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We consider the unbounded settling dynamics of a circular disk of diameter $d$ and finite thickness $h$ evolving with a vertical speed $U$ in a linearly stratified fluid of kinematic viscosity $\nu$ and diffusivity $\kappa$ of the stratifying agent, at moderate Reynolds numbers ($Re = Ud/\nu$). The influence of the disk geometry (diameter $d$ and aspect ratio $\chi = d/h$) and of the stratified environment (buoyancy frequency $N$, viscosity and diffusivity) are experimentally and numerically investigated. Three regimes for the settling dynamics have been identified for a disk reaching its gravitational equilibrium level. The disk first falls broadside-on, experiencing an enhanced drag force that can be linked to the stratification. A second regime corresponds to a change of stability for the disk orientation, from broadside-on to edgewise settling. This occurs when the non-dimensional velocity $U/\sqrt{\nu N}$ becomes smaller than some threshold value. Uncertainties in identifying the threshold value is discussed in terms of disk quality. It differs from the same problem in a homogeneous fluid which is associated with a fixed orientation (at its initial value) in the Stokes regime and a broadside-on settling orientation at low, but finite Reynolds numbers. Finally, the third regime corresponds to the disk returning to its broadside orientation after stopping at its neutrally buoyant level.

Key words: stratified flows, particle/fluid flow, instability.

1. Introduction

While the rising/settling dynamics of rigid objects in a homogeneous fluid have raised considerable interest, the influence of a density stratification of the fluid on the motion of a body is a much more complex problem which starts to be explored. Such a fluid environment is, however, encountered in many environmental and industrial situations, where transport and mixing processes are related to the settling of biomass or pollutants to the bottom of the ocean or in the atmosphere. In particular, in marine biology it is desirable to gain a better understanding of the dynamics of microorganisms in a stratified environment (MacIntyre et al. 1995), of their role in mixing (Wagner et al. 2014; Houghton et al. 2018) and of their potential influence on the general circulation of the oceans (Wunsch & Ferrari 2004). The influence of density gradients on the dispersal of pollutants is likewise an essential question for waste-water disposal in the ocean (Koh & Brooks 1975), as it is for the quality of the atmosphere (Fernando et al. 2001). Also, atmospheric events such as dust storms and volcanic eruptions (Carazzo & Jellinek 2012) have a strong impact on air traffic and require further investigation and modeling.

Among the variety of freely moving bodies investigated for a homogeneous fluid, the...
disk of finite thickness is particularly interesting. Anisotropy of the body plays an important role on the motion of the body, especially when the velocity of the body departs from its principal axis, corresponding to axial symmetry (Ern et al. 2012). For a disk of finite thickness $h$, diameter $d$, and density $\rho_d$ evolving in a homogeneous fluid of density $\rho$ and kinematic viscosity $\nu$, the motion of the body depends on three parameters. First we define the Archimedes number

$$Ar = \sqrt{(\rho_d/\rho - 1)ghd/\nu},$$

(1.1)

which is analogous to a Reynolds number $(Ud/\nu)$ based on the gravitational velocity $U_g = \sqrt{(\rho_d/\rho - 1)gh}$ ($g$ being the gravitational acceleration) instead of the observed velocity $U$. The two other parameters are the density ratio $\mathcal{R}$, and the geometrical aspect ratio $\chi$ given by

$$\mathcal{R} = \frac{\rho_d}{\rho} \quad \text{and} \quad \chi = \frac{d}{h}.$$  

(1.2)

The buoyancy-driven motion of a disk in a homogeneous fluid has been thoroughly investigated experimentally (Willmarth et al. 1964; Field et al. 1997; Fernandes et al. 2007), numerically (Auguste et al. 2010; Chrust et al. 2013) and theoretically (Fabre et al. 2008; Tchoufag et al. 2014). These works mainly focus on the different non-rectilinear and non-vertical paths, along with their characteristics which are observed above a critical Archimedes number $Ar_c(\mathcal{R}, \chi)$ that depends on the density ratio and the aspect ratio. Among these, periodic motions are observed and associated with an unsteady wake and the periodic release of vortices. Below $Ar_c$, the body follows a rectilinear vertical path associated to a stationary flow in the frame moving with the body. This path corresponds to a broadside-on type of motion, where the principal axis of the disk is aligned with the velocity. Disks dropped with different initial inclination angles tend to orient themselves with their face normal to the direction of motion, indicating that a single stable position exists for the disk in this regime. This behavior is observed for Reynolds numbers that are greater than approximately 0.1 (this critical value depends on the body shape), whereas in the Stokes regime, the body retains its original orientation as it falls (McNown & Malaika 1950).

From a general point of view, in the case of a continuously stratified fluid characterized by its Brunt-Väisälä frequency

$$N = \sqrt{\frac{g}{\rho_0} \frac{d\rho}{dz}},$$

(1.3)

where $\rho_0$ is a reference density for the background density $\rho(z)$, the dynamics of moving objects are governed by two additional parameters, the Froude number $Fr = U/Nd$ and the Schmidt (or Prandtl) number $Sc = \nu/\kappa$ (or $Pr = \nu/\kappa$) describing the competition between diffusive effects in the fluid, we denote $\kappa$ as the diffusion coefficient of the stratifying agent. Experimental (Yick et al. 2007; Birò et al. 2008a) and numerical studies (Yick et al. 2009; Doostmohammadi et al. 2014) have been devoted to the freely falling sphere, showing that the influence of the linearly stratified fluid can be considered as an increase of the drag coefficient due to a buoyancy driven jet at the rear of the object. The intensity and structure of this jet, and more generally of the object wake, depend on the Reynolds, Froude and Schmidt numbers. Similar results have been obtained for moving spheres at fixed velocities (Torres et al. 2000; Hanazaki et al. 2009a, b, 2015; Okino et al. 2017), although the first observations of the peculiar nature of the stratified wake for vertically moving spheres originates from much earlier work (Ochoa & Van Woert 1977; Mowbray & Rarity 1967). In the limit of low values of the Reynolds number ($Re < 1$), recent studies investigated the influence of stratification on the Stokes dynamics of a
sphere in sharply stratified (two-layer) fluids (Srdić-Mitrović et al. 1999; Camassa et al. 2010) and linearly stratified fluids (Ardekani & Stocker 2010), the history force experienced by a vertically moving idealized object (Candelier et al. 2014), the mixing induced in the stratification (Wagner et al. 2014; Wang & Ardekani 2015), or the specific case of porous objects (Kindler et al. 2010; Camassa et al. 2013; Prairie et al. 2013).

The evolution of a disk in a linearly stratified fluid has not been studied to date. To our knowledge, a single numerical study considers the case of an anisotropic object, an ellipsoid, settling in a stratified fluid (Doostmohammadi & Ardekani 2014). The main result associated to this study is a rotational influence of the stratification on an initially tilted object, for $Re \simeq 0.1$. Some studies on moving droplets in a stratified ambient have also been realized (Blanchette & Shapiro 2012; Bayareh et al. 2013; Martin & Blanchette 2017), but the influence of the anisotropic shape of such object has not been studied.

This article presents the settling dynamics of a disk of finite thickness evolving in a linearly stratified fluid, at moderate to low values of the Reynolds number. We limit ourselves to low enough Reynolds numbers so that the orientation of the disk in an homogeneous fluid would remain broadside-on. It provides a parametrization for the steady stratified drag on a disk stably falling broadside-on for Reynolds numbers ranging from 1 to 100, and Froude numbers varying from 0.01 to 10. It also discusses the change of stability of the orientation from horizontal to vertical. The next section (§2) provides the analytical guidelines for the problem description. The experimental and numerical approaches are described in §3, the results are presented in §4 and discussed in §5, and conclusions are outlined in §6.

2. Modeling of the settling dynamics in a stratified environment

We provide here a formulation for the description of a circular disk settling in a linearly stratified fluid. Here, we neglected vertical variations of the kinematic viscosity since it varies only on the order of two percent over the whole water-column. We do not consider the case of sharp density interfaces, with spatial variations of $N$ and an inherently unsteady process, although several studies have provided analytical models on the forces on solid or porous objects (Camassa et al. 2010; Kindler et al. 2010; Camassa et al. 2013).

If we consider the settling dynamics of a disk with a fixed orientation with gravity (broadside-on for instance), one can consider its dynamics to follow

$$\rho_d \frac{dU}{dt} = - (\rho_d - \rho(z)) g + \frac{1}{2h} \rho(z) C_{SD} S U^2 + \frac{4}{\pi d^2 h} (F_H + F_A), \quad (2.1)$$

where $F_H$ and $F_A$ correspond to forces due to history and added-mass effects (Srdić-Mitrović et al. 1999; Biró et al. 2008b; Doostmohammadi et al. 2014). In the case of a sharp density stratification, other modeling approaches based on perturbations of the Stokes flow problem have been successfully developed, but not generalized to continuously stratified fluids (Camassa et al. 2010). We consider the disk to have a quasi-steady dynamics when the buoyancy force is instantaneously balanced by the drag force at any given depth. We define a stratified drag coefficient $C_{SD}$, similarly to Yick et al. (2009), as

$$C_{SD} = \frac{2(\rho_d/\rho(z) - 1)gh}{U(z)^2}. \quad (2.2)$$

This is valid only if the temporal variations of the velocity are weak enough to neglect added-mass effect and history forces, as it will be justified later. It should be noted that

† After submission of this manuscript, another study focusing on the dynamics of a disk encountering a stratified two-layer fluid (Mrokowska 2018) has been published.
in the case of a stratified fluid, even for a quasi-steady fall, several quantities describing the dynamics vary with $z$ including

\[
\mathcal{R}(z) = \frac{\rho_d}{\rho(z)}, \\
Re(z) = \frac{|U(z)|d}{\nu}, \\
Ar(z) = \frac{\sqrt{(\rho_d/\rho(z) - 1)ghd}}{\nu}, \\
Fr(z) = \frac{|U(z)|}{Nd}.
\]

For a single experimental or numerical study, the different parameters can vary over a range of values, however the ratio $Re/Fr = Nd^2/\nu$ remains constant, and the Archimedes and Reynolds number can be related by $Ar^2 = Re^2C_H^D/2$ for a quasi-steady fall.

In the limit case of a homogeneous fluid ($1/Fr \to 0$), in the parameter range $Ar \leq 50$ associated to $Re \leq 130$ for which the disk has a stable steady evolution falling broadside-on, the drag coefficient must tend to (Pitter et al. 1973; Clift et al. 1978)

\[
C_{HD}^H = \frac{64}{\pi Re^2} \left(1 + 0.138 Re^{0.792}\right), \tag{2.7}
\]

valid for $1.5 \leq Re \leq 133$ and for disks with $\chi > 1$.

For any stratified fluid ($Fr$ being finite), in the same parameter range, the drag coefficient can be related to the one in a homogeneous fluid as follows

\[
C_{HD}^S = C_{HD}^H \{1 + \Pi(Fr, Re, R, \chi, Pr)\}, \tag{2.8}
\]

with $\Pi$ tending to zero when $Fr$ tends to infinity.

In the case of a spherical object evolving in a linearly stratified fluid, one can consider that eq. (2.8) should be written

\[
C_{SD}^S = C_{HD}^H (1 + aRe^{p}Fr^{q}), \tag{2.9}
\]

where $a$, $p$ and $q$ are constants. These constants can be different, depending on the nature of the stratified fluid through the Schmidt (or Prandtl) number, and on the range of the Reynolds and Froude numbers that are considered, as suggested by previous studies. Indeed, for a sphere at low enough values of the Reynolds number ($Re < 10$), analytical studies have found power-law dependency with $p = 1/3$, $q = -2/3$ (Zvirin & Chadwick 1975) or $p = 1/4$, $q = -1/2$ (Candelier et al. 2014) is possible. Experimental and numerical studies (Yick et al. 2009; Kindler et al. 2010) have matched the values of $p = 1/2$, $q = -1$ and $a \simeq 1.9$. As a remark, these studies present the Richardson number $Ri = Re/Fr^2$ as a more relevant parameter to describe the stratified drag for regimes where buoyancy and viscous forces are at play. Numerical studies (Bayarch et al. 2013) have found $q = -2.8$ and $a \simeq 21$ for a droplet rising at fixed values of the Reynolds number ($Re = 396$ and 792).

3. Experimental and numerical approaches

3.1. Experimental approach

We consider disks of finite-thickness (or short-length cylinders) whose densities range between $1020$ and $1025$ kg/m$^3$. Their diameters $d$ and heights $h$ range from $5$ to $20$ mm and from $1$ to $5$ mm, respectively. The corresponding aspect ratios $\chi = d/h$ are $3$, $6$ and $10$, determined with an accuracy of $1\%$. Disks have been manufactured from cylindrical
Settling disks in a linearly stratified fluid

bars of Nylon and have a density close to (but always larger than) the surrounding salt-stratified fluid. Based on the stratification realized, the parameter $R - 1$ varies from $10^{-2}$ to zero, when the disk reaches its neutral depth. Small imperfections in their design could lead to minor imperfections in the distribution of mass, as discussed in §4.4.

The disks are released in a large glass tank (50 cm high with a square cross-section of 56 cm width) containing salt-stratified water ($Sc \approx 700$). The linearly stratified fluid is obtained using the double-bucket method (Oster 1965; Economidou & Hunt 2009), the stratification is measured using a Microscale Conductivity-Temperature probe (from PME) displaced using a linear traverse which, after calibration, had a precision of 1% in conductivity and temperature. Small fluctuations of $\rho(z)$ compared to a linear profile can induce some uncertainties in the estimate of a mean value for $N$, of the order of 5 to 10% on average (see Fig.2(a) for instance), with the specific case of small value of $N$ reaching 40%. Plastic wrap placed over the top of the tank prevents perturbation induced by convection and evaporation.

The disks are released from a twizzer after being introduced just below the free surface, into a small cylinder at the center of the tank, immersed down to 5 cm below the free surface as indicated in Figure 1. The main reason for this process is to maintain the disks near the center of the tank and avoid large flow perturbations in the tank when introducing or removing the twizzer. However, it should be noted that the presence of the cylinder can also induce weak perturbations of the velocity and inclination of the disks when they escape from the cylinder and evolve in the large section of the tank. Below the cylinder, we consider their dynamics unbounded (the distance to the side-walls is larger than 15 diameters for the largest disk).

A pair of identical cameras (1280x1024 pixels) image two perpendicular fields of view of the tank at the sampling frequency between 1 and 5Hz. Camera 1 images the $(y, z)$-plane and camera 2 the $(x, y)$-plane, and their fields of view correspond to physical dimensions of roughly 32 cm x 26 cm in the center of the tank. This is sufficient to record most of the trajectory without moving the cameras, but the location of the cameras does not permit to visualize the bottom part of the cylinder, hence we cannot have a recording of

Figure 1. Schematic of the experimental setup. (a) Overall three-dimensional view, (b) horizontal and (c) side views.
the initial conditions of release for the disk orientation nor its speed. The cameras are located nearly 2 m away from the center of the tank and follow the contour of the object that appears dark over a white background, which allows for a recording of the center of mass of the disk, along with its contour and orientation. Optical setup allows for a depth of field resolving the three phases of the trajectory as discussed in §4.

The stratification can affect the detection of the object due to the vertical gradient of the optical index $n$ which is proportional to the gradient in density in a stratified fluid. As discussed in Dalziel et al. (2007), the variation in the observed position due to the stratification is proportional to $\delta z = (L^2/2)\partial \ln n/\partial z$ when observing the object located at a distance $L$ from the side of the tank. In our case, the weak and constant density gradient ($\partial \rho / \partial z \simeq 25 \text{ kg.m}^{-4}$) corresponds to an optical gradient $\partial n / \partial z \simeq 5 \times 10^{-3} \text{ m}^{-1}$, that leads to $\delta z \lesssim 2 \times 10^{-4} \text{ m}$. When compared to the disk thickness $h$, $\delta z/h \lesssim 0.22$ for the disks used, this indicates an absolute error of $0.22h$ in the position of the disk, which is constant over the whole domain. This does not affect the velocity measurements nor the estimate of the non-dimensional parameters defined in §2.

The technique used to extract the three-dimensional dynamics is similar to the approach described in Fernandes et al. (2007), and is described in Figure 1(b) and (c). The two cameras are calibrated in three dimensions. For each camera, by imaging a grid at several positions along the optical axis (the grid plane being perpendicular to it), we measure the evolution of the spatial resolution $\beta_i$ ($i = 1$ or 2) from pixel to meter along the optical axis, such that

$$
\beta_1(x) = \beta_0^0 (1 + \epsilon_1 x) \quad \text{and} \quad \beta_2(y) = \beta_0^0 (1 + \epsilon_2 y).
$$

(3.1)

As explained before, we neglect the influence of the stratification on the calibration, and $\beta_i$ does not depend on $z$. In practice, typical values are $\beta_0^0 \simeq 2.5 \times 10^{-4} \text{ m/pix}$, and their variation with position along the optical axis is $\epsilon_i \simeq 10^{-2} \text{ m}^{-1}$. The three-dimensional position of the disk is obtained by an iterative process. The position of the disk (center of mass) is first estimated in each image by using a first guess of the spatial resolution to use ($\beta_i = \beta_0^0$). It allows for a new estimate of the $(x, y, z)$-coordinates of the disk, which can be used to update the value of $\beta_i$. The position of the disk is then computed based on these new values of the spatial resolutions for each camera. The iterations end when the norm of the distance between the new position of the disk and the previous estimate is less than $10^{-5}\text{m}$.

A fluorescent dye and UV-lightning are used in some cases to visualize the structure of the stratified wake. The disks are immersed in a concentrated solution of fluoresceine with a density matching the density of the top fresh layer of the stratification, before being released in the stratified tank.

For all experimental runs, the parameters are described in Table 1.

### 3.2. Numerical approach

Using the Boussinesq approximation, the equations of motion for viscous, incompressible fluids with variable density in the entire computational domain are written as

$$
\nabla \cdot \mathbf{u} = 0, \quad (3.2)
$$

$$
\rho_0 \frac{D \mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + (\rho - \bar{\rho}) \mathbf{g} + \mathbf{f}, \quad (3.3)
$$

where $t$ is the time, $\mathbf{u}$ the velocity vector, $p$ the hydrodynamic pressure, $\mathbf{g}$ the gravitational acceleration, $\mu$ the dynamic viscosity of the fluid, $\rho_0$ the reference fluid density, and $\bar{\rho}$ the volumetric average of the density over the entire computational domain. The density $\rho$ can be written as $\rho = \rho_f + \phi (\rho_d - \rho_f)$, where $\rho_f$ is the fluid density that depends
on the fluid temperature or salinity; the indicator function \( \phi \), which represents the volume fraction of grid cells occupied by the solid disk, is a phase indicator to identify the disk and liquid phases with \( \phi = 1 \) inside the disk and \( \phi = 0 \) inside the fluid domain. The body force \( \mathbf{f} \) in the momentum equation, Eq. (3.3), accounts for the solid-fluid interaction by using a distributed Lagrange multiplier (DLM) method, which has been extensively employed to study the settling of rigid, general-shaped particles in both homogeneous fluids and stratified fluids (Ardekani et al. 2008; Doostmohammadi & Ardekani 2013; Doostmohammadi et al. 2014; Doostmohammadi & Ardekani 2014, 2015). The disk is impermeable to the stratifying agent and temporal evolution of the density field is governed by a convection-diffusion process described by

\[
\frac{D \rho}{Dt} = \kappa \nabla^2 \rho, \tag{3.4}
\]

where \( \kappa \) is the diffusivity of the stratifying agent. Equations (3.2)-(3.4) are discretized using a finite volume method on non-uniform fixed cartesian staggered grids. The time discretization is obtained using a first-order Euler method. The convection and diffusion terms in both Equations (3.3) and (3.4) are solved using the QUICK (quadratic upstream interpolation for convective kinetics) and central-difference schemes (Leonard 1979), respectively. The initial fluid density linearly varies with depth \( \rho_f = \rho_0 + \gamma(z - z_i) \), where \( \gamma \) is the vertical density gradient and \( z \) is the vertical component of spatial coordinates. The periodic boundary conditions for density and velocity components are used on side boundaries of the rectangular computational domain. The boundary conditions for density and velocity on the top and bottom boundaries are defined as \( \frac{\partial \rho}{\partial z} = \gamma \) and \( \frac{\partial u}{\partial z} = 0 \), respectively.

We would like to highlight that stratified flows develop thin boundary layers at high Prandtl numbers that are computationally expensive to resolve. In this work, we use a horizontal and vertical resolution of 128 grid points per diameter for all numerical studies. While the density distribution in the jet would not be resolved with grid sizes below \( d/128 \), the settling velocity and CD would be still accurate for grid sizes as coarse as \( d/64 \). While the grid resolution used in this manuscript is not enough to resolve the thickness of a steady jet, the disk orientation instability occurs well before the steady jet is developed; and during this transient behavior studied in the present work, the jet thickness is much wider than the theoretical estimation for the steady jet (see Appendix A.1 for details). Consequently, the settling velocity and the orientation of the disk can be computed accurately, in agreement with the experiment. Finally, increasing the value of \( N \) generates a narrower jet, and requires a finer resolution, leading to computationally expensive simulations.

The size of the computational domain is \( 4d \times 4d \times 25d \), this lateral extent does not modify the onset of the instability compared to larger domains (see Appendix A.2).

For all numerical runs, the parameters are described in Table 1. Additional runs for direct comparison with experiments are also presented in Appendix A.3.

4. Results

We first present the overall generic trajectory for a disk settling in a linearly stratified fluid, before investigating more thoroughly the three distinct phases. As mentioned in the introduction, we limit ourselves to low enough Reynolds numbers so that the
The overall trajectory of a settling disk is shown in Figure 2. Figure 2(a) indicates the density stratification $\rho(z)$ and the corresponding profile of $N(z)$, which can be considered nearly constant with $N \simeq 0.5$ rad/s over the entire column in this case. Figure 2(b) displays the trajectory of the center of the disk, along with the disk orientation (gray rectangle). Finally, Figure 2(c) provides the corresponding evolution of the Archimedes and Froude numbers during the settling process.

Three regimes for the settling dynamics have been identified before the disk reaches its neutrally buoyant level. First, the disk experiences a quasi-steady phase when the stratification enhances the steady drag experienced by the disk falling broadside-on. We identify this first phase based on the inclination of the disk ($\theta \approx 0$). Along this phase, we still have a variation of the velocity and non-dimensional parameters with $z$, as can be noticed in Fig. 2(c). Then, there is a change of stability for the disk orientation (from

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<td></td>
<td>1.20</td>
<td>1.2</td>
<td>1.0221</td>
<td>20</td>
<td>$\neq 0$</td>
<td>1.020</td>
<td>1.0</td>
<td>1.43</td>
<td>0.18</td>
<td>10</td>
<td>700</td>
<td>16</td>
</tr>
<tr>
<td>E14</td>
<td></td>
<td>1.06</td>
<td>3.7</td>
<td>1.0225</td>
<td>19</td>
<td>$\neq 0$</td>
<td>1.021</td>
<td>1.0</td>
<td>1.43</td>
<td>0.50</td>
<td>2.9</td>
<td>700</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 1. Summary of experiments and numerical simulations (labeled EX and NX with X being a number, respectively). For experiments, $Ar_i$ is the value estimated from the start of the trajectory.

orientation of the disk in a homogeneous fluid would remain broadside-on, typically $Re \lesssim 100$ (Willmarth et al. 1964).

4.1. Overall trajectory of a settling disk

The overall trajectory of a settling disk is shown in Figure 2. Figure 2(a) indicates the density stratification $\rho(z)$ and the corresponding profile of $N(z)$, which can be considered nearly constant with $N \simeq 0.5$ rad/s over the entire column in this case. Figure 2(b) displays the trajectory of the center of the disk, along with the disk orientation (gray rectangle). Finally, Figure 2(c) provides the corresponding evolution of the Archimedes and Froude numbers during the settling process.
broadside-on to edgewise) when the value of the disk’s velocities become ‘sufficiently’ small. Similarly to phase 1, we identify the start of this second phase based on the inclination of the disk, when it becomes non-zero (we set an arbitrary threshold $\theta > 5^\circ$).

Finally, the last phase corresponds to the disk settling edgewise until stopping at its gravitational equilibrium level $z_0$, defined by $\rho(z_0) = \rho_d$. At the end of its settling, it returns towards a horizontal orientation. We identify the start of this final phase based on $\theta \simeq 90^\circ$ or by the maximum value reached along the trajectory, some disks ending their trajectory without completing the whole phase 2.

It should be noted that for both the experimental and numerical cases studied, the three phases of the dynamics occurs in a plane (no helicoidal motion is observed). Furthermore, for a given experiment or numerical simulation, all the non-dimensional parameters are not independent, more specifically the Reynolds $Re$ and Froude $Fr$ numbers follow this relationship $Re/Fr = N^2/d^2/\nu$ which is constant for a single experiment. $N$ and $d$ are different for different experiments.

4.2. Phase 1: Broadside-on settling

We consider the part of the trajectory with broadside-on settling, corresponding to phase 1 in Fig. 2(b). We first study the assumption of quasi-steady fall for the disk broadside-on, by comparing the importance of acceleration with drag and buoyancy forces. As can be seen in Figure 3(a), it is for non-dimensional times $Nt/2\pi$ smaller than 2 that the velocity of the disk changes the most. This is confirmed in Figure 3(b) where we measure $dU/dt$ to be of the order of $10^{-3}$ m/s$^2$ at most. For the problem considered, buoyancy forces are of order of $10^2$ N/m$^3$ whereas an estimate of the stratified drag coefficient is at least as large as in homogeneous case ($C_D^H \sim 1$ for $Re \in [10,100]$), leading to $\rho(z)C_D^H U^2/2h \sim 10^2$ N/m$^3$ in our experiments. If we compare the values of
these terms of opposite signs on the right-hand-side of eq. (2.1) with acceleration forces which are of order 1 N/m$^3$, this confirms our assumption of quasi-steady fall where buoyancy and drag forces are the dominant forces, and the small differences between the two is associated to small acceleration forces. Similar results have been obtained from the numerical simulations, although not shown here.

We seek for a generic expression for the stratified drag coefficient of a disk, thus we consider the trajectories of many disks of various aspect ratios and diameters ($d$, $h$ and $\chi$ varied), evolving in different stratified environments ($N$, $\nu$ and $\kappa$ changed). The parameters for each experimental or numerical run are listed in Table 1.

The numerical simulations provide a more complete description of the flow around the disk in this quasi-steady phase. Here we present results associated to a steady broadside-on settling for disks initially released with no velocity and $\theta_i = 0$ (runs N10 and 11). As the disk settles across a linearly stratified fluid, lighter fluid is drawn down from its original position. It is visible in Figure 4 that the vertical displacement of isopycnals, denoted as $\Gamma = (z_\rho(t) - z_\rho(0))/d$, can be as large as 3. Similar observations have been reported for a sphere moving downward in a salt-stratified fluid ($Sc = 700$), where isopycnals are displaced for a large extent (Hanazaki et al. 2009b; Doostmohammadi et al. 2014; Hanazaki et al. 2015). One can notice a bell-shape structure along the jet axis, in the far wake as described in Hanazaki et al. (2015). This structure is related to internal waves generated in the wake of the settling object. Internal waves generated upon settling can affect phase 3, as discussed later. Nevertheless, during the quasi-steady phase, since the flow structure around the disk remains axisymmetric (see Figure 4(a) and (b)), the disk settles under the broadside-on configuration.

Based on the disk velocity and the background fluid density profiles along the trajectories in Figure 3, we can compute the stratified drag coefficient $C_{SD}$ using eq. (2.2). To compare it to the homogeneous case and seek for an expression similar to eq. (2.8), we display $C_{SD}/C_{HD} - 1$ in Figure 5(a) as a function of the Froude number. The same quantity is also obtained from numerical runs. Although the observations seem to exhibit similar trends in Figure 5(a), the results do not all collapse onto a unique curve and we seek for a better description of the normalized drag coefficient with parameters. Yick et al. (2009) suggest that the origin of the added drag on a settling sphere in a linearly stratified fluid is due to the extra buoyancy force generated by a spherical shell of light fluid around the sphere, entrained by viscosity over some distance while settling. Any other fluid motion in the wake of the object is a remnant of the sphere’s passage but does not contribute to the stratified drag. Thus we can express the extra stratified drag normalized with the drag for a homogeneous fluid as a buoyancy force corresponding to the fluid displaced
Figure 4. Colormaps of (a) velocity $|\mathbf{u}|$ and (b) vorticity $|\omega|$ fields for run N12. (c) Zoom on the velocity vector field around the disk. Black solid lines are isopycnals for $\rho$ equals $1017.5 \text{kg/cm}^3$ to $1020.6 \text{kg/cm}^3$ by step of $0.5 \text{kg/cm}^3$. In (c), isopycnals with $\rho = 1017.5 \text{kg/cm}^3$ and $\rho = 1020.6 \text{kg/cm}^3$ are not visible. The images correspond to $U = 0.18 \text{cm/s}$.

Figure 5. Stratified drag coefficient $C_{SD}^S$ minus the homogeneous drag coefficient $C_{SD}^H$ in (a), and normalized by a function of the aspect ratio $\chi$ in (b), as a function of the Froude numbers during quasi-steady settling for trajectories identified in Figure 3(c). Symbols correspond to numerical simulations. The dashed line in (b) is a fit of the power law as given in eq. (4.3).

with the object,

$$\frac{C_{SD}^S}{C_{SD}^H} - 1 \simeq \frac{\Delta \rho g V}{\frac{1}{4} C_{DH} U^2 S},$$  \hspace{1cm} (4.1)

with $S$ the apparent section of the object, $V$ the volume of light fluid displaced and $\Delta \rho$ its density difference with the fluid at the depth of the object. When applied to the disk geometry, we can consider that a shell of light fluid of width $\delta$ has a volume $V$ which scales like $(\pi/2)d^2\delta(1 + 2/\chi)$ when $\delta/d \ll 1$, compared to the case of the sphere with $V$ that scales like $\pi d^2 \delta$. Observations from numerical simulations zoomed on the proximity of the disk, as shown in Figure 4(c), confirm that a thin shell of light fluid moves with the object (see also Fig. 7). For both geometries, $S$ scales like $(\pi/4)d^2$. The light fluid
composing the shell has been displaced over some distance $\Gamma d$, as already discussed when presenting the numerical results in Figure 4, and we can estimate its density based on the density gradient such that $\Delta \rho = \Gamma d(N^2 \rho/g)$. In the end, the extra drag due to the stratification normalized with the homogeneous fluid becomes

$$
\frac{C^S_D}{C^H_D} - 1 \approx \frac{1}{\frac{C^H_D}{Fr^2} \Gamma d \left(1 + \frac{2}{\chi}\right)},
$$

and we must consider proper scalings for $\delta$ and $\Gamma$.

It has been shown in previous studies (Yick et al. 2009; Hanazaki et al. 2015) that the width $\delta$ of the light fluid shell entrained by viscous effects should scale like $\sqrt{\rho/\Delta \rho}$, which leads to $\delta/d \sim \sqrt{Fr/Re}$. It should be noted that for a single experiment, $\delta/d$ is constant, but depends on the fluid and disk properties. The scaling law for the vertical entrainment of fluid $\Gamma$ depends on the Froude number, but is not universal. Typically, one considers $\Gamma \sim Fr^\alpha$, and the constant $\alpha$ depends on the range of values encountered for $Re$. Observations led to $\alpha = 1/2$ for a sphere when $Re < 1$ (Yick et al. 2009) and $\alpha = 1$ for spheres (Hanazaki et al. 2015) with $Re \geq 200$ and $Fr \geq 1$, or for cylinders with $Re \sim O(10^4)$ (Higginson et al. 2003). As displayed in Figure 5 (b), by looking at $(C^S_D - C^H_D)^2/Re/Fr/(1 + 2/\chi)$ we seek for a rescaled evolution of the drag coefficient that depends on $Fr$ only, and the data nicely collapses into a single curve that can be fitted as

$$
C^S_D = C^H_D \left[1 + \frac{a}{C^H_D} \left(1 + \frac{2}{\chi}\right) \sqrt{Fr/Re} Fr^\alpha \right]
$$

with the fitting parameters in bold, $a = 14 \pm 2$ and $q = -1.7 \pm 0.1$. This validates the discussion above with a value of $\alpha = 0.3 \pm 0.1$ for the case of a disk.

In the end, the drag model in eq. (4.3) is consistent with eq. (2.8), with only $\chi$, $Re$ and $Fr$ playing a key part in this parameter space for values of $Re \in [5,130]$, $Fr \in [0.1,3]$, $Pr \in [70,700]$ and $\chi \in [3,10]$. One must notice the good agreement for both the experimental and numerical results with the same modeling approach. These results can be furthermore validated by investigating the relation between the Archimedes and Reynolds number. As mentioned in §2, one expects $2(Ar/Re)^2 = C^S_D$ in a quasi-steady regime. If we compare $2(Ar/Re)^2$ to the expression in eq. (4.3), we do find a good correlation for phase 1 (cf. Figure 16(b) in Appendix B).

Compared to the known results for the sphere (Yick et al. 2009) in the same range of parameters $(C^S_D/C^H_D - 1 \sim Re^{1/2} Fr^{-1})$, here we find $C^S_D/C^H_D - 1 \sim (1 + 2/\chi) Re^{1/2} Fr^{-1/2}$ if we consider the drag on the disk to be $C^H_D \sim Re^{-1}$. This indicates that the disk is more strongly affected by the stratification (stronger added drag) than a sphere of similar diameter. We will come back on the implication of this aspect (cf. Table 2) in the conclusion.

### 4.3 Phase 2: Instability of the broadside-on settling

When the vertical velocity of the disk decreases, there is a change of stability for the disk orientation from broadside-on to edgewise settling, this part of the trajectory corresponds to phase 2 in Fig. 2 (b). As we will see, this transition in the orientation of the disk from 0 to $\pi/2$ seems to be a robust feature, for values of the Reynolds number in the range 10 to 50, and of the Froude number in the range 0.1 to 2 and various stratified environments. It should be noted that some of the disks do not reach the $\pi/2$-orientation before initiation of phase 3 (when the disk reaches its equilibrium depth). Furthermore, although not shown here, it is important to stress that the dynamics of phase 2 takes place in a plane, no helicoidal motion has been observed in experiments or numerics.
Figure 6 displays the evolution of the orientation as a function of the vertical velocity $U$ normalized by $\sqrt{\nu N}$ for both experimental and numerical runs. One should notice that the dynamics of all disks starts in phase 1, corresponding to a large velocity with $\theta \sim 0$, then its rotation starts (phase 2) at some lower velocity where $\theta$ suddenly increases; each trajectory in Fig. 6 reads from right to left. In the numerical simulations displayed (runs N1 to 10), it should be noted that we set the initial inclination at $\theta_i = 5^\circ$. For numerical simulations of a single disk released with $\theta_i = 0$ (runs N11 to 12, used in phase 1), the analysis shows that the orientation angle of these disks remains constant at $\theta \simeq 0$ during the whole settling motion computed.

One can notice that all disks start rotating below a threshold non-dimensional velocity of $3.9 \pm 1.1$ for experimental runs (Fig. 6(a)), and of $1.8 \pm 0.2$ for numerical runs (Fig. 6(b)). Since there is some dispersion of the values for the non-dimensional velocity at which the angle starts to increase, we defined the threshold value for $U_c$ to be the velocity when $\theta = 5^\circ$. We also denote $\rho_c$ as the density of the fluid for this instant in the trajectory. We did not find such a collapse of the evolution of $\theta$ when searching for the influence of $Pr$, $Ar$ or $Fr$ on this threshold. We discuss the dynamics of the change of stability in more details in §5.

Finally, we emphasize that the dynamics of phase 2 is associated with a complicated process coupling the stratified flow structure in the wake of a freely rotating object of anisotropic shape. However, some important features can be identified in the experiments and numerical simulations. In Figure 7, we present snapshots of the stratified wake visualized using dye for experiments, and by tracking the magnitude of the density perturbations for the numerics (see colormap in Figure 7(a)), for two different runs. Similarly to the case of a sphere studied in Hanazaki et al. (2009a), the stable wake of a moving object in a stratified fluid is mainly composed of a narrow vertical jet. This jet is generated by the up-going fluid that is lighter than its surroundings, initially entrained by viscosity within a thin shell around the object as already discussed in §4.2. When the disk starts to rotate, this narrow jet slowly drifts from the center of the back face of the disk inducing a torque on the disk that amplifies the rotation. Both the experimental and the numerical approach are able to get this dynamics with very similar trends between the two.

This change of the stratified wake behind the disk could depend on its aspect ratio, its settling speed, its orientation and the stratified environment. We discuss the stability...
Figure 7. Examples of wake visualizations during phase 2 in experiments and numerics, the parameters chosen for each run are different. Maps of (a) density variations $\rho' = \rho - \rho_f$ for the numerical simulation (run N10), and (b) fluorescent dye for the experiment (run E14). At the first frame, $Re \simeq 40$ and $Fr \simeq 0.75$ for experiments and $Re \simeq 17$ and $Fr \simeq 0.23$ for numerics; each frame is separated by $\Delta t = 2s$, i.e. $N \Delta t = 1$.

and the dynamics of the rotation in §5, but we anticipate that more studies are needed to characterize the vertical dynamics of a tilted disk, when its orientation is fixed since it is impossible to discriminate what effect occurs first, the modification of the orientation of the disk or the position of the jet (if any order is applicable).

4.4. Phase 3: Reaching a neutrally buoyant depth

Finally, when the disk ends up settling edgewise and gets close to its neutrally buoyant depth $z_0$, it returns to a nearly horizontal orientation, which we characterize by an equilibrium angle $\theta(z_0) = \alpha$. In the case of a perfectly designed disk, with the center of buoyancy at the center of the geometry, one expects $\alpha = 0$. However, as described in appendix C, minor imperfections in the distribution of mass (or in the shape equivalently) can lead to non-zero equilibrium inclination angles. We investigate the impact of this observation in §5.4.

This last phase is associated to a nearly zero horizontal drift while rotating as shown in Figure 8(a) and (b), although the spatial resolution might not be sufficient to conclude. Similarly to the change of stability at the initiation of phase 2, the rotation from a nearly vertical disk to the equilibrium angle seems to be similar for all disks, as shown in Figure 6(b), with a clear change of stability from vertical to horizontal. It is interesting to notice that this rotation can be towards 0 or $\pi$, suggesting that the initiation of phase 2
Figure 8. (a) The settling trajectories, (b) the orientation angle and (c) the temporal evolution of the vertical position in logarithmic scale for experimental disks to emphasize on phase 3.

of the trajectory is not automatically correlated with the imperfection in the distribution of mass (heavy face can be up or down).

The temporal evolution during this last phase is very slow compared to phases 1 and 2, with values of the Reynolds number smaller than 1. Numerical simulations of this last phase have not been obtained due to very long computational times. As shown in Figure 8(c), most disks reach $z_0$ smoothly. However, some disks oscillate around the equilibrium position $z_0$. We observed that all the oscillatory cases are associated with a maximum value for the Reynolds number larger than 80, whereas the maximum values of the Froude number could be smaller or larger than 1. These oscillations are due to internal motion of the fluid (internal waves) generated while releasing the disks or during phase 1 as mentioned in §4.2, since the frequency of this oscillatory motion is near $N$.

This is similar to observations made for a sphere settling in a linearly stratified fluid from the surface down to a neutrally buoyant depth $z_0$ (Birò et al. 2008a), where oscillations around $z_0$ were observed due to internal waves generated by the wake of the settling body, with initial values of the Reynolds number larger than 300.

5. Discussion on the change of stability of the disk orientation

Based on the experimental and numerical results, several points have been identified in the dynamics of a disk in a linearly stratified fluid with various values for the disk and the fluid properties. First of all, the broadside-on settling at moderate Reynolds number ($Re > 10$) is a stable regime for the disk, for Froude numbers larger than 0.1 to 1 (depending on the value of the Reynolds number). An instability of the broadside-on settling occurs when the non-dimensional velocity $U/\sqrt{\nu N}$ becomes smaller than a threshold value ($3.9 \pm 1.1$ for experiments and $1.8 \pm 0.2$ for numerics) leading to the edgewise orientation as a new stable orientation, as shown in Fig. 6. The threshold for experimental and numerical runs have similar values and the instability in orientation is a robust effect of the stratified environment. However more information can be provided on the disk dynamics and on the origin of the instability, in order to help modeling this instability. We provide here complementary analyses and results by means of numerical simulations to investigate the change of stability of the orientation of the disk.

5.1. Possible orientations for settling

We first focus on the perfectly balanced disk, and discuss the stability of specific orientation for settling. As described in §4.2, we have investigated the stability of the broadside-
on settling in a linearly stratified fluid for a perfectly balanced disk. All numerical simulations initiated in a fluid at rest, with no velocity \((U_i = 0)\) and no tilt for the disk \((\theta_i = 0)\), lead to a steady vertical settling for the disk. More precisely, during the same time \((Nt/2\pi < 3)\), variations in the orientation angle remain very weak \((\theta < 10^{-1} \text{ rad})\), and have a very weak tendency to increase. Hence we conclude that the settling dynamics in a linearly stratified fluid at rest, with an initial orientation \(\theta_i = 0\), can be considered an equilibrium position for the disk. A complementary study investigates the stability of the edgewise settling. Numerical simulations of a disk (similar to run N11) are realized with an initial orientation \(\theta_i \approx \pi/2\) and different values of the viscosity. In a homogenous fluid, the disk eventually settles under the broadside-on configuration regardless of initial release condition. When settling in a linearly stratified fluid, a disk with an initial orientation \(\theta_i = 85^\circ\) can be stable at low values of the Reynolds and Froude numbers with \(Re \leq 11.2, Fr \leq 1.14\) \((\nu = 10^{-5}\text{m}^2/\text{s})\), and is unstable for larger values tested such as \(Re = 304, Fr = 3.1\) \((\nu = 10^{-6}\text{m}^2/\text{s})\). This is in good agreement with the experimental observations. Here again we can conclude that the settling dynamics in a linearly stratified fluid at rest, with an initial orientation \(\theta_i = \pi/2\), can be considered an equilibrium position for the disk at low enough velocities. We must now study the stability of these equilibrium positions.

5.2. Stability of the orientation for settling

To better understand disk changing stability in the orientation (phase 2), we present in §4.3 runs realized with an initial inclination of the disk, \(\theta_i = 5^\circ\). Although the initial perturbation can have a different nature for experiments and numerical simulations, the dynamics of the instability are similar for both approaches. A useful tool to follow the change of equilibrium orientation is the phase diagram \((\theta, \dot{\theta}/N)\) which is shown in Figure 9 for experiments and numerics. All trajectories of the disks in the phase diagram exhibit very similar trends. First there is a convergence towards \((0, 0)\), the stable broadside-on settling. A spiral associated to damped pitching oscillations with an amplitude of \(\dot{\theta}/N\) near \(\theta = 0\) being \(O(1)\) is observed, with differences in amplitude between experiments and numerical simulations that are due to the different initial conditions of release. In the case of a homogeneous fluid, the damped pitching oscillations with a frequency \(f\) are related to the natural frequency of oscillation of the body \(U/d\), corresponding to a Strouhal number.
Figure 10. Temporal evolution of the orientation of the disk $\theta$ for (a) experiments and (b) numerical simulations, along with the exponential growth using $\sigma_m$ (2.7 in (a), 2.8 in (b)).

$St = fU/d$ of order 0.1 (Willmarth et al. 1964). In the stratified environment, similar oscillations occur but could be affected by the stratification too. Nevertheless, we cannot discriminate the role of $U/d$ from $N$, the natural frequency in the stratified fluid, since their ratio $U/Nd = Fr$ is $O(1)$ in our study.

Then the $(0, 0)$ equilibrium position becomes unstable and all trajectories follow a positive or negative branch (black dashed line, discussed in §5.3) towards a new stability point $(\pi/2, 0)$, with a clear change of stability towards edgewise settling. It should be noted that some trajectories do not reach this point before the end of settling, when the disk reaches its equilibrium depth before a complete change in orientation.

The final branch of the trajectories in Fig. 9(a) correspond to phase 3, when the disk at its equilibrium depth rotates towards its non-zero equilibrium inclination angle in a quasi-static manner ($\dot{\theta}/N \sim 0$).

5.3. Temporal evolution for the change of orientation

We focus now on the temporal evolution of the orientation of the disk for the same trajectories shown in Figure 9. The results are presented in Figure 10, with the same display for the experimental and numerical results. We set a common origin of time for an arbitrary orientation $\theta = 5^\circ$ corresponding to a time $t = t_c$. There is a good collapse of early dynamics for the orientation of all disks in (a) experiments and (b) numerical simulations. Furthermore, it is possible to model this early dynamics by an exponential growth, as expected for an instability mechanism, with $\theta(t) = 5 \exp \left[ \sigma N(t - t_c)/2\pi \right]$. Values for the growth rate $\sigma$ of each trajectory have been extracted and the mean value ($\sigma_m \simeq 2.7$) is similar for experiments and numerical simulations (2.7 and 2.8 respectively). No significant influence of the aspect ratio of the disk on $\sigma$ have been observed. At fixed value of $\chi$, a small increase in $\sigma$ of 6% and 10% can also be observed when the viscosity increases by a factor 3 and 7 respectively (runs N1 to N3). The growth rate does not depend on the Archimedes, Froude, Reynolds or Richardson numbers either, and can be considered nearly constant in the range of values spanned.

It can be noted that the part of the trajectory corresponding to an exponential growth, is also reported in the phase diagrams in Figures 9 by adding a dashed line for $\theta(t) = 5 \exp \left[ \sigma_m N(t - t_c)/2\pi \right]$.

5.4. Parameters influencing the observed threshold

The identification of the threshold for the broadside-on orientation to become unstable has shown some variations around a mean trend in terms of velocity, that can have several origins. The disks being manufactured from a long bar, there could be some imperfections
in their design (uneven distribution of mass, asymmetry between the faces, etc.); fluid perturbations in the tank can also initiate some tilting of the disk, or there could be an influence of the initial release conditions of the disk.

Since the control of the initial conditions of release for the experimental disks is difficult, we have tested to let a numerical disk (run N13) settle from different initial angles $\theta_i$ (0.6°, 5° and 11°), with no initial velocity. The disk changes its orientation at the same critical velocity for all cases, suggesting that the release conditions do not influence the identification of the threshold. Furthermore, especially for experimental cases, the long duration of the steady phase 1 makes it unlikely to influence the transition from broadside-on to edgewise settling. The influence of the confinement of the disk has also been tested numerically (see appendix §A.2), without any noticeable effect on the values of the threshold for the vertical velocity. We now focus on the geometrical aspects.

All experimental disks are slightly unbalanced, it can affect the way this instability occurs and/or its corresponding threshold. A possible scenario could be that the imperfections in the mass distribution of the disk are initiating the change of stability in its orientation. A model for an unbalanced disk is presented in appendix C, with the overall geometry being unchanged but the density distribution being uneven, with one half of the disk having a density $\rho_d$ and the other half $\rho_d(1+\epsilon)$. Typical values for $\epsilon$ can be estimated from observations made in §4.4 and using eq.(C 14), since it relates it to the angle $\alpha$ of the disk with the horizontal when ending at its neutrally buoyant depth. For instance, with $\alpha \simeq 15^\circ$ for run E7 or $35^\circ$ for run E3, the model predicts $\epsilon \simeq 7.3 \times 10^{-5}$ or $1.2 \times 10^{-4}$ respectively. Numerical simulations of an unbalanced disk as modeled in appendix C with different values for $\epsilon$ are presented in Figure 11. Other settling parameters are similar to run N1 except for $N = 0.495$ rad/s. The evolution of the orientation angle with the settling velocity is shown in Fig. 11(a). The case with $\epsilon = 5 \times 10^{-4}$ (red thick line) corresponds to an unbalanced disk that should sit vertical when reaching its neutral depth. One can observe that with increasing values of $\epsilon$, the rotation of the disk is initiated at an earlier time in the settling process, hence at a higher velocity. In Fig. 11(b), the threshold for the normalized velocity is modified by 25% with $\epsilon = 2.4 \times 10^{-5}$ and is more than double for $\epsilon = 2.4 \times 10^{-4}$. It should be noted that no modification of the temporal evolution of $\theta$ has been observed.

These results are in good agreement with the experimental estimate for the threshold velocity that is almost twice larger than the value for the numerical disk, since the experimental disks are not perfectly made.
Figure 12. Threshold values shown as $Fr_c$ as a function of $d/\sqrt{\nu/N}$ for all unstable cases. Two additional results are shown, run N4 is initially unstable, and “ellipsoid” refers to parameters in Doostmohammadi & Ardekani (2014).

5.5. Stability domains for settling disks

Finally, from all these results, one can estimate the critical values for the parameters ($Ar_c$, $Re_c$ and $Fr_c$) when the disks start to rotate by considering the values of the velocity of the disk $U_c$ and the density of the surrounding fluid $\rho_c$, obtained at $t = t_c$ ($\theta_c = 5^\circ$), which are close to the values at which the rotation starts. Since $Re/Fr$ is constant for each run, only $Ar_c = \sqrt{((p_d/\rho_c - 1)g\rho_d/\nu)}$ and $Fr_c = U_c/Nd$ are relevant here.

When discussing this threshold in §4.3, results led to the conclusion that $U_c$ scales like $\sqrt{\nu N}$ although some dispersion of the threshold could be observed. The main reason is due to the quality of the experimental disks that are slightly unbalanced, as discussed in §5.4, which could lead to a modification of $U_c$ (and hence in $Fr_c$) up to a factor 2. Based on the scaling found for the velocity, the Froude number should follow $Fr_c \sim \sqrt{\nu N}/d$, which is tested in Figure 12. Here the experimental and numerical results have similar trends, which can be associated to $1/Fr_c \approx 0.50(d/\sqrt{\nu N})$ (dashed line) for the numerical simulations, and $1/Fr_c \approx 0.25(d/\sqrt{\nu N})$ (dotted line) for experimental values. These results are in reasonably good agreement with experiments and numerical simulations. They are also in agreement with the results in §5.4 which showed that imperfect disks rotates earlier than perfect ones. Due to the dispersion of the extracted data, it must be noted that other models could match the data. For instance experimental results could be better described by a power law of the type $1/Fr_c \approx 0.16(d/\sqrt{\nu N})^{1.25}$ and numerical results by $1/Fr_c \approx 0.78(d/\sqrt{\nu N})^{0.85}$. The physical implications for such threshold estimates remain unclear.

Nevertheless, these predictions provide a clear separation of domains in the $(d/\sqrt{\nu N}, Fr)$ parameter space associated to a stable broadside-on or edgewise settling. We also include extra numerical results in Figure 12 that confirm these stability domains. Run N4 in a very viscous stratified fluid initiated with $\theta_i = 5^\circ$ revealed the broadside-on settling to be immediately unstable, the corresponding estimate for the threshold values reported in the figure is indeed located in the stable region for edgewise settling, i.e. $1/Fr > 0.5(d/\sqrt{\nu N})$. Similarly, we include results from Doostmohammadi & Ardekani (2014) obtained for an ellipsoid at low Reynolds and Froude numbers released in a stratified fluid with $\theta_i = 20^\circ$. The ellipsoid immediately rotates towards an edgewise settling, showing the stability of this orientation for an ellipsoid with $R = 1.1$, $\chi = 2$, $Ar = 15$, $Re = 0.1$, $Fr = 1.03$, $Pr = 10^3$; indicated by a black diamond in Fig. 12.

Finally, we verified that the parameters of the threshold, expressed as $Fr_c d/\sqrt{\nu N}$,
do not depend on $Ar_c$ (not shown). This suggest that there is no influence of the depth (or the local density $\rho_c$) at which the transition occurs, and that the main physics of the change of stability is indeed in the critical Froude number.

6. Conclusion

We presented experimental and numerical results on the settling dynamics of a disk of finite thickness in a linearly stratified fluid. We characterized its dynamics at low and intermediate values of the Archimedes or Reynolds numbers ($Ar$ and $Re$ in 0.1 to 150), and for Froude numbers in the range 0.01 to 4, showing the existence of two separate domains in the parameter space associated to stable broadside-on settling (high $Re/Ar$, high $Fr$) and edgewise settling (low $Re/Ar$, low $Fr$).

We provided a parametric model for the stratified drag of the disk when settling broadside-on in a quasi-steady regime, inspired by a similar study based on buoyancy effects acting on the settling dynamics of a sphere in a linearly stratified fluid (Yick et al. 2009). The relevant parameters for the modification of the drag are the Reynolds and Froude numbers and the aspect ratio of the disk. This parametric description can be of great interest to predict the settling speed of anisotropic objects evolving in a stratified fluid. For instance, when considering plankton dynamics in the ocean, a discoid shape is a common feature for various oceanic species (Hillebrand et al. 1999; Clavano et al. 2007) such as diatoms. Typical specifications for a microorganism are a radius of $a \approx 2$ mm, with aspect ratio $\chi = 3$ and a nearly buoyant density $\rho_p \approx 1040$ kg/m$^3$, while the stratification can vary from $N \approx 10^{-3}$ rad/s in the deep ocean to peak values greater than $2.0 \times 10^{-2}$ rad/s at sharp discontinuities (MacIntyre et al. 1995). When modeling the settling speed of such microorganism by a disk in a stratified environment, it is thus important to take into account both the effect of the geometry and of the stratification. The settling speed of a disk can be 2.5 times smaller in a linearly stratified fluid than in the homogeneous case when the stratification is important ($N \approx 10^{-2}$). It can be 2 to 5 times smaller than estimates made with the model of a sphere Yick et al. (2009) in the same environment, as summarized in Table 2.

When discussing the change of stability for the orientation of the disk, we provided a strong evidence of the instability mechanism, occurring at a threshold value for the vertical settling speed that can be discussed in terms of the Froude number $Fr$, compared to intrinsic properties of the fluid and the disk. In the case of a ‘perfect’ disk, the threshold verifies $1/Fr_c \approx 0.50(d/\sqrt{\nu/N})$. This formula shall be considered due some scattering

<table>
<thead>
<tr>
<th>object</th>
<th>$N$ (rad/s)</th>
<th>$U$ (m/s)</th>
<th>$T_S$ (s)</th>
<th>$Re$</th>
<th>$Fr$</th>
<th>comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>sphere</td>
<td>0</td>
<td>3.00 $\times 10^{-4}$</td>
<td>13.33</td>
<td>6.00 $\times 10^{-1}$</td>
<td>$\infty$</td>
<td>stable</td>
</tr>
<tr>
<td></td>
<td>1.0 $\times 10^{-3}$</td>
<td>2.98 $\times 10^{-4}$</td>
<td>13.42</td>
<td>5.96 $\times 10^{-1}$</td>
<td>149</td>
<td>unstable</td>
</tr>
<tr>
<td></td>
<td>2.0 $\times 10^{-2}$</td>
<td>2.53 $\times 10^{-4}$</td>
<td>15.81</td>
<td>5.06 $\times 10^{-1}$</td>
<td>6.32</td>
<td></td>
</tr>
<tr>
<td>disk</td>
<td>0</td>
<td>1.85 $\times 10^{-4}$</td>
<td>21.62</td>
<td>7.40 $\times 10^{-1}$</td>
<td>$\infty$</td>
<td>stable</td>
</tr>
<tr>
<td>broadside-on</td>
<td>1.0 $\times 10^{-3}$</td>
<td>1.83 $\times 10^{-4}$</td>
<td>21.86</td>
<td>7.32 $\times 10^{-1}$</td>
<td>45.7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.0 $\times 10^{-2}$</td>
<td>4.80 $\times 10^{-5}$</td>
<td>83.33</td>
<td>1.92 $\times 10^{-1}$</td>
<td>0.60</td>
<td>unstable</td>
</tr>
</tbody>
</table>

Table 2. Settling parameters for an object of density $\rho_p = 1040$ kg/m$^3$, diameter $d = 4$ mm (and aspect ratio $\chi = 3$ for the disk) in a fluid with $\rho_0 = 1000$ kg/m$^3$. The settling time is $T_S = d/U$. 


observed in the data. The growth rate of the instability has also been measured and is controlled by the Brunt-Väisälä frequency for both experiments and numerics, with a weak influence of the geometrical properties of the disks and the viscosity of the fluid. From our observations of the wake of the disk, we suggest that the mechanism driving this instability is associated to the displacement of the strongly localized low-pressure point at the center of the back face of the disk when settling broadside-on, which induces a destabilizing torque on the disk in the horizontal direction. While reorientation of a disk in a homogeneous fluid at low Reynolds numbers is stabilized by a hydrodynamical torque (Fabre et al. 2012), this effect seems to weaken in a stratified fluid when the vertical velocity of the disk decreases. This stabilizing effect can be overreached by the destabilizing stratified effect at sufficiently low values of $Re/Ar$ and $Fr$. This should be applicable to plankton dynamics, as discussed before, since low values of $Re/Ar$ and $Fr$ can be reached in the ocean (cf. comments in Table 2).

There are numerous interesting perspectives for this study. The modeling of the stratification influence on the drag of an anisotropic bluff body, taking into account boundary layer effects would provide the basis for the understanding of the competing effect of vorticity and buoyancy. The threshold estimates could be more precisely studied although it is computationally very expensive to resolve such stratified flows at high Prandtl numbers. Furthermore a stability analysis for the change of preferential orientation for the disk is also required to understand the onset of instability, along with the temporal dynamics. A proper modeling of the torque on the disk (or on any object actually) in a stratified environment is still lacking. Finally, the instability of the orientation of the disk could be of great importance for the dynamics of a collection of anisotropic objects in a stratified medium, modifying the interactions and triggering instabilities in spatial distributions.

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Appendix A. Numerical validations

A.1. Grid resolution tests

In order to investigate the effect of grid resolution on the numerical modeling of the fluid perturbations (velocity and density) around the disk, we first consider two cases, a disk moving at a fixed speed with an imposed orientation ($\theta = 0$), and a sphere moving at a fixed speed as a reference case (already studied in Hanazaki et al. (2009b, 2015)). This configuration allows us to compare the density and velocity in the jet for the same conditions (i.e., angle, velocity, etc.) as higher number of points around the disk is used. Additional computations have been done for numerical validations, the details on these additional computations are described in Table 3. Since the objects have a fixed velocity, there is a precise value of the Froude and Reynolds numbers for each run, $Re = 40$ (resp. $25$) and $Fr = 0.2$ (resp. 4) for the disks (resp. sphere).

The results for the disk at two values of the Prandtl number are shown in Fig.13, at time $Nt/2\pi \simeq 0.8$ ($t = 20\pi$) after the start of motion. This time is chosen since it corresponds approximately to the time at which the instability in orientation occurs in the case of a freely-settling disk (see run Nv4 below). Three different fields are displayed (velocity, density perturbation and its Laplacian), extracted along a horizontal line which is half radius away from the center of the disk. As can be seen for all quantities, the grid resolution of $d/128$ is sufficient to get accurate values of velocity and density perturbation, although a resolution of $d/200$ is needed for the Laplacian of density to be properly resolved at $Pr = 700$.

As time evolves, the jet in the wake of the disk gets thinner until it reaches a steady state. To verify this behavior, we investigate the evolution of the jet radius which is defined as half-width at half-maximum (HWHM) of $\nabla^2 \rho'$ along a horizontal line one radius away from the center of the disk, similarly to Hanazaki et al. (2015). We find the jet radius reaches a steady state at time of $Nt/2\pi = 11$ for $Pr = 700$, and $Nt/2\pi = 3$ for $Pr = 7$, while the sphere case is steady at $Nt/2\pi \simeq 3$ which is in agreement with the observations in Hanazaki et al. (2015). Concerning the radius of the jet, the steady value reached $R/d = 6 \times 10^{-3}$ and $3 \times 10^{-2}$ for the disk in Nv1 and Nv2 respectively; and $R/d = 10^{-2}$ for the sphere, which is consistent with the prediction of $(Fr/2RePr)^{1/2}$ in Hanazaki et al. (2015).

We now compare those results with the case of a freely settling disk (Nv4) in the next section, with same physical parameters. In order to investigate the effect of grid resolution
on the disk’s settling velocity, various grid resolutions are utilized in both horizontal and vertical directions (see Fig. 14(a)) for run Nv4 described in Table 3. For the highest grid resolution, we use 256 grid points per disk’s diameter in the horizontal direction and 256 grid points per disk’s diameter in the vertical direction. This resolution is not sufficient to resolve fully the density perturbations developing in the jet at the rear of the disk when reaching a steady-state, but it should remain sufficient for this scenario due to the temporal evolution of the instability which occurs before the jet gets very thin as shown before. When $t \sim 20s$ ($Nt/2\pi \sim 2$), the instability in terms of disk rotation occurs. The simulation results from 128 grid points per disk’s diameter in the horizontal and vertical directions are almost identical to the highest resolution. This result is due to the orientation of the settling disk undergoing instability before the jet reaches steady state. Therefore, the jet thickness is wider than a steady jet and lower resolution are sufficient to describe the problem. Overall, for horizontal and vertical resolutions of $d/128$, the relative error observed is 4% for density perturbation and 2% for vertical velocity (not shown here).

As a final remark, in our simulation scheme, the central-difference scheme is considered as second order accurate whereas the QUICK scheme has a third order accuracy. Therefore, the overall accuracy remains as second order, which is consistent with our error test (not shown).

A.2. Domain dependence test

We investigate the effect of domain size on the problem for a run similar to N8 in the main part of the manuscript. The simulation results of angular velocity with time for a domain size of $4d \times 4d \times 20d$ is compared to that for a domain size of $8d \times 8d \times 20d$ and
16d × 16d × 20d (see Fig. 14(b)), where d is the disk’s radius. In addition, the simulations all provide a critical number $A_r$ of 152. Throughout entire simulations, we use a width of 4d.

We should note that the purpose of the numerical simulation results is to understand the flow physics associated with Phase 1 (quasi-steady state) and 2 (onset of the orientation instability). Here the corresponding Reynolds number is about 26 at the onset of the instability.

### A.3. Direct comparison with experiments

We have simulated three different runs with physical parameters identical to experiments. We show direct comparison of these results with experiments. Runs Nv5, 6 and 7 (see Table 3) correspond to experimental run E2, E5 and E7 (see Table 1). For each case plotted in Fig.15, we follow the dynamics in terms of the non-dimensional velocity $U/U_g$ and the angle as a function of $\rho^* = 1 - (\rho_d - \rho(z))/(\rho_d - \rho_0)$, with $U_g = \sqrt{(\rho_d/\rho_0 - 1)gh}$ a gravitational settling velocity. Each numerical run is initiated with the disk near the exact experimental release location, although not visible in experiment (red lines start at values of $\rho^* \approx 0.4$). Due to the long duration of these computations, only the early dynamics of the instability is reproduced (blue lines end before $\rho^* = 1$).

Overall, the agreement between numerical simulations and experiments is good. The quasi-steady settling (phase 1) is well reproduced, and the initiation of the orientation instability occurs at very similar locations and with the same dynamics. These results confirm the validity of our approach, they are also in agreement with the results obtained on the estimate of the stratified drag coefficient in §4.2, and on the instability dynamics of the orientation of the disk in §4.3 and §5.3.

### Appendix B. Relation between the Archimedes and Reynolds numbers

Although it is common to discuss the dynamics of moving objects according to the Reynolds number, the Archimedes number is a more convenient non-dimensional parameter since it can be prescribed without knowing the velocity of the object. Figure 16(a) provides the evolution of $Re$ and $Ar$ over the 3 phases of the overall dynamics of all experimental disks. Figure 16(b) validates the discussion in §4.2 leading to eq. (4.3) by testing the relation $2(Ar/Re)^2 = C_D^R$ in phase 1.
Appendix C. Equilibrium orientation of an unbalanced disk in a stratified fluid

We consider the unbalanced disk shown in Figure 17, with the prescribed orientation compared to the laboratory frame \((O, x, y, z)\) given by the intrinsic and precession angles, \(\psi\) and \(\alpha\), respectively, where the center of the geometry is located in \(O\). The axes of the frame associated with the disk, \((O, x_1, y_1, z_1)\), are parallel to the principal axes of the inertia tensor. Indeed, the density of the disk \(\rho_d\) is constant for each half of the disk \((\rho_d\) and \(\rho_d(1 + \epsilon)\) as indicated by the gray shading. We neglect the nutation of the disk due to the symmetry of the problem.
Figure 17. Uneven disk made of two half-disks in the laboratory frame \((x, y, z)\). The tilted frame \((x_1, y_1, z_1)\) fixed to the disk is aligned with the principal axes of the inertia tensor. The densities of the white and gray halves are \(\rho_d\) and \(\rho_d(1 + \epsilon)\), respectively.

We define the following rotation matrices

\[
A = \begin{pmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{pmatrix},
\] (C 1)

such that we can relate the two frames by

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = BA
\begin{pmatrix}
x_1 \\
y_1 \\
z_1
\end{pmatrix}.
\] (C 2)

The disk is immersed in a linearly stratified fluid whose density is defined as

\[
\rho(z) = \rho_0 \left(1 - \frac{N^2 z}{g}\right) = \rho_0 - \frac{N^2 \rho_0}{g} \left[z_1 \cos \alpha + \sin \alpha (x_1 \sin \psi + y_1 \cos \psi)\right]
\] (C 3)

where (C 2) is used to express \(z\) in terms of the tilted coordinates.

We study the torque resulting from the buoyancy force on the disk, which is given by

\[
\mathcal{M}_b = \iiint_{D} \mathbf{OM} \wedge [\rho_d(x, y, z) - \rho(z)] \mathbf{g} \, d^3V.
\] (C 4)

By considering the computation in the tilted frame and switching to cylindrical coordinates, we can rewrite (C 4) as

\[
\mathcal{M}_b = \int_{-h/2}^{h/2} \int_{d/2}^{d/2} \int_{0}^{2\pi} \mathbf{OM}_1 \wedge [\rho_d(\theta_1) - \rho(r_1, \theta_1, z_1)] \mathbf{g} \, r_1 \, dr_1 \, d\theta_1 \, dz_1.
\] (C 5)

where \(\mathbf{OM}_1 = r_1 \mathbf{e}_{r_1} + z_1 \mathbf{e}_{z_1}\) and \(x_1 = r_1 \cos \theta_1\) and \(y_1 = r_1 \sin \theta_1\). The density distribution of the disk \(\rho_d(\theta_1)\) is defined as

\[
\rho_d(\theta) = \begin{cases}
\rho_d(1 + \epsilon) & \text{for } \theta \in [0, \pi] \\
\rho_d & \text{for } \theta \in [\pi, 2\pi]
\end{cases}
\] (C 6)

The gravitational acceleration in the tilted frame is given as \(\mathbf{g} = -g(\cos \alpha \mathbf{e}_{z_1} + \sin \alpha \sin \psi \mathbf{e}_{x_1} + \sin \alpha \cos \psi \mathbf{e}_{y_1})\). This leads to the following expression for the vector product in (C 5)

\[
\mathbf{OM}_1 \wedge \mathbf{g} = g \left(r_1 \cos \alpha \mathbf{e}_{r_1} + r_1 \sin \alpha (\sin \psi \sin \theta_1 - \cos \psi \cos \theta_1) \mathbf{e}_{x_1} - z_1 \sin \alpha \mathbf{e}_{z_1}\right).
\] (C 7)

We now consider the integral in (C 5) which can be decomposed in the following man-
After some manipulations, one can notice small values of $\epsilon$ for a disk with $\epsilon, \alpha$ for the pair $(\epsilon, \alpha)$ respectively positive or negative values of $\epsilon$.

Based on the orientation in Figure 17, the stable orientation indeed occurs for negative (resp. positive) values of $\epsilon$ if $\epsilon$ is positive (resp. negative). Table 4 gives some examples for the pair $(\epsilon, \alpha)$ for a disk with $d = 14$ mm, $\chi = 8$ in a stratified fluid with $N = 0.5$ rad/s and $\rho_0 = \rho_d$. For this choice of parameters comparable to the values studied in the manuscript, one can notice small values of $\epsilon$ are required for the disk to be nearly
balanced. The disk used in our experiments have measured values of $\alpha$ in the range $1^\circ$ to $65^\circ$ (cf. Figure 8 in §4), corresponding to an uneven distribution of mass with $\epsilon \leq 10^{-4}$.

<table>
<thead>
<tr>
<th>$\alpha$ (in deg)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>25</th>
<th>45</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\epsilon \times 10^4$</td>
<td>0.187</td>
<td>0.372</td>
<td>0.555</td>
<td>0.907</td>
<td>1.52</td>
<td>2.11</td>
</tr>
</tbody>
</table>

Table 4. Solutions ($\epsilon, \alpha$) of eq. (C.14) for a disk with $d = 14\text{mm}$, $\chi = 8$ in a stratified fluid with $N = 0.5 \text{ rad/s}$ and $\rho_0 = \rho_d$ ($g = 9.81 \text{ m}/\text{s}^2$).