story for one round. But what happens when you do multiple rounds? Then, if the $i$-th covering number of the graph is greater than $i$, this means that in the next rounds, the minimum number of people who will hear the value of the $i$ initial processes is the $\gamma_k$-covering number. And if this number is greater than $\gamma_k$, this repeats.

Covering number sequences capture this process. One can also see them as the sequences of covering numbers for powers of the graph.

**Definition 6.6 (Covering number sequences).** Let $G$ be a graph. Then the $i$-th covering numbers sequence of $G$ is $\gamma_i = (s_i)$, such that $s_1 = \gamma_0(G)$ and $\forall k \geq 1 : s_{k+1} = \left( \frac{\left| \Pi \right|}{\gamma_k(G)} : \begin{cases} \gamma_k(G) & \text{if } s_k \geq \gamma_k(G) \\ \gamma_k(G) - s_k & \text{if } s_k < \gamma_k(G) \end{cases} \right)$

Armed with these sequences, we get an upper bound directly from $G$.

**Theorem 6.7 (Upper bounds on $k$-set agreement by covering numbers sequences).** Let $A$ be a simple closed-above model defined by the graph $G$ on $\Pi$. Then if the $i$-th covering sequence of $G$ reaches $n$ at some point, $i$-set agreement is solvable on the model $A$.

We can adapt this bound for general closed-above models by generalizing the covering numbers sequences to a set of graphs.

**Definition 6.8 (Covering numbers sequences for sets of graphs).** Let $S$ be a set of graphs. Then the $i$-th covering numbers sequence of $S$ is $s_i = (s_i)_{G \in S}$, such that $s_1 = \min_{G \in S} \gamma_0(G)$ and $\forall k \geq 1 : s_{k+1} = \left( \frac{n}{\min_{G \in S} \gamma_k(G)} : \begin{cases} \max_{G \in S} \gamma_k(G) & \text{if } s_k \geq \max_{G \in S} \gamma_k(G) \\ \max_{G \in S} \gamma_k(G) - s_k & \text{if } s_k < \max_{G \in S} \gamma_k(G) \end{cases} \right)$

**Theorem 6.9 (Upper bounds on $k$-set agreement by covering numbers sequences for general closed-above models).** Let $S$ be a set of graphs on $\Pi$. Then if the $i$-th covering sequence of $S$ reaches $n$ at some point, $i$-set agreement is solvable on the oblivious closed-above model generated by $S$.

**Proof.** If the $i$-th covering number sequence of $S$ reaches $n$ after step $r$, this means that every set of $i$ processes is heard by everyone after $r$ rounds. In particular, the $i$ processes with the smallest initial values will be heard by everyone.

Hence sending all the values heard for now for $r$ rounds, and then deciding the smallest value received, ensures that one of the $i$-th smallest values will be chosen, and thus solves $i$-set agreement.
6.3 Lower bounds for multiple rounds

Theorem 6.10 (Lower bound (multiple rounds) on k-set agreement for simple closed-above models). Let \( r > 0 \) and let \( A \) be a simple closed-above model generated by the graph \( G \). Then \((y(G)−1)\)-set agreement is not solvable on \( A \) in \( r \) rounds by an oblivious algorithm.

Theorem 6.11 (Lower bound (multiple rounds) on k-set agreement for general closed-above models). Let \( r > 0 \) and let \( A \) be a closed-above model generated by the set of graphs \( S \). Let \( l(R) = \min\{r+M_t(S')−2 \mid t \in \{1, y^\text{set}(S')−1\}\} \). Then \((l+1)\)-set agreement is not solvable on \( A \) in \( r \) rounds by an oblivious algorithm.

As a concrete applications of these bounds, we consider a classical family of subgraphs: stars.

Definition 6.12 (Star graphs). Let \( G \) be a graph. Then \( G \) is a star graph if \( \exists \Sigma \subseteq \Pi : G = (V, S \times \Pi) \).

Theorem 6.13 (Lower bound for stars). Let \( S \) be the set of graphs which are unions of \( j \) stars with different centers. Then \( n\)-set agreement is not solvable in the closed-above model generated by \( S \).

7 CONCLUSION

We provided upper and lower bounds on \( k \)-set agreement for closed-above models, the subset of round-based models defined by subgraphs that must be present in the communication graph at each round. These models encompass many message-passing models of distributed computing focused on safety properties.

Regarding the bounds themselves, although their proofs leverage combinatorial topology, all our bounds are expressed in terms of combinatorial numbers of the graphs. That is, these bounds can be used without any knowledge of combinatorial topology. Yet combinatorial topology was instrumental in showing such sweeping results.

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REFERENCES


