OATAO is an open access repository that collects the work of Toulouse researchers and makes it freely available over the web where possible.

This is an author’s version published in:
http://oatao.univ-toulouse.fr/22478

Official URL
DOI : https://doi.org/10.1145/3019612.3019714

To cite this version: Koutras, Costas D. and Moyzes, Christos and Rantsoudis, Christos A reconstruction of default conditionals within epistemic logic. (2017) In: Annual ACM Symposium on Applied Computing (SAC 2017), 3 April 2017 - 7 April 2017 (Marrakech, Morocco).

Any correspondence concerning this service should be sent to the repository administrator: tech-oatao@listes-diff.inp-toulouse.fr
A reconstruction of Default Conditionals within Epistemic Logic

Costas D. Koutras  
Dept. of Informatics & Telecommunications  
University of Peloponnese  
Tripolis, Greece  
c.koutras@uop.gr

Christos Moyzes  
KR Group  
Dept. of Computer Science  
University of Liverpool, Liverpool, UK  
c.Moyzes@liverpool.ac.uk

Christos Rantsoudis  
Institut de Recherche en Informatique de Toulouse (IRIT)  
Toulouse, France  
Christos.Rantsoudis@irit.fr

ABSTRACT
Default conditionals are statements that express a condition of normality, in the form ‘if $\varphi$ then normally $\psi$’ and are of primary importance in Knowledge Representation. There exist modal approaches to the construction of modal logics of normality. Most of them are built on notions of preference among possible worlds, corresponding to the semantic intuition that ($\varphi \Rightarrow \psi$) is true in a situation if in the most preferred (most ‘normal’) situations in which $\varphi$ is true, $\psi$ is also true. It has been noticed that there exist natural epistemic readings of a default conditional, but this direction has not been thoroughly explored. A statement of the form ‘something known to be a bird, that can be consistently believed to fly, does fly’ involves well-known epistemic attitudes and allows the possibility of defining defaults within the rich framework of Epistemic Logic. We pursue this direction here within KBE, a recently introduced S4.2-based modal logic of knowledge, belief and estimation. In this logic, knowledge is a normal S4 operator, belief is a normal KD45 operator and estimation is a non-normal operator interpreted as a ‘majority’ quantifier over the set of epistemically alternative situations. We define and explore various conditionals using the epistemic operators of KBE, capturing ($\varphi \Rightarrow \psi$) in various ways, including ‘it is known that assuming $\varphi$ allows us to assume $\varphi \land \psi$’ or ‘if $\varphi$ is known and there is no reason to believe $\neg \psi$ then $\psi$ can be plausibly inferred’. Overall, we define here two weak nonmonotonic default conditionals, one monotonic conditional and two stronger nonmonotonic conditionals without axiom ID. Our results provide concrete evidence that the machinery of epistemic logic can be exploited for the study of default conditionals.

Keywords
Conditional Logic; Default Conditionals; Epistemic Logic

http://dx.doi.org/10.1145/3019612.3019714

1. INTRODUCTION
Knowledge Representation has always been concerned with ‘normality conditionals’, also called ‘defeasible conditionals’ or ‘defaults’. These are statements that express a condition of normality such as ‘birds normally fly’ or ‘adults are normally employed’ (although the validity of the latter conditional is intensely disputed in the era of the debt crisis ...). There exist other forms of expressions considered to fall within this class, such as ‘most birds have feathers’ or ‘Christos usually walks home after his class’. Default conditionals are intimately related to the major concerns of Non-monotonic Reasoning and have been further investigated after the study of nonmonotonic consequence relations and the introduction of KLM logics [19].

Default statements admit various readings. A fundamental one corresponds to their principal use of ascribing default properties to individuals (‘Twenty flies since birds normally fly’), a function accomplished elegantly also in McCarthy’s Circumscription (via classical first-order logic) and Reiter’s Default Logic (via the rules of inference adjoined to first-order logic). Other readings of defeasible conditionals seem closer to statements about (mostly qualitative but also quantitative) probability: ‘birds generally (typically, mostly) fly’. It has been noticed however that “the reading ‘a bird that can be consistently assumed to fly does fly’ is clearly epistemic in nature” [5, p. 95]. A ‘normality statement’ of the form ‘every Tuesday afternoon, you can find Jimmy taking a beer in the corner pub’ allows one to infer that on a ‘regular’ Tuesday s/he can meet Jimmy there and this default inference involves facts known (‘it is a Tuesday’), facts observed and believed (‘Jimmy frequents this place on Tuesday afternoon’), facts considered to be consistent with the belief base (‘there is no reason to believe this is an ‘irregular’ Tuesday’) and facts plausibly inferred (‘most probably I will meet him there’). Although difficult to agree on the subtle details of the epistemic attitudes involved, it seems that there is an agreement on the fact that such an epistemic description is quite reasonable. Cognitive statements of this kind are implicit in Reiter’s normal defaults [24] and the conditional entailment of H. Geffner and J. Pearl [12]: ‘a rule $\frac{a}{b}$ may be seen as a soft constraint for believing $b$ when $a$ is known, while a conditional rule $a \Rightarrow b$ can be viewed as a hard constraint to believe $b$ in a limited context defined by $a$ and possibly some background knowledge’ [9, p. 220].

The study of the connection of defeasible conditionals with the area of Epistemic and Doxastic Logic has not been hitherto pursued in its full entirety. In general, the ‘conditionals-via-modal-logic’ technique is known and quite successful [23, 4]; yet, the technical and
philosophical step to the construction of conditionals via Epistemic Logic has not been fully taken. The relation of Epistemic Logic to conditionals mainly revolves around the famous Ramsey test and this is also apparent in the earlier works of Lamarr & Shoham [21] or Friedman & Halpern [11] where an interesting notion of conditional belief is based on the semantics of default conditionals (see also [1, p. 107]). Modal approaches to defeasible conditionals [3, 20, 7] are mostly based on the model-theoretic intuition of ‘preference’ among possible worlds or propositions. The conditional \( \varphi \Rightarrow \psi \) is true in a possible world if \( \psi \) is true in the most ‘normal’ or ‘preferred’ \( \varphi \)-worlds accessible; equivalently, given the context of \( \varphi \), the proposition expressed by \( \varphi \land \neg \psi \) is preferred over the one expressed by \( \varphi \land \neg \neg \psi \) [7]. It is natural to consider that normality orderings are preorders (reflexive and transitive relations) and thus the modal approaches to defeasible conditionals usually employ the logic \( \text{S4} \) (or its extension \( \text{S4.3} \)) within which the defeasible conditional is modally defined [3]. Another modal construction of defeasible conditionals employs the notion of ‘size’: in [18] (\( \varphi \Rightarrow \psi \) becomes true whenever \( \psi \) is true in an ‘overwhelming majority’ of \( \varphi \)-worlds; assuming that we work within \( \omega \) (the first infinite ordinal), we can interpret ‘overwhelming majority’ as a cofinite subset of \( \omega \) and proceed to define modally the conditional within \( \text{K4DLZ} \) which is the modal logic of \( (\omega, <) \).

In this paper, we amplify the epistemic interpretation of defeasible conditionals and proceed to define them directly within Epistemic Logic. We work inside \( \text{KBE} \), a recently introduced epistemic logic [17] accounting for knowledge, belief and estimation (as a form of weak, complete belief, interpreted as ‘truth in most epistemic alternatives’). \( \text{KBE} \) comprises an \( \text{S4.2} \) framework for knowledge and belief, following the fundamental investigations of W. Lenzen [22] and R. Stalnaker [26]. The non-normal modal operator for estimation is interpreted as a ‘majority’ quantifier over the set of epistemic alternatives of a given possible world. The formal apparatus is that of a ‘weak ultrafilter’, which is an upwards-closed collection of sets, with pairwise non-disjoint members and such that exactly one out of a set and its complement occurs in the collection; the notion extends the weak filters introduced independently in [25, 15]. We define two nonmonotonic conditionals by capturing a size-oriented version of the fundamental intuition of normality conditionals: \( (\varphi \Rightarrow \psi) \) is set to mean that \( (\varphi \land \psi) \) is more normal compared to \( (\varphi \land \neg \psi) \), as it holds in ‘most’ epistemically alternative worlds; this is achieved by exploiting the nature of \( \text{KBE} \)’s ‘estimation’ operator as a majority quantifier. The logics emerging are rather weak compared to the ‘conservative core’ of default reasoning (the system \( \text{P} \), [19]) but this is neither surprising nor discouraging: weak conditionals of this kind have been also introduced in [7, system \( \text{C} \) and system \( \Lambda \)] under a rule-based interpretation of defaults and it is well-known that conditionals based on the plausibility structures of Friedman & Halpern do not generally satisfy all the KLM properties [11, p. 266]. Another, very ‘natural’ (but rather strong in epistemic assumptions) translation leads to a monotonic conditional, and two other epistemic definitions give rise to nonmonotonic conditional logics which do not satisfy the axiom \( \text{ID} \) (reflexivity), but they capture very interesting conditional principles and one of them comes close to an ‘overwhelming majority’ conditional defined in [18]. Note that for all these definitions a recursive translation in the language of \( \text{KBE} \) provides direct access to the tableaux proof procedure for this logic [17], and thus a machinery for testing theoremhood is readily available.

Our primary concern in this research is rather typical of (one of) the way(s) Conditional Logic is used in Knowledge Representa-

2. BACKGROUND - THE LOGIC \( \text{KBE} \)

We assume that the reader has a working knowledge of Modal Logic and Conditional Logic and is acquainted with the Scott-Montague (neighborhood) semantics and the cluster analysis of transitive normal modal logics (see [14, 2, 6]). In this paper, we reserve \( \Rightarrow \) for the classical (material) implication and \( \Rightarrow \) for the normality conditional or any other non-classical conditional constructed. The names of the modal axioms and systems mentioned in this paper are firmly entrenched in the literature. Less entrenched is the terminology on the ‘bridge’ axioms relating knowledge to belief [13]; we follow the naming given by W. Lenzen[22] and R. Stalnaker [26]. The axioms and rules of Conditional Logic can be found in Table 1, last page of this extended abstract.

The logic \( \text{KBE} \) has been introduced in [17]. The language \( L_{\text{KBE}} \) comprises three modal operators: \( K \varphi \) read as ‘the agent knows \( \varphi \)’, \( B \varphi \) read as ‘the agent believes \( \varphi \)’ and \( E \varphi \) read as ‘the agent estimates that \( \varphi \) is true’. One way to view the epistemic attitudes involved is to consider \( K \varphi \) as an \( S4 \) operator, and \( B \varphi \) as a \( KD45 \) operator, interconnected with the bridge axioms \( B1. \ K \varphi \Rightarrow B \varphi \) \( B2.3. \ B \varphi \Rightarrow \neg B \neg K \varphi \) and \( B2.4. \ B \varphi \Rightarrow KB \varphi \). Equivalently, following the work of W. Lenzen and R. Stalnaker, we consider the logic \( S4.2 \) within which belief is just an abbreviation defined by \( B \varphi \equiv \neg \neg K \varphi \); this equivalent perspective is very convenient as we are able to work within the model theory of \( S4.2 \). The estimation operator added on top of \( S4.2 \) is a non-normal majority quantifier: the intended interpretation is that an agent estimates that \( \varphi \) is true if \( \varphi \) holds in ‘most’ epistemic alternatives. Following is the axiomatization of \( \text{KBE} \), including the abbreviation for belief: \( \text{DB.} \ E \varphi \equiv \neg K \neg B \varphi \) (Definitio

Several introspective properties are proved in [17], including the following principles which are valid in \( \text{KBE} \): \( E \varphi \Rightarrow KE \varphi \), \( E \varphi \Rightarrow BE \varphi \), \( E \varphi \Rightarrow EE \varphi \), \( E \varphi \Rightarrow B \neg K \varphi \), \( E \varphi \Rightarrow B \neg B \varphi \), \( E \varphi \Rightarrow E \neg K \varphi \) (non-estimation implies introspection w.r.t ignorance and ‘lack of certainty’) and \( E \varphi \land B (\varphi \Rightarrow \psi) \Rightarrow E \psi \). Of particular importance is that belief can be equivalently defined in \( \text{KBE} \) as ‘estimation that the agent knows’: \( E \varphi \equiv B \varphi \) and the fact that knowledge about estimation amounts exactly to estimation itself \( K E \varphi \equiv E \varphi \).

DEFINITION 2.1. \( \text{KBE} \) is the propositional bimodal logic axiomatized by \( K, T, 4, C \), \( B, \text{BE} \), \( \text{CCE} \), \( \text{EK} \), \( \text{PIE} \) and closed
The possible-worlds models of KBE. To construct the frames of KBE we focus on the fact that one of the frame classes that determines S4.2 is the class of reflexive, transitive frames with a final cluster $FC$ (note that such a frame is automatically directed); this follows from the results in [16]. These S4.2-frames are combined with Scott-Montague semantics, in which each neighborhood is a cluster $\phi \Leftrightarrow R \cap Y \neq 0$, (pairwise non-disjointness). We obtain a weak ultrafilter by strengthening condition (ii) to a bi-conditional: $X \not\subseteq F \Leftrightarrow (W \setminus X) \subseteq F$ (exactly one, out of a set and its complement, is large). Genuinely weak ultrafilters exist and are of interest to ‘size’-oriented accounts of nonmonotonic knowledge, belief, weak belief, disbelief, ignorance, plausible judgement. The class of all $\mathfrak{K}$-frames; $\mathcal{N}$ is a non-empty set, $\mathcal{R} = (\forall \mathcal{N} \in \mathcal{W}, \mathcal{R} : W \rightarrow \mathcal{P}(\mathcal{W}))$, $\mathcal{R}$ is a reflexive, transitive relation with a nonempty final cluster $FC = \{w \in W | (\forall u \in W) wRu\}$, $\mathcal{N}$ is such that $\forall u \in W (\forall u \in W (wRu) \supseteq N(u))$, (be) $FC \in N(u)$, (pie) $\forall W \subseteq N(u)$, (ece) $\forall X \subseteq N(u)$ ($X \in N(u) \Leftrightarrow (w \not\subseteq N(u))$, (ek) $\forall X, Y \subseteq \mathcal{R}(u)$ ($X \not\subseteq \mathcal{R}(u) \Leftrightarrow (\forall u \in \mathcal{N}(u) \& Y \subseteq X \Leftrightarrow Y \not\subseteq \mathcal{N}(u))$). $\mathfrak{K}$ is called a kbe-frame. $\mathfrak{N} = (\mathfrak{K}, V)$ is called a kbe-model, if it is based on a kbe-frame and $V : \mathcal{P} \rightarrow \mathcal{P}(\mathcal{W})$ is a valuation.

The class of all kbe-frames is nonempty. The reader can verify that the frame of Figure 1, is a kbe-frame: $\mathcal{W} = \{u_1, u_2, u_3\}$, $\mathcal{R}$ is the relation shown in Figure 1, $N(u) = \{X \not\subseteq W | |X| \geq 3\} \cup \{u_1, u_2\}$, $u_3 \cup \{u_1, u_2\}$, $\mathcal{N}(u_1) = \mathcal{N}(u_2) = \mathcal{N}(u_3) = \{u_1, u_2, u_3\}$, $FC = \{u_1, u_2, u_3\}$ is the final cluster of the structure $(W, \mathcal{R})$. KBE is determined by the class of kbe-frames [17].

DEFINITION 2.3. Consider the model $\mathcal{N} = (\mathcal{W}, \mathcal{R}, \mathcal{N}, V)$ for the language $L_{KBE}$. The function $\mathcal{V} : L_{KBE} \rightarrow \mathcal{P}(\mathcal{W})$ is defined recursively as follows: $\mathcal{V}(p) = V(p)$, $(\forall \mathcal{P} \in \Phi)$, $\mathcal{V}(\bot) = \{\phi \in \Phi \}$, and $(\forall \phi, \psi \in L_{KBE}) \mathcal{V}(\phi \rightarrow \psi) = (W \setminus \mathcal{V}(\phi)) \cup \mathcal{V}(\psi)$.

3. AN EPISTEMIC RECONSTRUCTION OF CONDITIONAL LOGIC

3.1 The role of epistemic operators in defining default conditionals

One important topic in Epistemic Logic is the relation between the various epistemic and doxastic attitudes or cognitive states: knowledge, belief, weak belief, disbelief, ignorance, plausible judgement, there exist a lot indeed. The deep and influential analysis of W. Lenzen [22] and R. Stalnaker [26] has revealed a minimal set of principles that an epistemologist should accept on the properties of knowledge, belief and their ‘bridging’ relations, resulting in the fundamental role of S4.2. The logic KBE builds on these results to accommodate an operator of plausible estimation, tailored for cases when necessarily either $\phi$ or $\neg\phi$ - but not both - should be ‘estimated’.

From the discussion in the introductory section 1 and the quotations therein, it should be clear that there exist important forms of the normality conditional that can - or should - be read epistemically. But, when it comes to a careful formal definition of a default conditional within an epistemic logic, the question arises: which ‘attitude’, in which ‘place’ of the default conditional? In the first place, drawing inspiration from the ‘archetypical’ normal default from Reiter’s logic: $\phi \rightarrow \psi$, it seems natural to consider an epistemic translation of a normality conditional $(\phi \Rightarrow \psi)$ (if $\phi$ then normally $\psi$) in the form: $K(\phi \& \neg\psi) \rightarrow \psi$, interpreting the justification $\psi$ of the normal (whose meaning in Default Logic is ‘it is consistent to assume $\psi$‘, usually denoted as $M\psi$) as ‘I have no reason to believe $\neg\psi$’, the prerequisite as ‘I know $\phi$’ and the conclusion of the default as ‘I estimate that $\psi$ is the case’. This definition gives rise to an interesting conditional, which is not a default conditional: it satisfies the principle of monotony (alias strengthening the antecedent); see Section 3.3. Our experimentation revealed that this is due to the strong influence of the knowledge operator in the ‘prerequisite’, a phenomenon partly persisting when knowledge is replaced with belief. Other definitions of the conditional which exhibit the same behaviour include $K\phi \& \neg B(\phi \rightarrow \neg\psi) \rightarrow \psi$, and $K\phi \& \neg B(\phi \& \neg\psi) \rightarrow \psi$ while other definitions, including $(K\phi \& \neg E(\phi \rightarrow \neg\psi) \rightarrow \psi)$ and $(K\phi \& \neg E(\phi \& \neg\psi) \rightarrow \psi)$ lead to triviality as they are valid KBE principles (check with the axiomatization in Section 2), indicating clearly that given knowledge in the ‘prerequisite’, estimation cannot replace belief in the ‘justification’. However, once we replace knowledge by estimation (which is a much weaker operator) defining $(\phi \Rightarrow \psi)$ as $E\phi \& \neg B(\phi \rightarrow \neg\psi) \rightarrow \psi$ we obtain a weak defeasible conditional, whose properties we discuss in Section 3.2. Another, perhaps more interesting defeasible conditional arises when the interplay between the antecedent (the ‘prerequisite’) and the consequent (the ‘conclusion’) is ‘controlled’ through knowledge and ‘computed’ via estimation as $K(E\phi \rightarrow E(\phi \& \psi))$, read as ‘I can plausibly conclude normally $\psi$ assuming $\phi$’ if I know that estimating $\phi$ allows me
to estimate $\varphi \land \psi'$. This view comes close to the intuition that 
$(\varphi \land \psi)$ seems more ‘normal’ than $(\varphi \land \neg \psi)$; note also that given
the interpretation of estimation in KBE, $(\varphi \land \psi)$ holds in ‘most’ situations.
Finally, we proceed to define conditionals by imposing epistemic operators on
the antecedents of rule Modus Ponens. We define
$(\varphi \Rightarrow \psi) = K_\varphi \land (E(\varphi \rightarrow \psi) \land K(\varphi \rightarrow \psi) \land E\varphi$.
These two definitions introduce stronger nonmonotonic conditional logics
than the previous ones, but they do not satisfy axiom ID (reflexivity).
This is not very convenient as ‘reflexivity seems to be satisfied universally by
any kind of reasoning based on some notion of consequence’ [19, p. 177] and
defeasible conditionals are designed to incorporate some form of defeasible consequence.
However, as observed also in [19], conditionals that do not satisfy it
‘probably express some notion of theory change’. The latter definition
introduces a conditional logic which comes close to the logic
$\Omega$ introduced in [18] as a modally-defined, ‘majority’ default
conditional logic, imposing the validity of $(\varphi \Rightarrow \psi)$ iff $(\varphi \land \psi)$ is true in
a cofinite subset of the $\omega$ many possible worlds available.

3.2 Default conditionals, epistemically defined
We are now going to provide in detail the results about the logic
arising when the normality conditional is defined as $K(E\varphi \rightarrow E(\varphi \land \psi))$.
The logic is called EC1 (for Epistemic Conditional). A recursive
transformation unpacks any conditional formula in the language of
KBE, giving access to the proof procedures of the logic. In the rest of
the paper, $C_{KBE}$ will denote the class of all kbe-frames.

Definition 3.1 (Conditional Logic EC1). We recursively define
the following translation $(\cdot)^* : \mathcal{L}_\varphi \rightarrow \mathcal{L}_{KBE}$: (i) $(p)^* = p$,
if $p \in \Phi$ ($p$ is a propositional variable), (ii) $(\varphi \land \psi)^* = (\varphi)^* \land (\psi)^*$
for $\varphi, \psi \in \mathcal{L}_{\varphi}$, (iii) $(\neg \varphi)^* = \neg (\varphi)^*$ and (iv) $(\varphi \Rightarrow \psi)^* = \{K(E\varphi \rightarrow E(\varphi \land \psi))\}$. The logic EC1 consists of all formulae
$\varphi \in \mathcal{L}_{\varphi}$, such that $\varphi \in \mathcal{L}_{EC1}$ iff $C_{KBE} \models \varphi^*$.

Let us proceed to check the properties of EC1. Throughout the
proofs, $\mathcal{E}$ refers to an arbitrary kbe-frame $(W, R, N, V)$ and $\mathcal{M}$ to an
arbitrary kbe-model $(W, R, N, V)$.

Theorem 3.2. The logic EC1: (i) is closed under the rules
RCEA, RCEC and RCE and (ii) contains the axioms ID, CUT, Loop
and CM.

Proof. Due to space limitations we provide only the proof for
CUT. We have to show that $\mathcal{E} = (\varphi \land \psi \Rightarrow z) \land (\varphi \Rightarrow \psi)$
Assume an arbitrary world $w \in W$, such that $\mathcal{M}, w \models (\varphi \land \psi \Rightarrow z)$
and $\mathcal{M}, w \models (\varphi \Rightarrow \psi)$. By Def. 3.1
$\mathcal{M}, w \models K(E(\varphi \land \psi) \rightarrow E(\varphi \land \psi \land z))$ and $\mathcal{M}, w \models K(E\varphi \rightarrow E(\varphi \land \psi))$. Let $u \in W$ be such that $wR\; u$. Then we have that:

\[ \mathcal{M}, u \models E(\varphi \land \psi) \rightarrow E(\varphi \land \psi \land z) \] (1)

2. If $\mathcal{M}, u \models \neg E\varphi$ then $\mathcal{M}, u \models E\varphi \rightarrow E(\varphi \land \psi)$ trivially.

Let $\mathcal{M}, u \models E\varphi$, then by (2) we have $\mathcal{M}, u \models E(\varphi \land \psi)$ and by (1)
we derive that $\mathcal{M}, u \models E(\varphi \land \psi \land z)$. By definition, this means that
$\mathcal{R}(u) \cap \{\varphi \land \psi \land z\} \subseteq N(u)$. But $\{\varphi \land \psi \land z\} \subseteq \{\varphi \land z\}$ and thus
by the definition of $N$ we also have that $\mathcal{R}(u) \cap \{\varphi \land z\} \subseteq N(u)$.
By definition then, $\mathcal{M}, u \models E(\varphi \land z)$.

So, if $\mathcal{M}, u \models E\varphi$, then $\mathcal{M}, u \models E(\varphi \land \psi)$, which gives that
$\mathcal{M}, u \models E\varphi \rightarrow E(\varphi \land \psi)$. The world $w \in W$ was arbitrarily chosen
such that $wR\; u$, so $\mathcal{M}, w \models K(E\varphi \rightarrow E(\varphi \land \psi))$. By Def. 3.1 again,
$\mathcal{M}, w \models (\varphi \Rightarrow \psi)$. So, if $\mathcal{M}, w \models (\varphi \land \psi \Rightarrow z) \land (\varphi \Rightarrow \psi)$,
then $\mathcal{M}, w \models (\varphi \Rightarrow \psi)$, which gives that $\mathcal{M}, w \models (\varphi \land \psi \Rightarrow z) \land (\varphi \Rightarrow \psi) \rightarrow (\varphi \Rightarrow \psi)$. Since the world $w$ and model $\mathcal{M}$ were arbitrarily
chosen, the proof is complete.

The following theorem presents the rules and axioms not present in
EC1. The counterexample constructed is based on the kbe-frame of Figure 1.

Theorem 3.3. The logic EC1: (i) is not closed under the rule
RCK, (ii) does not contain the axioms AC, CC, OR, CV,
CSO, MP, MOD, CA, CS, CEM, SDA, Transitivity, Monotonicity
and Contraposition.

Proof. We provide only the proof for axiom OR. Consider
the kbe-frame $\mathcal{S}_1$ and the model $\mathcal{M}$ of $\mathcal{S}_1$ based on the following valuation:
$V(\varphi) = \{u_1\}$, $V(\psi) = \{u_2\}$ and $V(\varphi \land \psi) = \emptyset$. It suffices to show that
$\mathcal{M}_1, w \models (\varphi \Rightarrow \psi)$ and $\mathcal{M}_1, w \models (\varphi \land \psi \Rightarrow z)$. Indeed,
we have that $\forall v \in W, \mathcal{M}_1 \models (\varphi \land \psi) \not\models \neg (\varphi \lor \psi)$ and thus
$\forall v \in W, \mathcal{M}_1, v \models \neg \varphi \lor \psi$. This also gives $\forall v \in W. \mathcal{M}_1, v \models (E\varphi \rightarrow E(\varphi \land \psi))$ and
$\mathcal{M}_1, w \models (E(\varphi \land \psi) \rightarrow (\varphi \lor \psi) \land z)$. Thus,
Similarly, $\mathcal{M}_1, w \models K(E\varphi \rightarrow E(\psi \land z))$, as $R(\psi) \cap |\psi| \not\models \neg N(\psi)$,
$\forall v \in W$. By Def. 3.1 $\mathcal{M}_1, w \models (\varphi \Rightarrow \psi)$ and $\mathcal{M}_1, w \models (\varphi \land \psi \Rightarrow z)$. Furthermore,
$\mathcal{M}_1, w \models (\varphi \Rightarrow \psi \land z)$ and thus $\mathcal{M}_1, w \models E(\varphi \land \psi) \lor \varphi \land \psi \Rightarrow z)$ as $\mathcal{M}_1, w \models (\varphi \Rightarrow \psi)$. So, $\mathcal{M}_1, w \models (\varphi \land \psi \Rightarrow z)$ and
$\mathcal{M}_1, w \models (\varphi \Rightarrow \psi)$, but $\mathcal{M}_1, w \models (\varphi \lor \psi \Rightarrow z)$ and the proof is complete.

We are going now to investigate an alternative definition of defeasible
conditional within KBE. Namely, we define the conditional
as $E\varphi \land \neg E(\varphi \rightarrow \psi) \rightarrow E(\varphi \land \psi)$; see also the comments at Section
3.1. For the rest of this section, we omit the obvious recursive
translation of the conditionals into KBE and proceed directly to
the results. The logic EC2 corresponding to the definition above is
rather weak and the details are provided in the following theorems
whose proof is omitted.

Theorem 3.4. The logic EC2: (i) is closed under the rules
RCEA, RCEC and RCE, (ii) contains the axiom ID.

Theorem 3.5. The logic EC2: (i) is not closed under the rule
RCK, (ii) does not contain the axioms AC, CC, Loop,
OR, CV, CSO, CM, MP, MOD, CA, CS, CEM, SDA, Transitivity, Monotonicity
and Contraposition.

3.3 A monotonic conditional within KBE
This section investigates a Reiter-style conditional: $(\varphi \Rightarrow \psi)$
is defined as $K(\varphi \land \neg E(\varphi \rightarrow \psi) \rightarrow E(\varphi \land \psi))$; see also the comments at Section
1) but otherwise possesses the axioms CA, CV, the
‘conditional excluded middle’ axiom CEM and axiom SDA.

Theorem 3.6. The logic EC3: (i) is closed under the rules
RCEA, RCEC and RCE, (ii) contains the axioms ID, AC,
CV, CA, CEM, SDA and Monotonicity.
THEOREM 3.7. The logic EC₃: (i) is not closed under the rule RCK, (ii) does not contain the axioms CUT, CC, Loop, OR, CSO, CM, MP, MOD, CS, Transitivity and Contraposition.

3.4 Two default conditionals without Reflexivity

The last two conditionals actually investigate the possibility of enforcing epistemic values on the antecedents of the classical inference rule Modus Ponens.

Definition 3.8. [Logics EC₄ and EC₅]. Let the logics EC₄ and EC₅ consist of all formulae ϕ ∈ Lₜ, defined in the same way as the previous definitions, accordingly as: EC₄ = (ϕ → ψ) as Kϕ ∧ E(ϕ → ψ), and EC₅: (ϕ → ψ) as Kϕ ∧ E(ϕ).

THEOREM 3.9. The logic EC₄: (i) is closed under the rules RCEA and RCEC, (ii) contains the axioms CUT, Loop, OR, CSO, CM, MOD, CA and Transitivity.

Proof. We provide only the proof for Transitivity. Observe that in kbe-models M, w |= Kϕ ∧ E(ϕ → ψ) iff M, w |= Kϕ ∧ Eψ. We have to show that M, w |= (ϕ → ψ) ∧ (ϕ → z) → (ϕ → z). Assume an arbitrary world w ∈ W such that M, w |= (ϕ → ψ) ∧ (ϕ → z), where M is a model of ψ. Obviously, M, w |= (ϕ → ψ) and M, w |= (ϕ → z). Then, M, w |= Kϕ ∧ Eψ and M, w |= Kϕ ∧ Ez. So M, w |= Kϕ ∧ Ez, which also gives M, w |= Kϕ ∧ ϕ, and thus M, w |= (ϕ → z). □

THEOREM 3.10. The logic EC₅: (i) is not closed under the rules RCK and RCE, (ii) does not contain the axioms ID, AC, CC, CV, MP, CS, CEM, SDA, Monotonicity and Contraposition.

THEOREM 3.11. The logic EC₅: (i) is closed under the rules RCEA, RCK and RCEC, (ii) contains the axioms CUT, AC, CC, Loop, OR, CSO, CM, MP, MOD and Transitivity.

THEOREM 3.12. The logic EC₄: (i) is not closed under the rule RCE, (ii) does not contain the axioms ID, CV, CA, CS, CEM, SDA, Monotonicity and Contraposition.

Proof. (SDA) Consider the kbe-frame F and the model M₁ of F, based on the following valuation: V(ϕ) = {u₁}, V(ψ) = {u₁} and V(z) = W. It suffices to show that M₁, w |= (ϕ ∨ ψ) → z) → (ϕ → z) ∧ (ψ → z). Indeed, we have that R(w) ∩ ||ϕ|| ∈ N(ϕ) and thus M₁, w |= E(ϕ ∨ ψ). Furthermore, ∀v ∈ W, M₁, v |= ψ = z and thus ∀v ∈ W, M₁, v |= (ϕ ∨ ψ) → z. This means that M₁, w |= K(ϕ ∨ ψ) → z). So M₁, w |= K(ϕ ∨ ψ) ∧ E(ϕ ∨ ψ). Using Definition 3.8.3 then, we get M₁, w |= (ϕ ∨ ψ) → z.

But M₁, w |= Eψ because R(w) ∩ ||ψ|| ∉ N(ψ). It follows that M₁, w |= K(ϕ → z) ∧ Eψ. Using Definition 3.8 then, we get M₁, w |= (ϕ → z). So M₁, w |= (ϕ ∨ ψ) → z), but M₁, w |= (ϕ → z) and the proof is complete. □

Table 1 summarizes our results and allows the reader to compare the relative strength of the conditional logics epistemically defined in this paper. A ‘tick’ implies that the system has the corresponding rule or axiom, a shaded box implies that the system does not have it and an empty box means that we have not checked this for the logics found in the literature. The last column describes logics EC₄, of this paper, CE is from [23], CT₄ from [3] and logic _PG from [18].

4. CONCLUSIONS

In this paper, we have pursued the possibility of defining defeasible conditionals syntactically, employing the machinery of Epistemic and Doxastic Logic. As argued in the introductory section, the epistemic reading of some forms of normality statements has been noticed within the KR community but the fine interplay between the epistemic attitudes and the default conditionals has not been thoroughly investigated up to now. We believe that our results have demonstrated the feasibility and the merits of such an approach which opens the possibility of employing a very fertile area and its modern extensions (like Dynamic Epistemic Logic) for the study of defeasible conditionals. One thing unique in this treatment of default conditionals is a kind of reverse engineering: which ‘species' of knowledge and ‘doxastic attitudes’ (belief, weak belief, disbelief, estimation, etc.) do we need in order to describe the phenomenon of ‘jumping to conclusions by default’? This question is important for defining the ‘correct’ translation of defaults in epistemic logic, yet, answering this will shed more light on the very 'nature' of default statements. Needless to say, this question has important philosophical repercussions and hinges on the delicate relation between epistemic and doxastic notions which is still debated in Philosophy.

The approach we have taken is qualitative. A quantitative approach will probably prove harder to design but more flexible in its use. A prime candidate for a quantitative epistemic treatment of defeasible conditionals is the area of epistemic probabilistic logic [8]. In general, one can imagine of an epistemic language that would combine epistemic operators with the ability of direct handling of statements about the probability of certain facts, along the lines of Fagin and Halpern [10]. It is conceivable that a language allowing statements of the form K(p(ϕ) ≥ p₁) → E(p(ψ) ≥ p₂) would allow for explicit reasoning about epistemic attitudes and probability, giving alternative ways for defining default conditionals. It seems that we are still far form understanding the cognitive process triggered and the epistemic attitudes involved in the procedure of Reasoning by Default. In hindsight, it is interesting to observe that the body of work in KR that relates knowledge and belief with nonmonotonic reasoning practically does not differentiate between the two notions and relies mainly on introspection. Yet, we believe that the connection of epistemic logic to default reasoning is much deeper and - for the time being - hardly known.

5. REFERENCES


Table 1: KLM systems, Axioms and Conditional Logics

<table>
<thead>
<tr>
<th>Axioms and Rules</th>
<th>System</th>
<th>KLM</th>
<th>CONDITIONAL LOGICS</th>
</tr>
</thead>
<tbody>
<tr>
<td>REF</td>
<td>$ A \vdash A $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>LLE</td>
<td>$ A \supset B \vdash A $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>RW</td>
<td>$ A \supset B \vdash A \supset C $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>CUT</td>
<td>$ A \vdash B \vdash A $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>CM</td>
<td>$ A \vdash B \vdash A \supset C $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>AND</td>
<td>$ A \vdash A \vdash B \vdash A \supset C $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>Loop</td>
<td>$ A \vdash A \vdash B \vdash C \vdash A \vdash A $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>OR</td>
<td>$ A \vdash A \vdash B \vdash A \vdash C $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>RM</td>
<td>$ A \vdash A \vdash B \vdash A \vdash A \vdash C $</td>
<td>✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
</tbody>
</table>

Note: ID, RCEA, RCCK, CUT, AC, CC, Loop, OR, CV, RCEC, RCE, CSO, CM, MP, MOD, CA, CS, CEM, SDA, Trans, Men, Contr.


