A reconstruction of Default Conditionals within Epistemic Logic

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ABSTRACT

Default conditionals are statements that express a condition of *normality*, in the form '*if* φ *then normally* ψ' and are of primary importance in Knowledge Representation. There exist modal approaches to the construction of conditional logics of normality. Most of them are built on notions of *preference* among possible worlds, corresponding to the semantic intuition that $(\varphi \Rightarrow \psi)$ is true in a situation if in the most preferred (most 'normal') situations in which φ is true, ψ is also true. It has been noticed that there exist natural epistemic readings of a default conditional, but this direction has not been thoroughly explored. A statement of the form 'something *known* to be a bird, that can be consistently *believed* to fly, does fly' involves well-known epistemic attitudes and allows the possibility of defining defaults within the rich framework of Epistemic Logic. We pursue this direction here within $\mathbf{KBE}$, a recently introduced $\mathbf{S4.2}$-based modal logic of *knowledge*, *belief* and *estimation*. In this logic, knowledge is a normal $\mathbf{S4}$ operator, belief is a normal $\mathbf{KD45}$ operator and estimation is a non-normal operator interpreted as a 'majority' quantifier over the set of epistemically alternative situations. We define and explore various conditionals using the epistemic operators of $\mathbf{KBE}$, capturing $(\varphi \Rightarrow \psi)$ in various ways, including '*it is* known *that* assuming φ *allows us to* assume $\varphi \wedge \psi$' or '*if* φ *is known and there is no reason to believe* $\neg\psi$ *then* ψ *can be plausibly inferred*'. Overall, we define here two weak nonmonotonic default conditionals, one monotonic conditional and two stronger nonmonotonic conditionals without axiom $\mathbf{ID}$. Our results provide concrete evidence that the machinery of epistemic logic can be exploited for the study of default conditionals.

Keywords

Conditional Logic; Default Conditionals; Epistemic Logic

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1. INTRODUCTION

Knowledge Representation has always been concerned with '*normality conditionals*', also called '*defeasible conditionals*' or '*defaults*'. These are statements that express a condition of 'normality' such as '*birds normally fly*' or '*adults are normally employed*' (although the validity of the latter conditional is intensely disputed in the era of the debt crisis ...). There exist other forms of expressions considered to fall within this class, such as '*most birds have feathers*' or '*Christos usually walks home after his class*'. Default conditionals are intimately related to the major concerns of Nonmonotonic Reasoning and have been further investigated after the study of *nonmonotonic consequence relations* and the introduction of KLM logics [19].

Default statements admit various readings. A fundamental one corresponds to their principal use of ascribing default properties to individuals ('*Tweety flies since birds normally fly*'), a function accomplished elegantly also in McCarthy's *Circumscription* (via classical first-order logic) and Reiter's *Default Logic* (via the rules of inference adjoined to first-order logic). Other readings of defeasible conditionals seem closer to statements about (mostly qualitative but also quantitative) probability: '*birds generally* (*typically, mostly*) *fly*'. It has been noticed however that "*the reading '*a bird that can be consistently assumed to fly does fly*' is clearly epistemic in nature*" [5, p. 95]. A 'normality statement' of the form '*every Tuesday afternoon, you can find Jimmy taking a beer in the corner pub*' allows one to infer that on a 'regular' Tuesday s/he can meet Jimmy there and this default inference involves facts *known* ('it is a Tuesday'), facts *observed* and *believed* ('Jimmy frequents this place on Tuesday afternoon'), facts considered to be consistent with the belief base ('there is no reason to believe this is an 'irregular' Tuesday') and facts *plausibly inferred* ('most probably I will meet him there'). Although difficult to agree on the subtle details of the epistemic attitudes involved, it seems that there is an agreement on the fact that such an epistemic description is quite reasonable. Cognitive statements of this kind are implicit in Reiter's normal defaults [24] and the conditional entailment of H. Geffner and J. Pearl [12]: '*a rule* $\frac{a\,:\,b}{b}$ *may be seen as a soft constraint for believing* b *when* a *is known, while a conditional rule* $a \Rightarrow b$ *can be viewed as a hard constraint to believe* b *in a limited context defined by* a *and possibly some background knowledge*' [9, p. 220].

The study of the connection of defeasible conditionals with the area of Epistemic and Doxastic Logic has not been hitherto pursued in its full entirety. In general, the '*conditionals-via-modal-logic*' technique is known and quite successful [23, 4]; yet, the technical and philosophical step to the construction of conditionals via Epistemic Logic has not been fully taken. The relation of Epistemic Logic to conditionals mainly revolves around the famous *Ramsey test* and this is also apparent in the earlier works of Lamarre & Shoham [21] or Friedman & Halpern [11] where an interesting notion of *conditional belief* is based on the semantics of default conditionals (see also [1, p. 107]). Modal approaches to defeasible conditionals [3, 20, 7] are mostly based on the model-theoretic intuition of 'preference' among possible worlds or propositions. The conditional $\varphi \Rightarrow \psi$ is true in a possible world if ψ is true in the most 'normal' or 'preferred' φ-worlds accessible; equivalently, given the context of φ, the proposition expressed by $\varphi \wedge \psi$ is preferred over the one expressed by $\varphi \wedge \neg\psi$ [7]. It is natural to consider that normality orderings are preorders (reflexive and transitive relations) and thus the modal approaches to defeasible conditionals usually employ the logic $\mathbf{S4}$ (or its extension $\mathbf{S4.3}$) within which the defeasible conditional is modally defined [3]. Another modal construction of defeasible conditionals employs the notion of 'size': in [18] $(\varphi \Rightarrow \psi)$ becomes true whenever ψ is true in an 'overwhelming majority' of φ-worlds; assuming that we work within ω (the first infinite ordinal), we can interpret 'overwhelming majority' as a cofinite subset of ω and proceed to define modally the conditional within $\mathbf{K4DLZ}$ which is the modal logic of $(\omega, <)$.

In this paper, we amplify the epistemic interpretation of defeasible conditionals and proceed to define them directly within Epistemic Logic. We work inside $\mathbf{KBE}$, a recently introduced epistemic logic [17] accounting for *knowledge, belief* and *estimation* (as a form of weak, complete belief, interpreted as '*truth in most epistemic alternatives*'). $\mathbf{KBE}$ comprises an $\mathbf{S4.2}$ framework for knowledge and belief, following the fundamental investigations of W. Lenzen [22] and R. Stalnaker [26]. The non-normal modal operator for *estimation* is interpreted as a 'majority' quantifier over the set of epistemic alternatives of a given possible world. The formal apparatus is that of a '*weak ultrafilter*', which is an upwards-closed collection of sets, with pairwise non-disjoint members and such that exactly one out of a set and its complement occurs in the collection; the notion extends the *weak filters* introduced independently in [25, 15]. We define two nonmonotonic conditionals by capturing a size-oriented version of the fundamental intuition of normality conditionals: $(\varphi \Rightarrow \psi)$ is set to mean that $(\varphi \wedge \psi)$ is more normal compared to $(\varphi \wedge \neg\psi)$, as it holds in 'most' epistemically alternative worlds; this is achieved by exploiting the nature of $\mathbf{KBE}$'s 'estimation' operator as a majority quantifier. The logics emerging are rather weak compared to the 'conservative core' of default reasoning (the system $\mathbf{P}$, [19]) but this is neither surprising nor discouraging: weak conditionals of this kind have been also introduced in [7, system $\mathbf{C}$ and system $\mathbf{\Lambda}$] under a rule-based interpretation of defaults and it is well-known that conditionals based on the plausibility structures of Friedman & Halpern do not generally satisfy all the KLM properties [11, p. 266]. Another, very 'natural' (but rather strong in epistemic assumptions) translation leads to a monotonic conditional, and two other epistemic definitions give rise to nonmonotonic conditional logics which do not satisfy the axiom $\mathbf{ID}$ (*reflexivity*), but they capture very interesting conditional principles and one of them comes close to an 'overwhelming majority' conditional defined in [18]. Note that for all these definitions a recursive translation in the language of $\mathbf{KBE}$ provides direct access to the tableaux proof procedure for this logic [17], and thus a machinery for testing theoremhood is readily available.

Our primary concern in this research is rather typical of (one of) the way(s) Conditional Logic is used in Knowledge Representation: the main objective is to introduce epistemically-driven theories of sentences expressing defeasible conditionals and provide a syntactic, yet intuitively justified, account of the notion of a default conditional, and *not* to directly provide a framework for default reasoning which would pin down the contingent conclusions that can be plausibly extracted given a background conditional theory [7, 6]. We focus on the 'epistemic connection' of defeasible conditionals and investigate the possibility of a direct syntactic definition within Epistemic Logic. Due to space limitations, we do not provide extended background material and the proofs are omitted or sketched.

2. BACKGROUND - THE LOGIC KBE

We assume that the reader has a working knowledge of *Modal Logic* and *Conditional Logic* and is acquainted with the Scott-Montague (neighborhood) semantics and the cluster analysis of transitive normal modal logics (see [14, 2, 6]). In this paper, we reserve $\rightarrow$ for the classical (material) implication and $\Rightarrow$ for the normality conditional or any other non-classical conditional constructed. The names of the modal axioms and systems mentioned in this paper are firmly entrenched in the literature. Less entrenched is the terminology on the 'bridge' axioms relating knowledge to belief [13]; we follow the naming given by W. Lenzen[22] and R. Stalnaker [26]. The axioms and rules of Conditional Logic can be found in Table 1, last page of this extended abstract.

The logic $\mathbf{KBE}$ has been introduced in [17]. The language $\mathcal{L}_{\mathbf{KBE}}$ comprises three modal operators: $\mathsf{K}\varphi$ read as '*the agent **knows** φ*', $\mathsf{B}\varphi$ read as '*the agent **believes** φ*' and $\mathsf{E}\varphi$ read as '*the agent **estimates** that φ is true*'. One way to view the epistemic attitudes involved is to consider $\mathsf{K}\varphi$ as an $\mathbf{S4}$ operator, and $\mathsf{B}\varphi$ as a $\mathbf{KD45}$ operator, interconnected with the bridge axioms $\mathbf{B1}$. $\mathsf{K}\varphi \rightarrow \mathsf{B}\varphi$, $\mathbf{B2.3}$. $\mathsf{B}\varphi \rightarrow \neg\mathsf{B}\neg\mathsf{K}\varphi$ and $\mathbf{B2.4}$. $\mathsf{B}\varphi \rightarrow \mathsf{KB}\varphi$. Equivalently, following the work of W. Lenzen and R. Stalnaker, we consider the logic $\mathbf{S4.2}$ within which belief is just an abbreviation defined by $\mathsf{B}\varphi \equiv \neg\mathsf{K}\neg\mathsf{K}\varphi$; this equivalent perspective is very convenient as we are able to work within the model theory of $\mathbf{S4.2}$. The estimation operator added on top of $\mathbf{S4.2}$ is a non-normal majority quantifier: the intended interpretation is that an agent estimates that φ is true if φ holds in 'most' epistemic alternatives. Following is the **axiomatization** of $\mathbf{KBE}$, including the abbreviation for belief: $\mathbf{DB}$. $\mathsf{B}\varphi \equiv \neg\mathsf{K}\neg\mathsf{K}\varphi$ (*Belief definition*), $\mathbf{K}$. $\mathsf{K}\varphi \wedge \mathsf{K}(\varphi \rightarrow \psi) \rightarrow \mathsf{K}\psi$ (*Knowledge is closed under logical consequence*), $\mathbf{T}$. $\mathsf{K}\varphi \rightarrow \varphi$ (*Only true things are known*), $\mathbf{4}$. $\mathsf{K}\varphi \rightarrow \mathsf{KK}\varphi$ (*Positive introspection, with respect to knowledge*), $\mathbf{CB}$. $\mathsf{B}\varphi \rightarrow \neg\mathsf{B}\neg\varphi$ (*Belief is consistent*), $\mathbf{BE}$. $\mathsf{B}\varphi \rightarrow \mathsf{E}\varphi$ (*Beliefs are estimations*), $\mathbf{CCE}$. $\mathsf{E}\varphi \equiv \neg\mathsf{E}\neg\varphi$ (*Estimation is consistent and complete*), $\mathbf{EK}$. $\mathsf{E}\varphi \wedge \mathsf{K}(\varphi \rightarrow \psi) \rightarrow \mathsf{E}\psi$ (*Estimation can be safely inferred through knowledge*), $\mathbf{PIE}$. $\mathsf{E}\varphi \rightarrow \mathsf{KE}\varphi$ (*Introspection with respect to estimation*).

Several introspective properties are proved in [17], including the following principles which are valid in $\mathbf{KBE}$: $\mathsf{E}\varphi \rightarrow \mathsf{KE}\varphi$, $\mathsf{E}\varphi \rightarrow \mathsf{BE}\varphi$, $\mathsf{E}\varphi \rightarrow \mathsf{EE}\varphi$, $\neg\mathsf{E}\varphi \rightarrow \mathsf{K}\neg\mathsf{K}\varphi$, $\neg\mathsf{E}\varphi \rightarrow \mathsf{B}\neg\mathsf{K}\varphi$, $\neg\mathsf{E}\varphi \rightarrow \mathsf{E}\neg\mathsf{K}\varphi$ (*non-estimation implies introspection wrt ignorance and 'lack of certainty'*) and $\mathsf{E}\varphi \wedge \mathsf{B}(\varphi \rightarrow \psi) \rightarrow \mathsf{E}\psi$. Of particular importance is that belief can be equivalently defined in $\mathbf{KBE}$ as '*estimation that the agent knows*': $\mathsf{EK}\varphi \equiv \mathsf{B}\varphi$ and the fact that knowledge about estimation amounts exactly to estimation itself $\mathsf{KE}\varphi \equiv \mathsf{E}\varphi$.

DEFINITION 2.1. $\mathbf{KBE}$ is the propositional bimodal logic axiomatized by $\mathbf{K}$, $\mathbf{T}$, $\mathbf{4}$, $\mathbf{CB}$, $\mathbf{BE}$, $\mathbf{CCE}$, $\mathbf{EK}$, $\mathbf{PIE}$ and closed

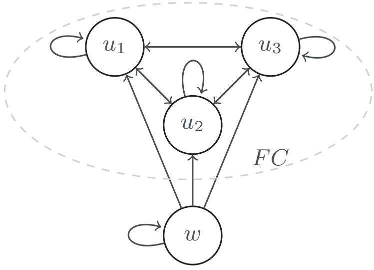

Figure 1: W **and** $\mathcal{R}$ **of frame** $\mathfrak{F}_1$

under the rule $\mathbf{RN}_{\mathsf{K}}$. $\dfrac{\varphi}{\mathsf{K}\varphi}$.

The possible-worlds models of $\mathbf{KBE}$. To construct the frames of $\mathbf{KBE}$ we focus on the fact that one of the frame classes that determines $\mathbf{S4.2}$ is the class of *reflexive, transitive* frames with a *final cluster* FC (note that such a frame is automatically directed); this follows from the results in [16]. These $\mathbf{S4.2}$-frames are combined with Scott-Montague semantics, in which each neighborhood is a complete collection of *large sets* on the epistemic alternatives of the world at hand, given our intention to interpret estimation as a generalized 'most' quantifier. To capture the notion of *large sets*, the abstract notion of a weak filter has been independently introduced in [25, 15]. A non-empty collection F of subsets of W is a *weak filter* iff (i) $X \in F$ and $X \subseteq Y \subseteq W$ implies $Y \in F$ (*upwards closure*) and (ii) $X \notin F$ or $(W \setminus X) \notin F$ (equivalently: $X \in F$ and $Y \in F$ implies that $X \cap Y \neq \emptyset$, *pairwise non-disjointness*). We obtain a *weak ultrafilter* by strengthening condition (ii) to a biconditional: $X \notin F \;\Leftrightarrow\; (W \setminus X) \in F$ (*exactly one, out of a set and its complement, is large*). Genuinely weak ultrafilters exist and are of interest to 'size'-oriented accounts of nonmonotonic reasoning [17]. The following definition introduces the class of $\mathbf{KBE}$-frames; $\mathcal{N}$ is a 'neighborhood function' assigning to each world a collection of large subsets of its $\mathcal{R}$-epistemic alternatives (see [17] for more details).

DEFINITION 2.2. Consider the triple $\mathfrak{F} = \langle W, \mathcal{R}, \mathcal{N}\rangle$, where W is a non-empty set, $\mathcal{R} \subseteq W \times W$, $\mathcal{N} : W \to \mathcal{P}(\mathcal{P}(W))$, $\mathcal{R}$ is a reflexive, transitive relation with a nonempty *final cluster* $FC = \{v \in W \mid (\forall w \in W)\; w\mathcal{R}v\}$. $\mathcal{N}$ is such, that $\forall w \in W$ (**nr**) $\mathcal{N}(w) \subseteq \mathcal{P}(\mathcal{R}(w))$, (**be**) $FC \in \mathcal{N}(w)$, (**pie**) $\forall X \subseteq \mathcal{R}(w)\; \forall u \in W\; \big(X \in \mathcal{N}(w)\; \&\; w\mathcal{R}u \Longrightarrow X \cap \mathcal{R}(u) \in \mathcal{N}(u)\big)$, (**cce**) $\forall X \subseteq \mathcal{R}(w)\; \big(X \in \mathcal{N}(w) \Longleftrightarrow \mathcal{R}(w) \setminus X \notin \mathcal{N}(w)\big)$, and (**ek**) $\forall X, Y \subseteq \mathcal{R}(w)\; \big(X \in \mathcal{N}(w)\; \&\; Y \supseteq X \Longrightarrow Y \in \mathcal{N}(w)\big)$. $\mathfrak{F}$ is called a **kbe-frame**. $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ is called a **kbe-model**, if it is based on a **kbe-frame** and $V : \Phi \to \mathcal{P}(W)$ (Φ is the set of propositional variables) is a valuation.

The class of all **kbe-frames** is nonempty. The reader can verify that the frame of Figure 1, is a **kbe-frame**: $W = \{w, u_1, u_2, u_3\}$, $\mathcal{R}$ is the relation shown in Figure 1, $\mathcal{N}(w) = \big\{X \subseteq W \mid |X| \geq 3\big\} \cup \big\{\{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}\big\}$, $\mathcal{N}(u_1) = \mathcal{N}(u_2) = \mathcal{N}(u_3) = \big\{\{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, FC\big\}$, where $FC = \{u_1, u_2, u_3\}$ is the final cluster of the structure $\langle W, \mathcal{R}\rangle$. $\mathbf{KBE}$ is determined by the class of **kbe-frames** [17].

DEFINITION 2.3. Consider the model $\mathfrak{M} = \langle W, \mathcal{R}, \mathcal{N}, V\rangle$ for the language $\mathcal{L}_{\mathbf{KBE}}$. The function $\overline{V} : \mathcal{L}_{\mathbf{KBE}} \to \mathcal{P}(W)$ is defined recursively as follows: $\overline{V}(p) = V(p)$, $(\forall p \in \Phi)$, $\overline{V}(\bot) = \varnothing$, and $(\forall \varphi, \psi \in \mathcal{L}_{\mathbf{KBE}})$ $\overline{V}(\varphi \to \psi) = (W \setminus \overline{V}(\varphi)) \cup \overline{V}(\psi)$, $\overline{V}(\mathsf{K}\varphi) = \{w \in W \mid \mathcal{R}(w) \subseteq \overline{V}(\varphi)\}$, $\overline{V}(\mathsf{E}\varphi) = \{w \in W \mid \mathcal{R}(w) \cap \overline{V}(\varphi) \in \mathcal{N}(w)\}$. As usual, we will write $\mathfrak{M}, w \models \varphi$ instead of $w \in \overline{V}(\varphi)$.

3. AN EPISTEMIC RECONSTRUCTION OF CONDITIONAL LOGIC

3.1 The role of epistemic operators in defining default conditionals

One important topic in **Epistemic Logic** is the relation between the various **epistemic and doxastic attitudes or cognitive states**: *knowledge, belief, weak belief, disbelief, ignorance, plausible judgement*, there exist a lot indeed. The deep and influential analysis of W. Lenzen [22] and R. Stalnaker [26] has revealed a minimal set of principles that an epistemologist should accept on the properties of knowledge, belief and their 'bridging' relations, resulting in the fundamental role of $\mathbf{S4.2}$. The logic $\mathbf{KBE}$ builds on these results to accommodate an operator of *plausible estimation*, tailored for cases when necessarily either φ or $\neg\varphi$ - but not both - should be 'estimated'.

From the discussion in the introductory section 1 and the quotations therein, it should be clear that there exist important forms of the normality conditional that can - or *should* - be read epistemically. But, when it comes to a careful formal definition of a default conditional within an epistemic logic, the **question** arises: **which 'attitude', in which 'place' of the default conditional?** In the first place, drawing inspiration from the 'archetypical' normal default from Reiter's logic: $\dfrac{\varphi : \psi}{\psi}$, it seems natural to consider an epistemic translation of a normality conditional ($\varphi \Rightarrow \psi$) (if φ then *normally* ψ) in the form: $\mathsf{K}\varphi \wedge \neg\mathsf{B}\neg\psi \;\to\; \mathsf{E}\psi$, interpreting the *justification* ψ of the normal default (whose meaning in Default Logic is 'it is consistent to assume ψ', usually denoted as $M\psi$) as '*I have no reason to believe* $\neg\psi$', the *prerequisite* as '*I know* φ' and the *conclusion* of the default as '*I estimate that* ψ *is the case*'. This definition gives rise to an interesting conditional, which is *not* a default conditional: it satisfies the *principle of monotony* (alias *strengthening the antecedent*); see Section 3.3. Our experimentation revealed that this is due to the strong influence of the *knowledge* operator in the 'prerequisite', a phenomenon partly persisting when *knowledge* is replaced with *belief*. Other definitions of the conditional which exhibit the same behaviour include $\mathsf{K}\varphi \wedge \neg\mathsf{B}(\varphi \to \neg\psi) \;\to\; \mathsf{E}\psi$ and $\mathsf{K}\varphi \wedge \neg\mathsf{B}(\varphi \wedge \neg\psi) \;\to\; \mathsf{E}\psi$ while other definitions, including $(\mathsf{K}\varphi \wedge \neg\mathsf{E}(\varphi \to \neg\psi) \to \mathsf{E}\psi)$ and $(\mathsf{K}\varphi \wedge \neg\mathsf{E}\neg\psi \to \mathsf{E}\psi)$ lead to triviality as they are valid $\mathbf{KBE}$ principles (check with the axiomatization in Section 2), indicating clearly that given *knowledge* in the 'prerequisite', *estimation* cannot replace *belief* in the 'justification'.

However, once we replace *knowledge* by *estimation* (which is a much weaker operator) defining ($\varphi \Rightarrow \psi$) as $\mathsf{E}\varphi \wedge \neg\mathsf{B}(\varphi \to \neg\psi) \;\to\; \mathsf{E}(\varphi \wedge \psi)$ we obtain a weak defeasible conditional, whose properties we discuss in Section 3.2. Another, perhaps more interesting defeasible conditional arises when the interplay between the *antecedent* (the 'prerequisite') and the *consequent* (the 'conclusion') is 'controlled' through *knowledge* and 'computed' via *estimation* as $\mathsf{K}(\mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge \psi))$, read as '*I can plausibly conclude 'normally* ψ *assuming* φ' *iff I know that estimating* φ *allows me* *to estimate* $\varphi \wedge \psi$'. This view comes close to the intuition that $(\varphi \wedge \psi)$ seems more 'normal' than $(\varphi \wedge \neg\psi)$; note also that given the interpretation of *estimation* in $\mathbf{KBE}$, $(\varphi \wedge \psi)$ holds in 'most' situations.

Finally, we proceed to define conditionals by imposing epistemic operators on the antecedents of rule **Modus Ponens**. We define $(\varphi \Rightarrow \psi)$ as $\mathsf{K}\varphi \wedge \mathsf{E}(\varphi \to \psi)$ and $\mathsf{K}(\varphi \to \psi) \wedge \mathsf{E}\varphi$. These two definitions introduce stronger nonmonotonic conditional logics than the previous ones, but they do not satisfy axiom $\mathbf{ID}$ (*reflexivity*). This is not very convenient as '*reflexivity seems to be satisfied universally by any kind of reasoning based on some notion of consequence*' [19, p. 177] and defeasible conditionals are designed to incarnate some form of defeasible consequence. However, as observed also in [19], conditionals that do not satisfy it '*probably express some notion of theory change*'. The latter definition introduces a conditional logic which comes close to the logic $\overset{\Rightarrow}{\mathbf{\Omega}}$ introduced in [18] as a modally-defined, 'majority' default conditional logic, imposing the validity of $(\varphi \Rightarrow \psi)$ iff $(\varphi \wedge \psi)$ is true in a cofinite subset of the ω many possible worlds available.

3.2 Default conditionals, epistemically defined

We are now going to provide in detail the results about the logic arising when the normality conditional is defined as $\mathsf{K}(\mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge \psi))$. The logic is called $\mathbf{EC}_1$ (for *Epistemic Conditional*). A recursive translation unpacks any conditional formula in the language of $\mathbf{KBE}$, giving access to the proof procedures of the logic. In the rest of the paper, $C_{\mathbf{KBE}}$ will denote the class of all kbe-frames.

DEFINITION 3.1 (CONDITIONAL LOGIC $\mathbf{EC_1}$). We recursively define the following translation $()^* : \mathcal{L}_{\Rightarrow} \to \mathcal{L}_{\mathbf{KBE}}$: **(i)** $(p)^* = p$, if $p \in \Phi$ (p is a propositional variable), **(ii)** $(\varphi \circ \psi)^* = (\varphi)^* \circ (\psi)^*$ for $\circ \in \{\wedge, \vee, \to, \equiv\}$, **(iii)** $(\neg\varphi)^* = \neg(\varphi)^*$ and **(iv)** $(\varphi \Rightarrow \psi)^* = \mathsf{K}(\mathsf{E}\varphi^* \to \mathsf{E}(\varphi^* \wedge \psi^*))$. The logic $\mathbf{EC_1}$ consists of all formulae $\varphi \in \mathcal{L}_{\Rightarrow}$, such that: $\varphi \in \mathbf{EC_1}$ iff $C_{\mathbf{KBE}} \models \varphi^*$ iff $\vdash_{\mathbf{KBE}} \varphi^*$

Let us proceed to check the properties of $\mathbf{EC_1}$. Throughout the proofs, $\mathfrak{F}$ refers to an arbitrary kbe-frame $\langle W, \mathcal{R}, \mathcal{N} \rangle$ and $\mathfrak{M}$ to an arbitrary kbe-model $\langle W, \mathcal{R}, \mathcal{N}, V \rangle$.

THEOREM 3.2. The logic $\mathbf{EC_1}$: **(i)** is closed under the rules $\mathbf{RCEA}$, $\mathbf{RCEC}$ and $\mathbf{RCE}$ and **(ii)** contains the axioms $\mathbf{ID}$, $\mathbf{CUT}$, $\mathbf{Loop}$ and $\mathbf{CM}$.

PROOF. Due to space limitations we provide only the proof for $\mathbf{CUT}$. We have to show that $\mathfrak{F} \models (\varphi \wedge \psi \Rightarrow z) \wedge (\varphi \Rightarrow \psi) \to (\varphi \Rightarrow z)$. Assume an arbitrary world $w \in W$, such that $\mathfrak{M}, w \models (\varphi \wedge \psi \Rightarrow z) \wedge (\varphi \Rightarrow \psi)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, w \models (\varphi \wedge \psi \Rightarrow z)$ and $\mathfrak{M}, w \models (\varphi \Rightarrow \psi)$. By Def. 3.1 $\mathfrak{M}, w \models \mathsf{K}(\mathsf{E}(\varphi \wedge \psi) \to \mathsf{E}(\varphi \wedge \psi \wedge z))$ and $\mathfrak{M}, w \models \mathsf{K}(\mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge \psi))$. Let $u \in W$ be such that $w\mathcal{R}u$. Then, we have that:

$$\mathfrak{M}, u \models \mathsf{E}(\varphi \wedge \psi) \to \mathsf{E}(\varphi \wedge \psi \wedge z) \quad (1)$$

$$\mathfrak{M}, u \models \mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge \psi) \quad (2)$$

If $\mathfrak{M}, u \models \neg\mathsf{E}\varphi$ then $\mathfrak{M}, u \models \mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge z)$ trivially.

Let $\mathfrak{M}, u \models \mathsf{E}\varphi$. Then by (2) we have $\mathfrak{M}, u \models \mathsf{E}(\varphi \wedge \psi)$ and by (1) we derive that $\mathfrak{M}, u \models \mathsf{E}(\varphi \wedge \psi \wedge z)$. By definition, this means that $\mathcal{R}(u) \cap \|\varphi \wedge \psi \wedge z\| \in \mathcal{N}(u)$. But $\|\varphi \wedge \psi \wedge z\| \subseteq \|\varphi \wedge z\|$ and thus by the definition of $\mathcal{N}$ we also have that $\mathcal{R}(u) \cap \|\varphi \wedge z\| \in \mathcal{N}(u)$. By definition then, $\mathfrak{M}, u \models \mathsf{E}(\varphi \wedge z)$.

So, if $\mathfrak{M}, u \models \mathsf{E}\varphi$, then $\mathfrak{M}, u \models \mathsf{E}(\varphi \wedge z)$, which gives that $\mathfrak{M}, u \models \mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge z)$. The world $u \in W$ was arbitrarily chosen such that $w\mathcal{R}u$, so $\mathfrak{M}, w \models \mathsf{K}(\mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge z))$. By Def. 3.1 again, $\mathfrak{M}, w \models (\varphi \Rightarrow z)$. So, if $\mathfrak{M}, w \models (\varphi \wedge \psi \Rightarrow z) \wedge (\varphi \Rightarrow \psi)$, then $\mathfrak{M}, w \models (\varphi \Rightarrow z)$, which gives that $\mathfrak{M}, w \models (\varphi \wedge \psi \Rightarrow z) \wedge (\varphi \Rightarrow \psi) \to (\varphi \Rightarrow z)$. Since the world w and model $\mathfrak{M}$ were arbitrarily chosen, the proof is complete. □

The following theorem presents the rules and axioms not present in $\mathbf{EC_1}$. The counterexample constructed is based on the kbe-frame of Figure 1.

THEOREM 3.3. The logic $\mathbf{EC_1}$: **(i)** is not closed under the rule $\mathbf{RCK}$, **(ii)** does not contain the axioms $\mathbf{AC}$, $\mathbf{CC}$, $\mathbf{OR}$, $\mathbf{CV}$, $\mathbf{CSO}$, $\mathbf{MP}$, $\mathbf{MOD}$, $\mathbf{CA}$, $\mathbf{CS}$, $\mathbf{CEM}$, $\mathbf{SDA}$, **Transitivity**, **Monotonicity** and **Contraposition**.

PROOF. We provide only the proof for axiom $\mathbf{OR}$. Consider the kbe-frame $\mathfrak{F}_1$ and the model $\mathfrak{M}_1$ of $\mathfrak{F}_1$ based on the following valuation: $V(\varphi) = \{u_1\}$, $V(\psi) = \{u_2\}$ and $V(z) = \emptyset$. It suffices to show that $\mathfrak{M}_1, w \not\models (\varphi \Rightarrow z) \wedge (\psi \Rightarrow z) \to (\varphi \vee \psi \Rightarrow z)$. Indeed, we have that $\forall v \in W$, $\mathcal{R}(v) \cap \|\varphi\| \notin \mathcal{N}(v)$ and thus $\forall v \in W$: $\mathfrak{M}_1, v \models \neg\mathsf{E}\varphi$. This also gives $\forall v \in W$: $\mathfrak{M}_1, v \models (\mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge z))$ and $\mathfrak{M}_1, w \models \mathsf{K}(\mathsf{E}\varphi \to \mathsf{E}(\varphi \wedge z))$ follows trivially. Similarly, $\mathfrak{M}_1, w \models \mathsf{K}(\mathsf{E}\psi \to \mathsf{E}(\psi \wedge z))$, as $\mathcal{R}(v) \cap \|\psi\| \notin \mathcal{N}(u)$, $\forall v \in W$. By Def. 3.1 $\mathfrak{M}_1, w \models (\varphi \Rightarrow z)$ and $\mathfrak{M}_1, w \models (\psi \Rightarrow z)$.

Furthermore, $\mathcal{R}(w) \cap \|\varphi \vee \psi\| \in \mathcal{N}(w)$ and thus $\mathfrak{M}_1, w \models \mathsf{E}(\varphi \vee \psi)$. But $\mathfrak{M}_1, w \not\models \mathsf{E}((\varphi \vee \psi) \wedge z)$ because $\mathcal{R}(w) \cap \|(\varphi \vee \psi) \wedge z\| \notin \mathcal{N}(w)$. It follows that $\mathfrak{M}_1, w \not\models \mathsf{E}(\varphi \vee \psi) \to \mathsf{E}((\varphi \vee \psi) \wedge z)$ and thus $\mathfrak{M}_1, w \not\models \mathsf{K}(\mathsf{E}(\varphi \vee \psi) \to \mathsf{E}((\varphi \vee \psi) \wedge z))$, as $w\mathcal{R}w$. We get $\mathfrak{M}_1, w \not\models (\varphi \vee \psi \Rightarrow z)$. So $\mathfrak{M}_1, w \models (\varphi \Rightarrow z)$ and $\mathfrak{M}_1, w \models (\psi \Rightarrow z)$, but $\mathfrak{M}_1, w \not\models (\varphi \vee \psi \Rightarrow z)$ and the proof is complete. □

We are going now to investigate an alternative definition of defeasible conditional within $\mathbf{KBE}$. Namely, we define the conditional as $\mathsf{E}\varphi \wedge \neg\mathsf{B}(\varphi \to \neg\psi) \to \mathsf{E}(\varphi \wedge \psi)$; see also the comments at Section 3.1. For the rest of this section, we omit the obvious recursive translation of the conditionals into $\mathbf{KBE}$ and proceed directly to the results. The logic $\mathbf{EC_2}$ corresponding to the definition above is rather weak and the details are provided in the following theorems whose proof is omitted.

THEOREM 3.4. The logic $\mathbf{EC_2}$: **(i)** is closed under the rules $\mathbf{RCEA}$, $\mathbf{RCEC}$ and $\mathbf{RCE}$, **(ii)** contains the axiom $\mathbf{ID}$.

THEOREM 3.5. The logic $\mathbf{EC_2}$: **(i)** is not closed under the rule $\mathbf{RCK}$, **(ii)** does not contain the axioms $\mathbf{CUT}$, $\mathbf{AC}$, $\mathbf{CC}$, $\mathbf{Loop}$, $\mathbf{OR}$, $\mathbf{CV}$, $\mathbf{CSO}$, $\mathbf{CM}$, $\mathbf{MP}$, $\mathbf{MOD}$, $\mathbf{CA}$, $\mathbf{CS}$, $\mathbf{CEM}$, $\mathbf{SDA}$, **Transitivity**, **Monotonicity** and **Contraposition**.

3.3 A monotonic conditional within KBE

This section investigates a Reiter-style conditional: $(\varphi \Rightarrow \psi)$ is defined as $\mathsf{K}\varphi \wedge \neg\mathsf{B}\neg\psi \to \mathsf{E}\psi$. The following two theorems verify that the logic $\mathbf{EC_3}$ of this conditional misses some of the principles found in the 'conservative core' of Default Reasoning (the system $\mathbf{P}$, see Table 1) but otherwise possesses the axioms $\mathbf{CA}$, $\mathbf{CV}$, the '*conditional excluded middle*' axiom $\mathbf{CEM}$ and axiom $\mathbf{SDA}$.

THEOREM 3.6. The logic $\mathbf{EC_3}$: **(i)** is closed under the rules $\mathbf{RCEA}$, $\mathbf{RCEC}$ and $\mathbf{RCE}$, **(ii)** contains the axioms $\mathbf{ID}$, $\mathbf{AC}$, $\mathbf{CV}$, $\mathbf{CA}$, $\mathbf{CEM}$, $\mathbf{SDA}$ and **Monotonicity**.

THEOREM 3.7. The logic $\mathbf{EC_3}$: **(i)** is not closed under the rule $\mathbf{RCK}$, **(ii)** does not contain the axioms $\mathbf{CUT}$, $\mathbf{CC}$, $\mathbf{Loop}$, $\mathbf{OR}$, $\mathbf{CSO}$, $\mathbf{CM}$, $\mathbf{MP}$, $\mathbf{MOD}$, $\mathbf{CS}$, **Transitivity** and **Contraposition**.

3.4 Two default conditionals without Reflexivity

The last two conditionals actually investigate the possibility of enforcing epistemic values on the antecedents of the classical inference rule **Modus Ponens**.

DEFINITION 3.8. **[Logics $\mathbf{EC_4}$ and $\mathbf{EC_5}$].** Let the logics $\mathbf{EC_4}$ and $\mathbf{EC_5}$ consist of all formulae $\varphi \in \mathcal{L}_{\Rightarrow}$ defined, in the same way as the previous definitions, accordingly as: $\mathbf{EC}_4$: $(\varphi \Rightarrow \psi)$ as $\mathsf{K}\varphi \wedge \mathsf{E}(\varphi \rightarrow \psi)$, and $\mathbf{EC}_5$: $(\varphi \Rightarrow \psi)$ as $\mathsf{K}(\varphi \rightarrow \psi) \wedge \mathsf{E}\varphi$.

THEOREM 3.9. The logic $\mathbf{EC_4}$: **(i)** is closed under the rules $\mathbf{RCEA}$ and $\mathbf{RCEC}$, **(ii)** contains the axioms $\mathbf{CUT}$, $\mathbf{Loop}$, $\mathbf{OR}$, $\mathbf{CSO}$, $\mathbf{CM}$, $\mathbf{MOD}$, $\mathbf{CA}$ and **Transitivity**.

PROOF. We provide only the proof for **Transitivity**. Observe that in kbe-models $\mathfrak{M}, w \models \mathsf{K}\varphi\wedge\mathsf{E}(\varphi \rightarrow \psi)$ iff $\mathfrak{M}, w \models \mathsf{K}\varphi\wedge\mathsf{E}\psi$. We have to show that $\mathfrak{F} \models (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow z) \rightarrow (\varphi \Rightarrow z)$. Assume an arbitrary world $w \in W$, such that $\mathfrak{M}, w \models (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow z)$, where $\mathfrak{M}$ is a model of $\mathfrak{F}$. Obviously, $\mathfrak{M}, w \models (\varphi \Rightarrow \psi)$ and $\mathfrak{M}, w \models (\psi \Rightarrow z)$. Then, $\mathfrak{M}, w \models \mathsf{K}\varphi \wedge \mathsf{E}\psi$ and $\mathfrak{M}, w \models \mathsf{K}\psi \wedge \mathsf{E}z$. So $\mathfrak{M}, w \models \mathsf{K}\varphi \wedge \mathsf{E}\psi \wedge \mathsf{K}\psi \wedge \mathsf{E}z$, which also gives $\mathfrak{M}, w \models \mathsf{K}\varphi \wedge \mathsf{E}z$. Then, we have that $\mathfrak{M}, w \models \mathsf{K}\varphi \wedge \mathsf{E}(\varphi \rightarrow z)$ and thus $\mathfrak{M}, w \models (\varphi \Rightarrow z)$. □

THEOREM 3.10. The logic $\mathbf{EC_4}$: **(i)** is not closed under the rules $\mathbf{RCK}$ and $\mathbf{RCE}$, **(ii)** does not contain the axioms $\mathbf{ID}$, $\mathbf{AC}$, $\mathbf{CC}$, $\mathbf{CV}$, $\mathbf{MP}$, $\mathbf{CS}$, $\mathbf{CEM}$, $\mathbf{SDA}$, **Monotonicity** and **Contraposition**.

THEOREM 3.11. The logic $\mathbf{EC_5}$: **(i)** is closed under the rules $\mathbf{RCEA}$, $\mathbf{RCK}$ and $\mathbf{RCEC}$, **(ii)** contains the axioms $\mathbf{CUT}$, $\mathbf{AC}$, $\mathbf{CC}$, $\mathbf{Loop}$, $\mathbf{OR}$, $\mathbf{CSO}$, $\mathbf{CM}$, $\mathbf{MP}$, $\mathbf{MOD}$ and **Transitivity**.

THEOREM 3.12. The logic $\mathbf{EC_5}$: **(i)** is not closed under the rule $\mathbf{RCE}$, **(ii)** does not contain the axioms $\mathbf{ID}$, $\mathbf{CV}$, $\mathbf{CA}$, $\mathbf{CS}$, $\mathbf{CEM}$, $\mathbf{SDA}$, **Monotonicity** and **Contraposition**.

PROOF. ($\mathbf{SDA}$) Consider the kbe-frame $\mathfrak{F}_1$ and the model $\mathfrak{M}_1$ of $\mathfrak{F}_1$ based on the following valuation: $V(\varphi) = \{u_1\}$, $V(\psi) = \{u_2\}$ and $V(z) = W$. It suffices to show that $\mathfrak{M}_1, w \not\models (\varphi \vee \psi \Rightarrow z) \rightarrow (\varphi \Rightarrow z)\wedge(\psi \Rightarrow z)$. Indeed, we have that $\mathcal{R}(w)\cap\|\varphi\vee\psi\| \in \mathcal{N}(w)$ and thus $\mathfrak{M}_1, w \models \mathsf{E}(\varphi \vee \psi)$. Furthermore, $\forall v \in W$, $\mathfrak{M}_1, v \models z$ and thus $\forall v \in W$, $\mathfrak{M}_1, v \models (\varphi\vee\psi \rightarrow z)$. This means that $\mathfrak{M}_1, w \models \mathsf{K}(\varphi\vee\psi \rightarrow z)$. So $\mathfrak{M}_1, w \models \mathsf{K}(\varphi\vee\psi \rightarrow z)\wedge\mathsf{E}(\varphi\vee\psi)$. Using Definition 3.8 then, we get $\mathfrak{M}_1, w \models (\varphi \vee \psi \Rightarrow z)$.

But $\mathfrak{M}_1, w \not\models \mathsf{E}\varphi$ because $\mathcal{R}(w) \cap \|\varphi\| \notin \mathcal{N}(w)$. It follows that $\mathfrak{M}_1, w \not\models \mathsf{K}(\varphi \rightarrow z) \wedge \mathsf{E}\varphi$. Using Definition 3.8 then, we get $\mathfrak{M}_1, w \not\models (\varphi \Rightarrow z)$. So $\mathfrak{M}_1, w \models (\varphi \vee \psi \Rightarrow z)$, but $\mathfrak{M}_1, w \not\models (\varphi \Rightarrow z)$ and the proof is complete. □

Table 1 summarizes our results and allows the reader to compare the relative strength of the conditional logics epistemically defined in this paper. A 'tick' implies that the system has the corresponding rule or axiom, a shaded box implies that the system does not have it and an empty box means that we have not checked this for the logics found in the literature. The last column describes logics $\mathbf{EC}_i$ of this paper, $\mathbf{CE}$ is from [23], $\mathbf{CT4}$ from [3] and logic $\overset{\Rightarrow}{\Omega}$ from [18].

4. CONCLUSIONS

In this paper, we have pursued the possibility of defining defeasible conditionals syntactically, employing the machinery of *Epistemic* and *Doxastic Logic*. As argued in the introductory section, the epistemic reading of some forms of normality statements has been noticed within the KR community but the fine interplay between the epistemic attitudes and the default conditionals has not been thoroughly investigated up to now. We believe that our results have demonstrated the feasibility and the merits of such an approach which opens the possibility of employing a very fertile area and its modern extensions (like *Dynamic Epistemic Logic*) for the study of defeasible conditionals. One thing unique in this treatment of default conditionals is a kind of reverse engineering: **which 'species' of knowledge and 'doxastic attitudes' (belief, weak belief, disbelief, estimation, etc.) do we need in order to describe the phenomenon of 'jumping to conclusions by default'?** This question is important for defining the 'correct' translation of defaults in epistemic logic, yet, answering this will shed more light on the very 'nature' of default statements. Needless to say, this question has important philosophical repercussions and hinges on the delicate relation between epistemic and doxastic notions which is still debated in Philosophy.

The approach we have taken is qualitative. A quantitative approach will probably prove harder to design but more flexible in its use. A prime candidate for a quantitative epistemic treatment of defeasible conditionals is the area of epistemic probabilistic logic [8]. In general, one can imagine of an epistemic language that would combine epistemic operators with the ability of direct handling of statements about the probability of certain facts, along the lines of Fagin and Halpern [10]. It is conceivable that a language allowing statements of the form $\mathsf{K}(p(\varphi) \geq p_1) \rightarrow \mathsf{E}(p(\psi) \geq p_2)$ would allow for explicit reasoning about epistemic attitudes and probability, giving alternative ways for defining default conditionals. It seems that we are still far form understanding the cognitive process triggered and the epistemic attitudes involved in the procedure of *Reasoning by Default*. In hindsight, it is interesting to observe that the body of work in KR that relates knowledge and belief with nonmonotonic reasoning practically does not differentiate between the two notions and relies mainly on *introspection*. Yet, we believe that the connection of epistemic logic to default reasoning is much deeper and - for the time being - hardly known.

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Table 1: KLM systems, Axioms and Conditional Logics

KLM

Axioms and Rules		Systems: C	CL	P	R
REF	$A \mid\!\sim A$	✓	✓	✓	✓
LLE	$\frac{\vdash A\equiv B \quad A \mid\!\sim C}{B \mid\!\sim C}$	✓	✓	✓	✓
RW	$\frac{\vdash A\rightarrow B \quad C \mid\!\sim A}{C \mid\!\sim B}$	✓	✓	✓	✓
CUT	$\frac{A\wedge B \mid\!\sim C \quad A \mid\!\sim B}{A \mid\!\sim C}$	✓	✓	✓	✓
CM	$\frac{A \mid\!\sim B \quad A \mid\!\sim C}{A\wedge B \mid\!\sim C}$	✓	✓	✓	✓
AND	$\frac{A \mid\!\sim B \quad A \mid\!\sim C}{A \mid\!\sim B\wedge C}$	✓	✓	✓	✓
Loop	$\frac{A_0 \mid\!\sim A_1 \quad \ldots \quad A_k \mid\!\sim A_0}{A_0 \mid\!\sim A_k}$		✓	✓	✓
OR	$\frac{A \mid\!\sim C \quad B \mid\!\sim C}{A\vee B \mid\!\sim C}$			✓	✓
RM	$\frac{A \mid\!\sim C \quad A\wedge B \not\mid\!\sim C}{A \mid\!\sim \neg B}$				✓

CONDITIONAL LOGICS

Axioms and Rules		Systems: CE	CT4	$\overset{\Rightarrow}{\Omega}$	1	2	3	4	5
ID	$A \Rightarrow A$	✓	✓		✓	✓	✓		
RCEA	$\frac{A \equiv B}{(A\Rightarrow C) \equiv (B\Rightarrow C)}$	✓	✓	✓	✓	✓	✓	✓	✓
RCK	$\frac{(A_1\wedge\ldots\wedge A_n)\rightarrow B}{(C\Rightarrow A_1\wedge\ldots\wedge C\Rightarrow A_n)\rightarrow(C\Rightarrow B)}$	✓		✓					✓
CUT	$(A\wedge B \Rightarrow C)\wedge(A\Rightarrow B)\rightarrow(A\Rightarrow C)$	✓	✓	✓	✓			✓	✓
AC	$(A\Rightarrow B)\wedge(A\Rightarrow C)\rightarrow(A\wedge B \Rightarrow C)$	✓	✓	✓			✓		✓
CC	$(A\Rightarrow B)\wedge(A\Rightarrow C)\rightarrow(A \Rightarrow B\wedge C)$	✓	✓	✓					✓
Loop	$(A_0\Rightarrow A_1 \wedge \ldots \wedge A_k\Rightarrow A_0)\rightarrow(A_0\Rightarrow A_k)$	✓		✓	✓			✓	✓
OR	$(A\Rightarrow C)\wedge(B\Rightarrow C)\rightarrow(A\vee B \Rightarrow C)$	✓	✓	✓				✓	✓
CV	$(A\Rightarrow B)\wedge\neg(A\Rightarrow\neg C)\rightarrow(A\wedge C \Rightarrow B)$						✓		
RCEC	$\frac{A \equiv B}{(C\Rightarrow A) \equiv (C\Rightarrow B)}$	✓		✓	✓	✓	✓	✓	✓
RCE	$\frac{A \rightarrow B}{A \Rightarrow B}$	✓			✓	✓	✓		
CSO	$(A\Rightarrow B)\wedge(B\Rightarrow A)\rightarrow((A\Rightarrow C)\equiv(B\Rightarrow C))$	✓	✓	✓				✓	✓
CM	$(A \Rightarrow B\wedge C)\rightarrow(A\Rightarrow B)\wedge(A\Rightarrow C)$	✓		✓	✓			✓	✓
MP	$(A\Rightarrow B)\rightarrow(A\rightarrow B)$								✓
MOD	$(\neg A\Rightarrow A)\rightarrow(B\Rightarrow A)$	✓						✓	✓
CA	$(A\Rightarrow B)\wedge(C\Rightarrow B)\rightarrow(A\wedge C \Rightarrow B)$	✓		✓			✓	✓	
CS	$(A \wedge B)\rightarrow(A\Rightarrow B)$								
CEM	$(A\Rightarrow B)\vee(A\Rightarrow\neg B)$						✓		
SDA	$(A\vee B \Rightarrow C)\rightarrow(A\Rightarrow C)\wedge(B\Rightarrow C)$						✓		
Trans.	$(A\Rightarrow B)\wedge(B\Rightarrow C)\rightarrow(A\Rightarrow C)$			✓				✓	✓
Mon.	$(A\Rightarrow B)\rightarrow(A\wedge C \Rightarrow B)$						✓		
Contr.	$(A\Rightarrow B)\rightarrow(\neg B\Rightarrow\neg A)$								

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