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Abstract—This work proposes a tractable evaluation of the maximum a posteriori (MAP) threshold of sparse-graph ensembles, by using an approximation for the extended belief propagation generalized extrinsic information transfer (EBP-GEXIT) function, first proposed by Measson et al. The approximation allows to find a MAP threshold in such numerically involved cases as the binary-input additive white Gaussian noise (AWGN) channel, graph ensembles with general component codes and/or irregularities. The paper contains examples of estimations of the MAP thresholds in the case of irregular low-density parity-check (LDPC), generalized LDPC, and doubly generalized LDPC codes ensembles. Our estimations are confirmed by numerical simulations.

I. INTRODUCTION

The maximum a posteriori (MAP) threshold of a given channel code ensemble is defined as the minimum value of the channel parameter for which the average conditional entropy of a transmitted codeword is bounded away from zero [1, Sec. 4.7]. The average is taken over all possible randomly chosen codes in the given code ensemble as the length of the code goes to infinity. For the capacity achieving code ensemble, its MAP threshold reaches the fundamental limit on the channel parameter given by Shannon’s channel coding theorem. It is known that performing the MAP decoding over any binary memoryless channel (BMS) is in general computationally intractable and this makes finding the MAP threshold a difficult problem [1]. Finding the MAP threshold is thus an important yet difficult problem in the field of channel coding theory.

Méasson et al. have proposed a method to find the MAP threshold of a given code ensemble [2], [3]. In this method, the MAP threshold is obtained by applying the Maxwell construction to the EBP-GEXIT chart of a given code ensemble [2]. For the binary erasure channel, when the close form expression for the density evolution equation is known, the EBP-GEXIT chart can be obtained analytically [1]. However for any other BMS channel, obtaining this EBP-GEXIT chart is in general difficult. In this case, Méasson et al. have proposed a numerical method to find the EBP-GEXIT chart, using which one can find an estimate of the MAP threshold [3, Sec. VIII].

In this paper, we consider the binary input additive white Gaussian noise (AWGN) channel for which we provide a simple numerical approximation method to obtain the EBP-GEXIT chart of the given code ensemble. We assume that the distribution of the messages exchanged during the belief propagation (BP) decoding is consistent Gaussian [4, Ch. 9]. Note that this Gaussian assumption was initially proposed by Chung et al. [5]. This Gaussian assumption simplifies the operations performed at the check and variable nodes and enables us to find the EBP-GEXIT chart in a computationally feasible manner. We next summarize the main contributions of this paper.

(1) We propose a simple numerical method to obtain the EBP-GEXIT chart of the given code ensemble that makes use of the above mentioned Gaussian assumption.

(2) We compare the EBP-GEXIT charts obtained by our method with the method given by Méasson’s et al. [3, Sec. VIII]. We observe that the EBP-GEXIT charts obtained by both the methods with Gaussian approximation are the same (see Fig. 1).

(3) Finally, we obtain the EBP-GEXIT chart of several generalized LDPC and doubly generalized LDPC codes and estimate their MAP thresholds.

The paper is organized as follows. We first recall some preliminaries related to the EBP-GEXIT chart in Section II. In Section III, we provide a numerical method to find the EBP-GEXIT chart of a given code ensemble. In Section IV section, we provide EBP-GEXIT chart of several generalized and doubly generalized LDPC codes and estimate their MAP-threshold. Finally, we discuss some future directions and conclude in Section V.

II. PRELIMINARIES AND NOTATIONS

Let $\lambda(x) = \sum \lambda_i x^{i-1}$ and $\rho(x) = \sum \rho_j x^{j-1}$ be the degree distribution pair of an LDPC code ensemble from the edge perspective and let $\Lambda(x)$ and $P(x)$ be the corresponding normalized degree distribution pair from the node perspective. For irregular LDPC codes, all constraint nodes correspond to single parity check codes and variable nodes correspond to repetition codes [4]. When some of the check nodes correspond to any other linear block code, the LDPC code is referred to as generalized LDPC (GLDPC) code and when both variable and check nodes correspond to any general linear block code, the code is referred to as
doubly generalized LDPC (DGLDPC) code [4]. We assume that variable nodes are unpunctured and have degree greater or equal to 2.

In this paper, we consider the transmission over a binary input AWGN channel where coded bits are modulated according to a binary phase shift keying and the additive noise is of zero mean and variance \( \sigma^2 \). The family of AWGN channels parameterized by \( \sigma \) will be denoted by \( \{\text{AWGN}(\sigma)\} \). Let \( X \) and \( Y \) be the channel input alphabet and output alphabets respectively. For the given \( \text{AWGN}(\sigma) \), the distribution of log likelihood ratios \( L := \log \frac{p(Y|X=x)}{p(Y|X=-1)} \) under the condition \( X = +1 \) is referred to as \( L \)-density and is denoted by \( c_\sigma \) [3, Sec. II]. The entropy \( H(c_\sigma) \) of \( \text{AWGN}(\sigma) \) is then defined as

\[
H(c_\sigma) = \int_{-\infty}^{\infty} c_\sigma(z) \log_2(1 + e^{-z})dz.
\]

It can be seen that \( H(c_\sigma) \in [0, 1] \). If \( H(c_\sigma) = h \), \( \text{AWGN}(\sigma) \) can be equivalently described by its entropy \( h \). Since \( h \) and \( \sigma \) can be obtained from one another, we can parametrize the family of AWGN channel either by \( h \) or \( \sigma \). In the remaining paper, we use \( c_\sigma \) and \( c_h \) to represent the same \( L \)-density if \( H(c_\sigma) = h \) and the corresponding AWGN channel is represented either by \( \text{AWGN}(\sigma) \) or by \( \text{AWGN}(h) \).

Let \( f_C \) and \( f_V \) be the functions corresponding to the operations performed at the check and variable nodes respectively while performing BP decoding. When the all-one codeword is transmitted, let \( a^{BP,l} \) be the density of the message transmitted by any randomly chosen variable node to check node in the \( l \)-th iteration of BP decoding. For the first iteration, \( a^{BP,0} \) is initialized to \( c_h \). For \( l \geq 1 \), \( a^{BP,l} \) can be obtained from \( a^{BP,l-1} \) as follows

\[
a^{BP,l} = c_h \ast f_V(f_C(a^{BP,l-1})),
\]

where the operator \( \ast \) is the convolution operator associated to the operations performed in one iteration of the BP decoding (for details refer [1, Sec. 4.1.4]). For an irregular LDPC code, \( f_C(\cdot) = \rho(\cdot) \) and \( f_V = \lambda(\cdot) \) [1, Theorem 4.97]. For GLDPC codes and DGLDPC codes, the operations performed at check and variable node are more complex.

We now recall the definition the EBP-GEXIT chart [3, Sec. VII-A]. Let us first define a complete fixed-point family. The family of densities \( \{a_x\}_x \) and \( \{c_x\}_x \) parameterized by \( x \in [0, 1] \) is called a complete fixed-point family if the following conditions are satisfied.

1. \( c_x \in \{\text{AWGN}(h)\}_h \) for some \( h \in [0, 1] \).
2. For any \( x \in [0, 1] \) we have \( a_x = c_x \ast f_V(f_C(a_x)) \), i.e., \( a_x \) is a fixed point density with respect to \( c_x \).
3. \( H(a_x) = x \).
4. \( \{a_x\}_x \) and \( \{c_x\}_x \) are smooth with respect to \( x \).

The EBP-GEXIT function \( g^{EBP}(x) \) for an LDPC code ensemble with degree distribution pair \( (\lambda, \rho) \) is then defined as [3, Sec. VII-A]

\[
g^{EBP}(x) := \int_{-\infty}^{\infty} \Lambda(f_C(a_x))(z)l(c_x(z))dz,
\]

where \( l(c_x(z)) \) is defined as follows [3, Example 7]

\[
l(c_x(z)) = \left( \int_{-\infty}^{\infty} e^{-\frac{(w-2z)^2}{4\sigma^2}}dw \right) / \left( \int_{-\infty}^{\infty} e^{-\frac{(w-1)^2}{4\sigma^2}}dw \right)
\]

The EBP-GEXIT chart is the curve obtained by plotting \( g^{EBP}(x) \) versus \( c_x \) for all possible values of \( x \in [0, 1] \).

### III. EBP-GEXIT Chart Over AWGN

In this section, we propose a simple numerical method to find the EBP-GEXIT chart of a given LDPC code ensemble. As explained in the previous section, in order to plot the EBP-GEXIT chart, we need to first find a fixed point density \( a \) for a given channel \( c_h \), i.e., we need to find a pair of densities \( a \) and \( c_h \) that satisfy the following equation

\[
a = c_h \ast f_V(f_C(a)).
\]

Note that the density \( a \) corresponds to a message transmitted by a variable node to a check node in the BP decoding. We assume that the distribution of \( a \) is a consistent normal distribution, i.e., for some real number \( m_a \), distribution of \( a \) is normal with mean \( m_a \) and variance \( 2m_a \), denoted by \( N(m_a, 2m_a) \). This Gaussian assumption is proposed by Chung et al. [5] and also used for classical EXIT charts analysis [6]. We shall next explain how this Gaussian assumption simplifies the operations required to find EBP-GEXIT curve.

First, we explain how \( c_h \ast f_V(f_C(a)) \) in (5) can be efficiently approximated using a classical approximation by an EXIT-like monodimensional fixed point equation. For a given density \( a = N(m_a, 2m_a) \), consider the function \( J(m_a) \) defined as follows

\[
J(m_a) := 1 - E_a[\log_2(1 + e^{-y})],
\]

where \( E_a \) denotes expectation with respect to \( a \). Approximate values of the functions \( J(\cdot) \) and \( J^{-1}(\cdot) \) can be deduced from [6]. Further, the function \( J(\cdot) \) is a one-to-one function and this implies that the density \( a \) can be uniquely determined from it. Using EXIT based monodimensional representation, the fixed point equation (5) can be equivalently stated using some abuse of notations as follows

\[
J(m_a) = c_h \oplus f_V(f_C(J(m_a))),
\]

where the operator \( \oplus \) corresponds to the change of operation occurred due to change from density \( a \) to \( J(m_a) \). For the irregular LDPC codes, the operations \( f_C(\cdot) \) and \( f_V(\cdot) \) can be simplified as follows [4]

\[
f_C(J(m_a)) = \sum_j \rho(j - 1)J^{-1}(1 - J(m_a)),
\]

\[
c_h \oplus f_V(f_C(m_a)) = \sum_i \lambda_i J^{-1}[(i - 1)J^{-1}[f_C(m_a)] + \frac{2}{\sigma^2}]
\]
where \(2/\sigma^2\) is the mean of the \(L\)-density \(c_h\). Note that, (7) can now be efficiently computed using (8). For GLDPC and DLDPC codes, the functions \(f_c(\cdot)\) and \(f_v(\cdot)\) are evaluated point-wise by means of Monte Carlo simulations (details are given in [7] and [8]).

We now explain how the EBP-GEXIT function can be computed efficiently. To this end, we first need to derive \(\Lambda(f_c(a))\) (see equation (3)) under our Gaussian assumption. Let \(b\) denote the density of the messages coming from the check nodes. Suppose this density is consistent Gaussian with mean \(m_b\). For a variable node of degree \(j\), the density obtained by taking the convolution of the input density \(j\) times is the consistent Gaussian density of mean \(jm_b\). Let us denote this density by \(b_j\). The density \(\Lambda(b)\) is thus the mixture of densities \(b_j\) given by

\[
\Lambda(b)(z) = \sum_j \Lambda_j b_j(z),
\]

where \(b_j(z)\) is given by,

\[
b_j(z) = \frac{1}{\sqrt{4\pi jm_b}} \exp\left(-\frac{z-jm_b}{4jm_b}\right).
\]

Substituting (9) in (3) we get,

\[
g^{EBP} = \int_{-\infty}^{\infty} \left[ \sum_j \Lambda_j b_j(z) \right] l(c_h(z)) dz = \sum_j \Lambda_j \int_{-\infty}^{\infty} b_j(z) l(c_h(z)) dz = \sum_j \Lambda_j E_{b_j}[l(c_h(z))].
\]

Remark 1. On contrary to the definition of complete fixed-point family, \(a\) and \(c_h\) pairs obtained using Algorithm 1 are not parameterized by some \(x \in [0, 1]\), since we find these pairs exhaustively. However it can be easily verified that \(H(a) = x\) for some \(x \in [0, 1]\) and the set of \(a\) and \(c_h\) obtained do form a complete fixed-point family.

### IV. Obtained Results

In this section, we first compare the EBP-GEXIT chart obtained using the proposed method with the method of [3] for various irregular LDPC codes. We then obtain EBP-GEXIT chart for various GLDPC and DGLDPC codes using the proposed algorithm and estimate their corresponding BP and MAP thresholds. An estimate of the MAP threshold is obtained by applying Maxwell’s construction to each EBP-GEXIT chart (the details about Maxwell’s construction can be found in [1, Sec. 3.20]).

A. EBP-GEXIT chart for irregular LDPC codes

Méasson et al. have proposed a numerical method to find the EBP-GEXIT chart for a given LDPC code ensemble [3, Sec. VIII]. In Figures 1 to 4 we plot the EBP-GEXIT charts obtained by Méasson’s method and our method for various regular and irregular LDPC codes. For plotting the EBP-GEXIT chart using Méasson’s method also we consider the Gaussian assumption explained in first paragraph of this section. It can be seen that the EBP-GEXIT charts obtained by both the methods are the same.

B. EBP-GEXIT chart for GLDPC and DGLDPC codes

In this section, several examples of GLDPC and DGLDPC codes are considered and their BP and MAP thresholds are estimated. To illustrate our approach, let us estimate BP and MAP thresholds of the following code examples:

- \(C_1\): (2, 7)-regular ensemble of design rate 1/7 based on the Hamming(7, 4) component code
- \(C_2\): (2, 15)-regular ensemble of design rate 7/15 based on the Hamming(15, 11) component code designed in [7]
• $C_3$: DGLDPC ensemble of rate 3/4 from [8]
• $C_4$: DGLDPC ensemble of rate 7/15 from [11]
• $C_5$: 3-regular GLDPC ensemble of rate 1/2 based on a TLDPC component code [12]
• $C_6$: irregular GLDPC ensemble of rate 1/2 from with a TLDPC component code [12]

The ensembles $C_3$ and $C_4$ have the following structure. Let the generator matrices $G_1$, $G_2$ and $G_3$ be $G_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$, $G_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

Then both $C_3$ and $C_4$, have variable nodes of constant degree 6. The variable nodes for $C_3$ correspond to repetition codes of length 6 (69% of all nodes), linear codes defined by $G_2$ (1%), linear codes defined $G_3$ (22%) and single parity check codes of length 6 denoted by SPC(6) (8%). On the check node side, nodes correspond to SPC(12). The variable nodes for $C_4$ correspond to repetition codes of length 6 (42.5% of all nodes), codes defined by $G_1$ (7.5%), codes defined by $G_3$ (7.5%), and SPC(6) (42.5%). Component codes for $C_4$ are Hamming (15, 11) codes. Codes $C_5$ and $C_6$ are GLDPC ensembles belonging to the class of TLDPC codes of type B, designed in [12]. $C_5$ is a 3-regular code, while $C_6$ has the following degree distribution of variable nodes $\lambda(x) = 0.5x + 0.182x^2 + 0.069x^3 + 0.249x^4$ which has been optimized in [12] to improve the BP threshold.

EBP-GEXIT charts of codes from $C_1$ to $C_6$ are given in Fig. 5, 6 and 7. Based on these curves, BP and MAP thresholds of the ensembles have been estimated (dashed lines in figures), and the results are reported to Table I.

Finally, in order to show the validity of our estimations, let us compare them with numerical simulations. Fig. 8 shows the bit error rates over the AWGN for spatially-coupled versions of codes from $C_5$ with a spatial coupling parameter $w$. It is known that [13], [14], the spatially-couple ensemble has a BP threshold that approaches the MAP threshold with $w$ (and it equals to the (non-coupled) BP threshold for $w = 0$). Referring to Table I, the BP threshold for $C_5$ is 1.4264 dB ($h = 0.4035$) and the MAP threshold = 0.5484 dB ($h = 0.4719$). Referring to Fig.8, the thresholds of spatially-coupled ensembles with $w = 1$ and $w = 3$ are around 0.9 – 0.95 dB, this is consistent with Table I.

![Fig. 1. EBP-GEXIT chart of LDPC code ensemble with $\lambda(x) = x^2$ and $p(x) = x^3$ is illustrated for our method (left side) and for the method of [3] (right side).](image1)

![Fig. 2. EBP-GEXIT chart of LDPC code ensemble with $\lambda(x) = 2/5x + 3/5x^3$ and $p(x) = x^6$ is illustrated for our method (left side) and for the method of [3] (right side).](image2)

![Fig. 3. EBP-GEXIT chart of LDPC code ensemble with $\lambda(x) = 3x + 6x^2 + 11x^7$ and $p(x) = x^9$ is illustrated for our method (left side) and for the method of [3] (right side).](image3)

![Fig. 4. EBP-GEXIT chart of LDPC code ensemble with $\lambda(x) = 0.17120x + 0.21035x^2 + 0.00273x^3 + 0.00090x^6 + 0.15269x^7 + 0.099227x^8 + 0.02802x^9 + 0.01126x^{10} + 0.07212x^{29} + 0.2583x^{19}$ and $p(x) = 0.33620x^5 + 0.08883x^7 + 0.57497x^{10}$ is illustrated for our method (left side) and for the method of [3] (right side). This is a capacity achieving ensemble [10].](image4)

![Fig. 5. EBP-GEXIT chart for $C_1$ (left) and for $C_2$ (right).](image5)
This paper proposes a tractable and fast MAP threshold evaluation for graph ensembles, based on the Gaussian approximation. It works well for cases where using the method from [3] is too involved (e.g., over the AWGN channel), and numerical results are tight. Our method can be extended to ensembles with punctured bits and to ensembles having bits of degree 1. This will make object of our future work.

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### References


### Table I

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<tr>
<th>LDPC Ensemble</th>
<th>BP (literature)</th>
<th>Upper Bound on MAP</th>
<th>Our BP estimate</th>
<th>Our MAP Estimate</th>
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<td>GLDPC C₁ (Hamn(7))</td>
<td>0.756 [15]</td>
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**Fig. 6.** EBP-GEXIT charts for $C_3$ (left) and $C_4$ (right).

**Fig. 7.** EBP-GEXIT charts for $C_5$ (left) and $C_6$ (right).

**Fig. 8.** Bit error rate vs. SNR (dB) for $(n, L, w)$ spatially-coupled codes from the ensemble $C_{3}$ with total code length $nL = (3000, 6000, 12000)$, coupling parameter $w = (1, 3)$ and the two-side termination.