Table 1
Closed-form expression for $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}\right]$.

$\frac{\mathrm{Var}\left[\widehat{\alpha}_{qq}\right]}{(\log_2 \mathrm{e})^2}$	$2\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j,n_j'} \mathrm{r}_{qq}^2(j,k;j',k')$
$\frac{\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]}{(\log_2 \mathrm{e})^2}$	$\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j,n_j'} \frac{\mathrm{r}_{q_1q_2}(j,k;j',k')\mathrm{r}_{q_1q_2}(j',k';j,k)+\mathrm{r}_{q_1q_1}(j,k;j',k')\mathrm{r}_{q_2q_2}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_1q_2}(j',k';j',k')}$
$\frac{\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1},\widehat{\alpha}_{q_2q_2}\right]}{(\log_2 \mathrm{e})^2}$	$2\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j,n_j'} \frac{\mathrm{r}_{q_1q_2}^2(j,k;j',k')}{\mathrm{r}_{q_1q_1}(j,k;j,k)\mathrm{r}_{q_2q_2}(j',k';j',k')}$
$\frac{\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1},\widehat{\alpha}_{q_1q_2}\right]}{(\log_2 \mathrm{e})^2}$	$2\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j,n_j'} \frac{\mathrm{r}_{q_1q_1}(j,k;j',k')\mathrm{r}_{q_1q_2}(j,k;j',k')}{\mathrm{r}_{q_1q_1}(j,k;j,k)\mathrm{r}_{q_1q_2}(j',k';j',k')}$

where

$$n_j := \frac{n}{2^j}.$$

Note that the approximation (4.5) is a finite sample one, and ignores the potential asymptotic decorrelation effect stemming from the shifting scaling factor $a(n)$ (see Proposition 3.1, (v), and (4.3)). Consider the normalization

$$\mathrm{r}_{q_1q_2}(j,k;j',k') := \frac{\Phi_{q_1q_2}^{jj'}(2^j k - 2^{j'} k')}{\sqrt{\Phi_{q_1q_1}^{jj}(0)\Phi_{q_2q_2}^{j'j'}(0)}} \in [-1, 1]. \tag{4.6}$$

Expressions (3.21), (4.5) and (4.6) yield the covariance approximation

$$\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right] \approx (\log_2 \mathrm{e})^2 \sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k=0}^{n_j-1} \sum_{k'=0}^{n_{j'}-1} \frac{\mathrm{r}_{q_1q_3}(j,k;j',k')\mathrm{r}_{q_2q_4}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j',k';j',k')} + \frac{\mathrm{r}_{q_1q_4}(j,k;j',k')\mathrm{r}_{q_2q_3}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j',k';j',k')}. \tag{4.7}$$

In particular, (4.7) further allows us to compute

$$\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right] = \frac{1}{4}\left(\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right] + \mathrm{Var}\left[\widehat{\alpha}_{q_2q_2}\right]\right) + \mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right] + \frac{1}{2}\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right] - \mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}\right] - \mathrm{Cov}\left[\widehat{\alpha}_{q_2q_2}, \widehat{\alpha}_{q_1q_2}\right]. \tag{4.8}$$

The variances (4.8) will be used in Section 5 in the construction of a test for fractal connectivity, i.e., for the hypothesis $H_0 : \delta_{q_1q_2} \equiv 0$. Table 1 summarizes the closed-form approximations for $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]$, $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$, $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}\right]$ and $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right]$ established in (4.7).

4.2.2. Impact of inter- and intra-scale correlations

The expression of $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right]$ can further be split into three terms, namely,

$$\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right] \approx (\log_2 \mathrm{e})^2 \sum_{j=j_1}^{j_2} \left[\frac{w_j^2}{n_j}\left(1 + \frac{1}{\mathrm{r}_{q_1q_2}(j,0;j,0)\mathrm{r}_{q_3q_4}(j,0;j,0)}\right) + \frac{w_j^2}{n_j^2} \sum_k \sum_{k'\neq k} \frac{\mathrm{r}_{q_1q_3}(j,k;j,k')\mathrm{r}_{q_2q_4}(j,k;j,k') + \mathrm{r}_{q_1q_4}(j,k;j,k')\mathrm{r}_{q_2q_3}(j,k;j,k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j,k';j,k')} + \sum_{j'\neq j} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_k \sum_{k'} \frac{\mathrm{r}_{q_1q_3}(j,k;j',k')\mathrm{r}_{q_2q_4}(j,k;j',k') + \mathrm{r}_{q_1q_4}(j,k;j',k')\mathrm{r}_{q_2q_3}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j',k';j',k')} \right], \tag{4.9}$$

where the first term in the sum over j reflects the variance only of wavelet coefficients, the second term the covariance of wavelet coefficients at a given scale, and the third term, the covariance of wavelet coefficients at different scales. In other words, if wavelet coefficients were independent, the second and third terms would equal zero.

The relative contributions of the three terms to the final variances are quantified by means of Monte Carlo simulations conducted following the same protocol and settings as those described in Section 3.3.3. Table 2, reporting the relative contributions of each of the three terms for various sample sizes under fractal connectivity (i.e., $\delta_{q_1q_2} \equiv 0$), clearly shows that the second and third terms (intra- and inter-scale covariances) cannot be neglected, namely, one cannot use only the first term (variance) in the construction of confidence intervals for fBm, as proposed in [9]. Identical conclusions, not shown here, are drawn under departures from fractal connectivity (i.e., $\delta_{q_1q_2} > 0$).

4.2.3. First order approximations for the variances and covariances of $\widehat{\alpha}_{q_1q_2}$

It is of interest to further examine the leading order approximations for the variances and covariances of $\widehat{\alpha}_{q_1q_2}$ and $\delta_{q_1q_2}$, corresponding to neglecting all intra- and inter-scale correlations amongst wavelet coefficients. The first order approximations neglecting all inter- and intra- scale correlations amongst wavelet coefficients of $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right]$ and of $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ are summarized in Table 3.

The results show that $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$ and $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]$, $q_1 \neq q_2$, do not depend on the actual values of the scaling exponents ($\alpha_{q_1q_1}$, $\alpha_{q_2q_2}$, $\alpha_{q_1q_2}$), which corroborates the numerical performance reported in Section 3.3.4.

Table 2
Relative contributions of the three terms in (4.9) to $\mathrm{Var}\left[\widehat{\alpha}_{(\cdot)}\right]$, $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right]$ and $\mathrm{Var}\left[\widehat{\delta}_{(\cdot)}\right]$ for various sample sizes. ($(\alpha_{11}, \alpha_{22}, \delta_{12}, \rho) = (0.2, 0.6, 0, 0.9)$, $j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$).

n	2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}
	$\mathrm{Var}[\widehat{\alpha}_{11}]$				$\mathrm{Var}[\widehat{\alpha}_{22}]$				$\mathrm{Var}[\widehat{\alpha}_{12}]$				$\mathrm{Var}[\widehat{\delta}_{12}]$			
var$\times 10^3$	65.33	9.51	1.87	0.42	69.27	10.05	1.97	0.44	18.76	2.73	0.54	0.12	12.42	1.81	0.35	0.08
term 1	0.74	0.68	0.66	0.66	0.69	0.65	0.63	0.62	0.72	0.67	0.65	0.64	0.71	0.66	0.65	0.64
term 2	0.15	0.14	0.14	0.14	0.20	0.19	0.18	0.18	0.17	0.16	0.16	0.16	0.18	0.17	0.16	0.16
term 3	0.11	0.17	0.20	0.20	0.11	0.17	0.19	0.19	0.11	0.17	0.19	0.20	0.11	0.17	0.19	0.2
co $\times 10^3$	$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}]$				$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{12}]$				$\mathrm{Cov}[\widehat{\alpha}_{22}, \widehat{\alpha}_{12}]$							
	54.47	7.92	1.55	0.35	33.12	4.82	0.95	0.21	34.1	4.95	0.97	0.22				
term 1	0.72	0.66	0.65	0.64	0.73	0.67	0.66	0.65	0.71	0.66	0.64	0.63				
term 2	0.18	0.17	0.16	0.16	0.16	0.15	0.15	0.15	0.19	0.18	0.17	0.17				
term 3	0.11	0.17	0.19	0.20	0.11	0.17	0.19	0.20	0.11	0.17	0.19	0.2				

Table 3
First-order approximations in (4.7) for the variances and covariances of $\widehat{\alpha}_{q_1q_2}$ and $\delta_{q_1q_2}$ neglecting all inter- and intra-scale correlations amongst wavelet coefficients.

$\frac{\mathrm{Var}[\widehat{\alpha}_{qq}]}{(\log_2 e)^2}$	$2\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}$
$\frac{\mathrm{Var}[\widehat{\alpha}_{q_1q_2}]}{(\log_2 e)^2}$	$\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}(1 + \frac{1}{\mathrm{r}^2_{q_1q_2}(j,0;j,0)})$
$\frac{\mathrm{Cov}[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}]}{(\log_2 e)^2}$	$2\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j} \mathrm{r}^2_{q_1q_2}(j, 0; j, 0)$
$\frac{\mathrm{Cov}[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}]}{(\log_2 e)^2}$	$2\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}$
$\frac{\mathrm{Var}[\widehat{\delta}_{q_1q_2}]}{(\log_2 e)^2}$	$\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}\left((\mathrm{r}^2_{q_1q_2}(j, 0; j, 0) + \frac{1}{\mathrm{r}^2_{q_1q_2}(j,0;j,0)}) - 2\right)$

Note that for an ideal-HfBm with fractal connectivity, it is straightforward to show that $\mathrm{r}_{qq}(j, 0; j, 0) \equiv 1$ and $\mathrm{r}_{q_1q_2}(j, 0; j, 0) \equiv \rho_{q_1q_2}$ when $q_1 \neq q_2$. While $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$ does not depend on correlations $\rho_{q_1q_2}$, as expected, $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]$, $q_1 \neq q_2$ does vary with $\rho_{q_1q_2}$ according to $1/\rho^2_{q_1q_2}$, showing that $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right] \to +\infty$ when $\rho_{q_1q_2} \to 0$. This can be interpreted as the fact that when $\rho_{q_1q_2} \to 0$, the scaling exponent $\alpha_{q_1q_2}$, $q_1 \neq q_2$, loses its meaning. Furthermore, $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right]$ depends on ρ as $\rho^2_{q_1q_2}$, not surprisingly indicating that when $\rho_{q_1q_2} \to 0$ (no correlation amongst components), $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right] \to 0$ (no correlation amongst estimates).

Moreover, the first order approximation of $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ is observed not to depend on the actual value of $\delta_{q_1q_2}$, while $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ clearly depends on $\delta_{q_1q_2}$ based on the numerical simulations reported in Section 3.3.4. This can be interpreted as the fact that, for $\widehat{\delta}_{q_1q_2}$, the first order approximation (neglecting all intra- and inter-scale correlations amongst wavelet coefficients) is not sufficient to approximate well $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$, as opposed to what is observed for $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$ and $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]$, $q_1 \neq q_2$.

To finish with, $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ varies with $\rho_{q_1q_2}$ as $\rho^2_{q_1q_2} + 1/\rho^2_{q_1q_2} - 2$. This shows again that when $\rho_{q_1q_2} \to 0$, parameter $\delta_{q_1q_2}$ becomes irrelevant. Moreover, it also shows that when $\rho_{q_1q_2} \to \pm 1$, $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right] \to 0$. This can be understood as the fact that when $\rho_{q_1q_2} \to \pm 1$, a departure from fractal connectivity is no longer permitted, as indicated by (2.12). Thus, $\rho_{q_1q_2} \to \pm 1$ implies $\delta_{q_1q_2} \to 0$, which is then no longer a random variable.

4.3. Practical computation of the variances and covariances of $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\delta}_{q_1q_2}$

4.3.1. Computation of $\mathbb{E}\left[d_{q_1}\right](j, k)d_{q_2}(j', k')$ and $\mathrm{r}_{q_1q_2}(j, k; j', k')$

Evaluation of (4.7) requires knowledge of the covariance between wavelet coefficients (4.6), which will be developed here explicitly for the discrete and the dyadic wavelet transforms. Let $h(k)$ and $g(k)$, $k = 1, \ldots, L$, be the coefficients of the high pass and low pass filters of the discrete wavelet transform, respectively, and let $\uparrow_2[\cdot]$ and $\downarrow_2[\cdot]$ be the dyadic upsampling and decimation operators. For any $q = 1, \ldots, m$, the wavelet transform of the discrete time process component $B_H(k)_q$ yields, at each scale $j = 1, \ldots, J$, sequences of approximation coefficients $a_q(j, k)$ and detail coefficients $d_q(j, k)$, $q = 1, \ldots, m$. The corresponding dyadic coefficients are given by $\tilde{a}_q(j, k) = a_q(j, 2^j k)$ and $\tilde{d}_q(j, k) = d_q(j, 2^j k)$, respectively. Pick the initialization $a_q(0, k) = B_H(k)_q$ (see [55] for a discussion of the initialization of the discrete wavelet transform). At scale $j = 1$, $a_q(1, \cdot) = h*a_q(0, \cdot)$ and $d_q(1, \cdot) = g*a_q(0, \cdot)$, at scale $j = 2$, $a_q(2, \cdot) = \uparrow_2[h]*a_q(1, \cdot) = \uparrow_2[h]*h*a_q(0, \cdot)$ and $d_q(2, \cdot) = \uparrow_2[g] * a_q(1, \cdot) = \uparrow_2[g] * h * a_q(0, \cdot)$. By iteration, we obtain the sequences of detail coefficients at each scale $j = j'$, i.e.,

$$d_q(j', k) = \left(g_{j'} * a_q(0, \cdot)\right)(k), \quad \tilde{d}_q(j', k) = d_q(j', 2^{j'} k), \tag{4.10}$$

where

$$g_{j'} = \uparrow_{2^{j'-1}}[g] * \left(\mathop{*}_{j=0}^{j'-2} \uparrow_{2^{j'}}[h]\right).$$

Now let $\gamma_{h_{q_1q_2}}(s, t)$, $1 \leq q_1 \leq q_2 \leq m$, be (univariate) fBm covariance functions with indices $h_{q_1q_2} = \alpha_{q_1q_2}/2$, i.e.,

$$\gamma_{q_1q_2}(s, t) = \rho_{q_1q_2}\sigma_{q_1}\sigma_{q_2}\{|s|^{\alpha_{q_1q_2}} + |t|^{\alpha_{q_1q_2}} - |s - t|^{\alpha_{q_1q_2}}\}. \tag{4.11}$$

Then (cf. [18]; note that a change in time scale would only result in a multiplicative constant, which we assume to be absorbed in $\rho_{q_1q_2}$ and which cancels out in the final expression for $\mathrm{r}_{q_1q_2}$)

$$\begin{aligned}
&\mathbb{E}[d_{q_1}(j,k)d_{q_2}(j',k+\tau)]/(\sigma_{q_1}\sigma_{q_2})\\
&\quad= \sum_p\sum_q g_j(p)g_{j'}(q)\mathbb{E}[a_{q_1}(0,k-p)a_{q_2}(0,k+\tau-q)]\\
&\quad= -\rho_{q_1q_2}\sum_p\sum_q g_j(p)g_{j'}(q)|-\tau+q-p|^{\alpha_{q_1q_2}}\\
&\qquad+\rho_{q_1q_2}\sum_p g_j(p)\sum_q g_{j'}(q)|k-p|^{\alpha_{q_1q_2}}+\rho_{q_1q_2}\sum_q g_{j'}(q)\sum_p g_j(p)|k+\tau-q|^{\alpha_{q_1q_2}}\\
&\quad= -\rho_{q_1q_2}\sum_p\sum_q g_j(p)g_{j'}(q)|\tau-q+p|^{\alpha_{q_1q_2}} \qquad\qquad (p'=p-q)\\
&\quad= -\rho_{q_1q_2}\sum_{p'}\sum_q g_j(p'+q)g_{j'}(q)|\tau-p'|^{\alpha_{q_1q_2}} = -\rho_{q_1q_2}\sum_{p'}\sum_q g_j(p'-q)\check{g}_{j'}(q)|\tau-p'|^{\alpha_{q_1q_2}}\\
&\quad= -\rho_{q_1q_2}\sum_p (g_j*\check{g}_{j'})(p)|\tau-p|^{\alpha_{q_1q_2}} = -\rho_{q_1q_2}((g_j*\check{g}_{j'})*\eta_{q_1q_2})(\tau),
\end{aligned}$$

where

$$\begin{aligned}
\check{g}_j(k) &= g_j(L-k), \quad k=1,\ldots,L,\\
\eta_{q_1q_2}(\tau) &= |\tau|^{\alpha_{q_1q_2}}.
\end{aligned}$$

Consequently,

$$\mathrm{r}_{q_1q_2}(j,k;j',k') = \rho_{q_1q_2}\frac{((g_j*\check{g}_{j'})*\eta_{q_1q_2})(k'-k)}{\sqrt{((g_j*\check{g}_j)*\eta_{ii})(0)\,((g_{j'}*\check{g}_{j'})*\eta_{ll})(0)}} \tag{4.12}$$

and, for the dyadic wavelet transform,

$$\tilde{\mathrm{r}}_{q_1q_2}(j,k;j',k') = \mathrm{r}_{q_1q_2}(j,2^jk;j',2^{j'}k'). \tag{4.13}$$

For given values of $\alpha_{q_1q_2}$ and $\rho_{q_1q_2}$, these expressions can be easily evaluated numerically.

4.3.2. *Practical estimation of (co)variances of $\widehat{\alpha}_{q_1q_2}$ and of $\widehat{\delta}_{q_1q_2}$*

Evaluating (4.7) and (4.8) for HfBm in practice requires the unknown parameter values $\alpha_{q_1q_2}$ and $\rho_{q_1q_2}$ in (4.11), and hence we replace them by their estimates $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\rho}_{q_1q_2}$. The former are defined in (3.21), and estimates $\widehat{\rho}_{q_1q_2}$ for $\rho_{q_1q_2}$ for $q_1 \neq q_2$ can be readily obtained as the cross-correlation coefficients of the first difference processes $Y_H(t)_{q_1}$ and $Y_H(t)_{q_2}$ (HfGn; see Remark 2.2). However, note that the expressions for the (co)variances of $\widehat{\alpha}_{q_1q_2}$ in the previous sections are derived assuming knowledge of the true parameter values and can only be expected to be approximations when these are replaced by estimates. This will be studied numerically in the next section.

4.3.3. *Assessment of the estimated (co)variances of $\widehat{\alpha}_{q_1q_2}$ and of $\widehat{\delta}_{q_1q_2}$ by means of Monte Carlo experiments*

Monte Carlo studies were conducted following the protocol and settings described in Section 3.3.3, aiming to evaluate the quality of the estimated approximations (4.7) and (4.8) for the (co)variances of $\widehat{\alpha}_{q_1q_2}$ and of $\widehat{\gamma}_{q_1q_2}$. The simulations involved 1000 independent realizations of each of two general instances of HfBm with $m = 2$ components, one using the true values of the parameters $\alpha_{q_1q_2}$ and $\rho_{q_1q_2}$, and the other, their estimates $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\rho}_{q_1q_2}$. Four different sample sizes, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$ and three different values $\rho_{12} = \{0.3, 0.6, 0.9\}$ are investigated for the set of exponents $[\alpha_{11}, \alpha_{22}, \alpha_{12}] = [0.2, 0.6, 0.4]$.

Table 4 summarizes the square roots of the ratios of the averages over realizations of (co)variance estimates and of the Monte Carlo (co)variances. The first four columns, labeled "theo/MC", report results obtained when using theoretical parameter values and yield the following conclusions. First, even for small sample size $n = 2^{10}$ and weak correlation $\rho_{12} = 0.3$, the quality of the approximations (4.7) and (4.8) is very good for the variances of exponents α_q, $q = 1, 2$, and satisfactory for the cross-exponent α_{12}, the covariance parameters and the connectivity parameter δ_{12}. Second, when the sample size n and correlation level ρ_{12} increase, the approximation of variances and covariances becomes excellent, with maximum errors of the order of 5% for $n = 2^{16}$ and strong correlation $\rho_{12} = 0.9$. Finally, the last four columns of Table 4, labeled "est/MC", report results obtained when using estimates $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\rho}_{12}$. They indicate that replacing the true parameter values $\alpha_{q_1q_2}$ and ρ_{12} with estimates has very little impact on the quality of approximations (4.7) and (4.8). Indeed, the average values of the (co)variance estimates are essentially equal to those obtained when using true parameter values.

5. Statistical test for fractal connectivity

5.1. *Procedure*

The mathematical and computational results in Sections 3 and 4 enable us to construct component-wise fractal connectivity tests, i.e., for the hypotheses

$$H_0:\ \delta_{q_1q_2} = \frac{\alpha_{q_1q_1}+\alpha_{q_2q_2}}{2} - \alpha_{q_1q_2} = 0, \qquad q_1 \neq q_2.$$

Table 4
Estimation of $\mathrm{Var}\left[\alpha_{(\cdot)}\right]$. Square roots of ratios of mean of (co)variances computed using (4.7) and of Monte Carlo (co)variances: (4.7) evaluated using theoretical values $\alpha_{11}, \alpha_{22}, \alpha_{12}, \rho_{12}$ (left columns, labeled "theo/MC") and estimates $\widehat{\alpha}_{11}, \widehat{\alpha}_{22}, \widehat{\alpha}_{12}, \widehat{\rho}_{12}$ (right columns, labeled "est/MC"). ($(\alpha_{11}, \alpha_{22}, \delta_{12}, \rho_{12}) = (0.2, 0.6, 0, \rho_{12}), j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$).

n		2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}
ρ_{12}	Ratio of $\sqrt{-/-}$	theo/MC				est/MC			
0.3	$\mathrm{Var}[\widehat{\alpha}_{11}]$	0.95	0.97	0.95	0.99	0.94	0.97	0.94	0.99
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	0.82	0.85	0.90	0.93	0.83	0.86	0.90	0.93
	$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}]$	1.12	1.07	0.97	1.32	1.09	1.06	0.97	1.31
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	0.78	0.82	0.88	0.91	0.80	0.82	0.88	0.91
0.6	$\mathrm{Var}[\widehat{\alpha}_{11}]$	0.98	0.98	0.96	0.98	0.98	0.97	0.96	0.98
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	0.94	0.92	1.00	1.00	0.93	0.92	1.00	1.00
	$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}]$	1.00	0.93	0.97	1.05	0.99	0.92	0.97	1.05
	$\mathrm{Var}[\widehat{\delta}_{12}]$	0.88	0.87	0.98	0.97	0.88	0.87	0.98	0.97
0.9	$\mathrm{Var}[\widehat{\alpha}_{11}]$	0.97	0.95	0.98	0.98	0.96	0.95	0.98	0.98
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	1.00	0.98	1.00	1.01	0.99	0.98	1.00	1.01
	$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}]$	1.01	0.99	1.01	1.02	1.00	0.98	1.01	1.01
	$\mathrm{Var}[\widehat{\delta}_{12}]$	0.93	0.92	0.95	0.97	0.95	0.93	0.96	0.99

Recall that we assume throughout that $\rho_{q_1q_2} \neq 0$ for $q_1 \neq q_2$ (see (2.8)). As a consequence of Theorem 3.1, the distribution of $\widehat{\delta}_{q_1q_2}$ under H_0 can be approximated over finite samples by

$$\widehat{\delta}_{q_1q_2} \sim \mathcal{N}\left(0, \mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]\right) \text{ under } H_0,$$

where, in turn, $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ can be approximated by (4.8). Therefore, a simple two-sided test with significance level $\mathsf{s} = P(\text{reject } H_0 | H_0 \text{ true})$ is given by

$$d_{\mathsf{s}} = \begin{cases} 1, & \text{if } |\widehat{\delta}_{q_1q_2}| > \sqrt{\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]}\phi^{-1}(1 - \mathsf{s}/2); \\ 0, & \text{otherwise,} \end{cases} \tag{5.1}$$

where $\phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard Normal distribution. In addition, the p-value of the test statistic (i.e., the probability of observing an absolute value at least as large as $|\widehat{\delta}_{q_1q_2}|$ for the test statistic under H_0) is given by

$$p(|\widehat{\delta}_{q_1q_2}|) := 2\phi\left(-|\widehat{\delta}_{q_1q_2}| \Big/ \sqrt{\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]}\right). \tag{5.2}$$

This test can be performed by evaluating (5.1) with an estimate for $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ obtained from the procedure detailed in Section 4.1.

5.2. *Monte Carlo assessment of the test performance*

We assess the performance of the test by applying it to 1000 independent realizations of HfBm with exponent values $[\alpha_{11}, \alpha_{22}] = [0.2, 0.6]$ and exponent values α_{12} detailed below for sample sizes $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$ and correlation levels $\rho_{12} = \{0.5, 0.7, 0.9\}$. For simplicity of illustration and without loss of generality, we consider again HfBm with $m = 2$ components. For each realization, the test decision (5.1) and the p-value (5.2) are evaluated using (4.8) with approximations (4.7) to obtain an estimate of the $\mathrm{Var}\left[\widehat{\delta}_{12}\right]$. Estimates of the expected values of the test decisions and p-values, denoted by $\widehat{d}_{\mathsf{s}}$ and $\widehat{p}$, are then obtained as the averages over realizations of test decisions and p-values (5.1) and (5.2).

We now compare the performance of the proposed test, denoted hereinafter HFBM (not to be confused with the stochastic process HfBm), to that of the test put forward in [26] (cf., [56] for preliminary comparative results). The latter relies on the intuition that the wavelet coherence function of two components of a multivariate Gaussian scale invariant random process approximately behaves as

$$\Gamma_{q_1q_2}(j) = S_{n_j}^{(q_1q_2)}(j) \Big/ \sqrt{S_{n_j}^{(q_1q_1)}(j) S_{n_j}^{(q_2q_2)}(j)} \simeq \rho_{q_1q_2} 2^{j(\alpha_{q_1q_2} - \alpha_{q_1q_1} - \alpha_{q_2q_2})}.$$

The test itself, denoted WCF (for wavelet coherence function), is formulated without the rigorous statistical framework developed above. Rather, it is built on the observation that $\Gamma_{q_1q_2}(j)$ is the Pearson product-moment correlation coefficient of the time series $d_{q_1}(j, \cdot)$ and $d_{q_2}(j, \cdot)$, and hence that the Fisher's z statistics of $\Gamma_{q_1q_2}(j), j = j_1, \ldots, j_2$, are approximately Gaussian, with known variances and, in the case of fractal connectivity, with equal means across scales. The test for fractal connectivity is then formulated as the UMPI test for the equality of means of Gaussian random variables, cf. [26] for details.

5.2.1. *Performance under H_0*

We first consider the case that H_0 is true, i.e., $(\alpha_{11} + \alpha_{22})/2 = \alpha_{12} = 0.4$. The significance level is set to $\mathsf{s} = 0.1$, and results are reported in Table 5 for the proposed test (top) and for the test in [26] (bottom). Note that, under H_0, averages of test decisions $\widehat{d}_{\mathsf{s}}$ should equal the preset significance level s, and averages of p-values $\widehat{p}$ should equal $\frac{1}{2}$. HFBM rejects H_0 with slightly larger probability than the prescribed value $\mathsf{s} = 0.1$, yet the differences between empirical significance levels $\widehat{d}_{\mathsf{s}}$ and s never exceed 5%; similarly, average p-values are slightly below 0.5. For large sample size and large ρ_{12}, average test decisions and p-values are very close to the theoretical values $\mathsf{s} = 0.1$ and $p = \frac{1}{2}$. These remarks are consistent with the results reported in Table 4, where a small but systematic underestimation of $\mathrm{Var}\left[\widehat{\delta}_{12}\right]$ for small sample sizes and ρ_{12} is observed.

In contrast, the empirical significances $\widehat{d}_{\mathsf{s}}$ of WCF strongly differ from the preset value s by values of up to 16%, and this difference is especially pronounced for large sample sizes for which one would expect the test to perform better. One reason for this poor performance

Table 5
Test significance. Mean test decisions and p-values for different values of n and ρ_{12} under $H_0: \delta_{12} = (\alpha_{11} + \alpha_{22})/2 - \alpha_{12} = 0$ ($[\alpha_{11}, \alpha_{22}] = [0.2, 0.6]$, $j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$) for the proposed test (top) and for the test in [26] (bottom).

HFBM – $H_0: \delta_{12} \equiv 0$, s = 0.1						
	$\rho_{12} = 0.5$		$\rho_{12} = 0.7$		$\rho_{12} = 0.9$	
	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$
$n = 2^{10}$	13.2	0.45	14.0	0.48	14.5	0.45
$n = 2^{12}$	13.8	0.45	10.9	0.47	11.1	0.45
$n = 2^{14}$	13.8	0.45	11.2	0.46	11.0	0.46
$n = 2^{16}$	12.3	0.46	10.9	0.48	11.0	0.47
WCF – $H_0: \delta_{12} \equiv 0$, s = 0.1						
	$\rho_{12} = 0.5$		$\rho_{12} = 0.7$		$\rho_{12} = 0.9$	
	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$
$n = 2^{10}$	14.7	0.44	15.3	0.46	16.0	0.44
$n = 2^{12}$	20.7	0.42	16.0	0.43	17.2	0.42
$n = 2^{14}$	20.9	0.40	18.6	0.41	22.0	0.39
$n = 2^{16}$	26.1	0.36	22.5	0.36	22.1	0.38

Table 6
Test power for adjusted significance $\widehat{d}_s = 0.1$. Mean test decisions and p-values for different values of n, ρ_{12} and α_{12}/alternative hypotheses $H_1: \delta_{12} = (\alpha_{11}+\alpha_{22})/2-\alpha_{12} \neq 0$ ($[\alpha_{11}, \alpha_{22}] = [0.2, 0.6]$, $j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$) for the proposed test (top) and for the test in [26] (bottom).

HFBM – $H_1: \delta_{12} \neq 0, \widehat{d}_s = 0.1$												
δ_{12}	$\rho_{12} = 0.5$				$\rho_{12} = 0.7$				$\rho_{12} = 0.9$			
	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2
$n = 2^{10}$	16.5	29.9	40.6	50.0	25.4	43.5	66.1	84.5	74.8	98.6	100.0	99.9
$n = 2^{12}$	28.8	60.8	81.9	93.3	63.6	96.8	99.8	100.0	100.0	100.0	100.0	100.0
$n = 2^{14}$	67.1	97.9	99.9	100.0	99.2	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$n = 2^{16}$	99.4	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
WCF – $H_1: \delta_{12} \neq 0, \widehat{d}_s = 0.1$												
δ_{12}	$\rho_{12} = 0.5$				$\rho_{12} = 0.7$				$\rho_{12} = 0.9$			
	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2
$n = 2^{10}$	10.5	15.4	16.6	22.3	14.1	21.7	36.9	52.9	43.7	81.0	94.2	99.2
$n = 2^{12}$	17.8	35.9	52.5	69.4	34.7	78.9	95.8	99.4	98.2	100.0	100.0	100.0
$n = 2^{14}$	43.4	86.2	99.0	100.0	89.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$n = 2^{16}$	93.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

may lie in the fact that the test in [26] was designed for fGn, rather than fBm. Note that the asymptotic calculations developed above can be adapted to the easier case of fGn (and, in principle, any other Gaussian process with stationary increments) without difficulty by simply changing the covariance function $\gamma_{q_1q_2}(s, t)$ in the calculations leading to the expressions (4.12) and (4.13).

5.2.2. Test power

We assess the power of the test under the alternative hypotheses $H_1: \delta_{12} = (\alpha_{11}+\alpha_{22})/2-\alpha_{12} \neq 0$ with $\delta_{12} = \{0.05, 0.1, 0.15, 0.2\}$. Yet, a direct power comparison of HFBM and WCF is only meaningful for identical rejection probabilities under H_0, since a test for which $\widehat{d}_s > s$ under H_0 is expected to display an artificially large power. In view of the distinct performances of HFBM and WCF, as discussed above (cf. Table 5), for each sample size and correlation level we adjusted the prescribed significance to the value $\tilde{s}$ for which the average rejection rate under H_0 equals $\widehat{d}_{\tilde{s}} = s = 0.1$. Using this adjusted level of significance $\tilde{s}$, the power of the test is then estimated as the average of the test decisions $d_{\tilde{s}}$ when H_1 is true. Results are reported in Table 6 and yield the following conclusions. First, the power of each test systematically increases with the magnitudes of the deviation from $\delta_{12} = 0$, of the correlation level ρ_{12} and of the sample size n, as expected. Second, HFBM is systematically and significantly more powerful. Indeed, it enables us to detect a non-zero value for δ_{12} up to two times as often as WCF. For instance, for the small sample size of $n = 2^{10}$ and the low correlation level of $\rho_{12} = 0.5$, it permits the detection of a deviation of 0.2 from the null value $\delta_{12} = 0$ with probability 0.5, as compared to a probability of 0.22 for the test in [26].

Overall, these results confirm that the proposed methods can be relevantly applied in the assessment of scaling and fractal connectivity in multivariate time series.

6. Conclusion

The present contribution introduces a versatile class of multivariate stochastic processes called Hadamard fractional Brownian motion (HfBm). HfBm provides a stochastic framework for scale invariance modeling within which cross-component scaling laws are not directly controlled by the scaling laws along the main diagonal. In other words, HfBm is not necessarily fractally connected.

Interestingly, the theoretical study of HfBm reveals that exact entry-wise scaling on both auto- and cross-components and departures from fractal connectivity are mathematically incompatible. In other words, there is a dichotomy in multivariate scaling modeling: either there is exact entry-wise scaling in every component combined with fractal connectivity, or departures from fractal connectivity are allowed at the price of approximate (i.e., asymptotic) scaling on the cross-components.

Our main mathematical results consist of an asymptotically normal, wavelet-based linear regression estimator for the scaling exponents, as well as asymptotically valid confidence intervals with convenient mathematical expressions. Furthermore, the Taylor expansions used in the development of the asymptotic confidence intervals lead to the construction of practical procedures for the numerical calculation of the variance of the estimates. These approximate calculations enable the study of the ubiquitous issue of the impact of neglecting intra- or inter-scale correlations amongst wavelet coefficients in the computations of variances and covariances for the estimates. We also devised an asymptotically normal hypothesis test for fractal connectivity. Again, a major feature of the designed test procedure is the fact that it can be applied to a single observed HfBm data path.

For both fractally and non-fractally connected instances, simulations demonstrate the satisfactory performance of the estimators of the scaling and fractal connectivity parameters, even for small sample size data. The estimation bias is shown to be negligible, and the variance decreases according to the inverse of the sample size. In addition, the practical computations of approximated variances and covariances of the estimates are shown to be of excellent quality, irrespective of sample size, and the Monte Carlo significance levels and powers are very close to their theoretical counterparts.

The tools developed in the present contribution pave the way for novel analysis and modeling perspectives on multivariate scaling in real-world data, in the spirit of [6]. Routines for the synthesis of HfBm, as well as for estimation, computation of confidence intervals and fractal connectivity testing will be made publicly available at the time of publication.

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Appendix. Proofs

This appendix comprises three parts, Appendix A, Appendix B and Appendix C, which contain the proofs for Sections 3.2, 3.3 and 4.1, respectively. In Appendix B and Appendix C, we assume throughout that the assumptions of Theorems 3.1 and 4.1, respectively, hold.

In the proofs, whenever convenient we will use the shorthand

$$a = a(n). \tag{.1}$$

For notational simplicity, we will assume throughout that $\sigma_q = 1, q = 1, \dots, m$. Since the main diagonal entries of an HfBm behave like a perturbed (univariate) fBm, throughout the appendix we only provide proofs for cross-components, i.e., when the indices q_l are pairwise distinct, $l = 1, 2, 3, 4$. Whenever convenient we will use $l = 1, 2, 3, 4$ in place of q_l, respectively, and also write

$$h_{q_l q_l} = h_l, \quad h_{q_l q_p} = h_{lp}.$$

In addition, without loss of generality it will be assumed that

$$\varrho^{(12)}(a2^j) > 0, \quad n \in \mathbb{N}, \quad j \in \mathbb{N} \tag{.2}$$

(see (3.8)), since otherwise we can consider $-\varrho^{(12)}(a2^j)$ instead (see also (3.13) and Remark C.1). We will also write log instead of $\log_2$, for visual clarity. In the proofs, C represents a generic constant whose value may change from one line to the next.

Appendix A. Section 3.2

Proof of Proposition 3.1. For simplicity, we will write $\Xi^{jj'}(\cdot)$ instead of $\Xi^{jj',a}(\cdot)$ throughout the proof.

To show (i), the change of variable $s = 2^{-j}t - k$ in (3.5) and the harmonizable representation of HfBm yield

$$\Xi^{jj'}(a(2^j k - 2^{j'} k')) = \int_{\mathbb{R}} e^{ia(2^j k - 2^{j'} k')x} f(x) \widehat{\psi}(a2^j x) \overline{\widehat{\psi}(a2^{j'} x)} dx, \tag{A.1}$$

where $f(x) := \Big(|x|^{-2(h_{q_1 q_2}+1/2)} g_{q_1 q_2}(x)\Big)_{q_1,q_2=1,\dots,m}$. Statement (ii) is a consequence of the formula (A.1) with

$$k = k' = 0, \quad j = j'. \tag{A.2}$$

Statement (iii) follows from Lemma C.1(ii), below under condition (3.6).

Turning to (iv), establishing (3.16) is equivalent to showing that

$$\frac{2^{-(j+j')/2}}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^{j} n_*} \Big(\frac{\Xi^{jj'}_{12}(a(2^j k - 2^{j'} k'))}{a^{2h_{12}}} - \Phi^{jj'}_{12}(2^j k - 2^{j'} k') \Big)$$

$$+ \frac{2^{-(j+j')/2}}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^{j} n_*} \Phi^{jj'}_{12}(2^j k - 2^{j'} k') \to 2^{-(j+j')/2} \mathrm{gcd}(2^j, 2^{j'}) \sum_{z=-\infty}^{\infty} \Phi^{jj'}_{12}(z\,\mathrm{gcd}(2^j, 2^{j'})), \tag{A.3}$$

$n \to \infty$.

Consider the entry $q_1 = 1, q_2 = 2$ of the matrix-valued function $\Xi^{jj'}$ as in (A.1). Based on a change of variable $y = ax$, recast the first term of the sum on the left-hand side of (A.3) as

$$\int_{\mathbb{R}} e^{iyz} |y|^{-(2h_{12}+1)} \rho_{12} \Big\{ g_{12}\Big(\frac{y}{a}\Big) - 1 \Big\} \widehat{\psi}(2^j y) \overline{\widehat{\psi}(2^{j'} y)} dy =: \int_{\mathbb{R}} e^{iyz} h_a(y) dy. \tag{A.4}$$

Fix $A > 0$ and denote $\widehat{\psi}^{(l)}(x) = \frac{d^l}{dx^l}(\Re\widehat{\psi}(x) + i\Im\widehat{\psi}(x))$, $l \geq 0$. By (3.1) and (3.4) and a Taylor expansion with Lagrange residual of the real and imaginary parts of $\widehat{\psi}$, there exist functions λ_1, λ_2 on $[-A, A]$ such that

$$\widehat{\psi}(x) = \Big(\frac{d^{N_\psi}}{dx^{N_\psi}}\Re\widehat{\psi}(x)\Big|_{\lambda_1(x)} + i\frac{d^{N_\psi}}{dx^{N_\psi}}\Im\widehat{\psi}(x)\Big|_{\lambda_2(x)}\Big)\frac{x^{N_\psi}}{N_\psi!}.$$

Therefore, and extending this reasoning to $\widehat{\psi}'(x)$, $\widehat{\psi}''(x)$,

$$|\widehat{\psi}^{(l)}(x)| = O(|x|^{N_\psi - l}), \quad x \to 0, \quad l = 0, 1, 2. \tag{A.5}$$

For h_a as in (A.4), we now show that

$$h_a, h_a', h_a'' \text{ are differentiable and } h_a(0) = h_a'(0) = h_a''(0) = 0. \tag{A.6}$$

We will only develop expressions for $y > 0$, since analogous developments hold for $y < 0$. For mathematical convenience, rewrite h_a as in (A.4) as

$$h_a(y) = y^{-(2h_{12}+1)}\rho_{12}\,\vartheta_a(y). \tag{A.7}$$

Hence,

$$h_a'(y) = \rho_{12}\{-(2h_{12}+1)y^{-(2h_{12}+2)}\vartheta_a(y) + y^{-(2h_{12}+1)}\vartheta_a'(y)\}, \tag{A.8}$$

$$h_a''(y) = \rho_{12}\{(2h_{12}+1)(2h_{12}+2)y^{-(2h_{12}+3)}\vartheta_a(y) - 2(2h_{12}+1)y^{-(2h_{12}+2)}\vartheta_a'(y) + y^{-(2h_{12}+1)}\vartheta_a''(y)\}. \tag{A.9}$$

Note that, by conditions (3.1) and (2.7),

$$2N_\psi + \varpi_0 - (2h_{12} + 1 + l) - 1 \geq 2 + \varpi_0 - 2h_{\max} - l > 0, \quad l = 0, 1, 2. \tag{A.10}$$

Then, h_a is also smooth around zero and $h_a(0) = 0$. Moreover, by (A.5) and (A.10) with $l = 0$, for fixed n and $|y| \leq a\varepsilon_0$,

$$\frac{|h_a(y)|}{y} \leq \frac{C}{y}y^{-(2h_{12}+1)}\Big(\frac{y}{a}\Big)^{\varpi_0}y^{2N_\psi} = \frac{C}{a^{\varpi_0}}y^{2N_\psi+\varpi_0-(2h_{12}+2)} \to 0, \tag{A.11}$$

as $y \to 0^+$, where the constant $C > 0$ does not depend on n. Similarly, by (2.6), (A.5) and (A.10) with $l = 1$,

$$\frac{|h_a'(y)|}{y} \leq \frac{C'}{y}\Big\{y^{-(2h_{12}+2)}\Big(\frac{y}{a}\Big)^{\varpi_0}y^{2N_\psi} + y^{-(2h_{12}+1)}\Big[\Big(\frac{y}{a}\Big)^{\varpi_0-1}\frac{1}{a}y^{2N_\psi} + \Big(\frac{y}{a}\Big)^{\varpi_0}y^{2N_\psi-1}\Big]\Big\} \leq C'\frac{y^{2N_\psi-(2h_{12}+2)+\varpi_0-1}}{a^{\varpi_0}} \to 0. \tag{A.12}$$

This proves (A.6), as desired. Next, note that

$$\frac{\partial}{\partial x}e^{itx}\psi(t) = ite^{itx}\psi(t), \quad \frac{\partial^2}{\partial x^2}e^{itx}\psi(t) = -t^2e^{itx}\psi(t)$$

and $it\psi(t), t^2\psi(t) \in L^1(\mathbb{R})$ by the continuity of ψ and condition (3.2). Therefore, by the dominated convergence theorem, $\widehat{\psi}'(x) = C\int_{\mathbb{R}} e^{itx}it\psi(t)dt$, $\widehat{\psi}''(x) = C'\int_{\mathbb{R}} e^{itx}(-t^2)\psi(t)dt$ for appropriate constants $C, C' \in \mathbb{R}$. Consequently, by (3.2),

$$\max_{l=0,1,2}\sup_{x\in\mathbb{R}}|\widehat{\psi}^{(l)}(x)| \leq C\int_{\mathbb{R}}|t|^l|\psi(t)|dt < \infty. \tag{A.13}$$

So, fix $z \neq 0$. By (2.5) and (A.13),

$$\lim_{|y|\to\infty}\Big|h_a(y)\Big| = 0 = \lim_{|y|\to\infty}\Big|h_a'(y)\Big|. \tag{A.14}$$

Thus, in view of (A.4), (A.6), (A.8), (A.9), (A.10) (with $l = 2$) and (A.14), by integrating by parts twice we obtain

$$\Big|\frac{\Xi_{12}^{jj'}(az)}{a^{2h_{12}}} - \Phi_{12}^{jj'}(z)\Big| = \Big|\frac{1}{z^2}\int_{\mathbb{R}} e^{izy}h_a''(y)dy\Big|$$

$$\leq \frac{C}{z^2}\int_{\mathbb{R}}\{|y|^{-(2h_{12}+3)}|\vartheta_a(y)| + |y|^{-(2h_{12}+2)}|\vartheta_a'(y)| + |y|^{-(2h_{12}+1)}|\vartheta_a''(y)|\}dy \leq \frac{C'}{z^2}, \tag{A.15}$$

where the last inequality is a consequence of (2.5), (A.5) and (A.13).

Now consider the first summation term in (A.3). We proceed as in the proof of proposition 3.3, (i), in [53] to establish that

$$\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^jn_*}\Big(\frac{\Xi_{12}^{jj'}(a(2^jk - 2^{j'}k'))}{a^{2h_{12}}} - \Phi_{12}^{jj'}(2^jk - 2^{j'}k')\Big) \to 0, \quad n \to \infty. \tag{A.16}$$

We outline the main steps for the reader's convenience. By Theorem 1.8 in Jones and Jones [57], p. 10, the set of values $r \in \mathbb{Z}$ to the equation $a2^jk - a2^{j'}k' = r$, $k, k' \in \mathbb{Z}$, is given by $\gcd(a2^j, a2^{j'})\mathbb{Z} =: \mathcal{R}$. Therefore, a pair $(k, k') \in \mathbb{Z}^2$ is a solution to

$$a2^jk - a2^{j'}k' = \gcd(a2^j, a2^{j'})w \tag{A.17}$$

for some $w \in \mathbb{Z}$ if and only if it is a solution to

$$2^j k - 2^{j'} k' = \gcd(2^j, 2^{j'})w \tag{A.18}$$

for the same w. Therefore, we can replace n with n_* in Lemmas B.2 and B.3, [53], and reexpress the first summation term on the left-hand side of (A.16) as

$$\sum_{r \in \mathcal{R} \cap B_{jj'}(n_*)} \frac{\xi_r(n_*)}{n_*} \Big(\frac{\Xi^{jj'}_{12}(ar) - \Phi^{jj'}_{12}(ar)}{a^{2h_{12}}} \Big). \tag{A.19}$$

In (A.19), $B_{jj'}(n_*)$ is the range for r such that the pairs (k, k') satisfying (A.18) lie in the region

$$1 \le k \le 2^{j'} n_*, \quad 1 \le k' \le 2^j n_*, \tag{A.20}$$

and $\xi_r(n_*)$ is the number of such solution pairs (k, k') given some r. Moreover, for any sufficiently large n, let $k_0 \in \{1, \dots, 2^{j'} n_*\}$ be the smallest number such that $(k_0, k'(k_0)) \in \mathbb{N}^2$ solves (A.17) (for some $w \in \mathbb{Z}$), where

$$k'(k) := \frac{2^j}{2^{j'}} k - \frac{\gcd(2^j, 2^{j'})w}{2^{j'}}. \tag{A.21}$$

From the proof of Lemma B.2, [53], the set $\mathcal{A}$ of such solutions to (A.18) has the form

$$\mathcal{A} = \Big\{ (k, k') \in \mathbb{Z}^2 : k = k_0 + \frac{2^{j'}}{\gcd(2^j, 2^{j'})} \mathbb{Z}, \quad k' \text{ is given by (A.21)} \Big\}. \tag{A.22}$$

In light of (A.22), define the function $k(z) = k_0 + \frac{2^{j'}}{\gcd(2^j, 2^{j'})} z$, $z \in \mathbb{Z}$. In particular, $(k(0), k'(k(0)))$ is a solution pair for (A.18). Let $\mathbb{R} \ni x = \gcd(2^j, 2^{j'})(n_* - k_0/2^{j'})$. Then, by (A.22), $(k(\lfloor x \rfloor), k'(k(\lfloor x \rfloor)))$ is the rightmost solution for (A.17) within the first-entry range $k = 1, \dots, 2^{j'} n_*$. Moreover, given r, the number of solution pairs in the region (A.20) is $\xi_r(n_*) = \lfloor x \rfloor + 1$, where

$$\lfloor x \rfloor n_*^{-1} \to \gcd(2^j, 2^{j'}), \quad n \to \infty. \tag{A.23}$$

In addition, by (A.15), and (3.14),

$$\Big| \frac{\Xi^{jj'}_{12}(ar) - \Phi^{jj'}_{12}(ar)}{a^{2h_{12}}} \Big| \le \frac{C}{r^2}, \quad r \ne 0, \quad \lim_{n \to \infty} \Big| \frac{\Xi^{jj'}_{12}(ar) - \Phi^{jj'}_{12}(ar)}{a^{2h_{12}}} \Big| = 0, \quad r \in \mathbb{Z}. \tag{A.24}$$

By expression (A.19), the dominated convergence theorem and (A.23), the limit (A.16) holds.

Next recall that, up to a change of sign, $\Phi^{jj'}_{12}(z)$ corresponds to the cross moment of the wavelet transform of a fBm. Thus, by an analogous procedure, we also obtain

$$\frac{1}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \Phi^{jj'}_{12}(2^j k - 2^{j'} k') \to \gcd(2^j, 2^{j'}) \sum_{z=-\infty}^{\infty} \Phi^{jj'}_{12}(z \gcd(2^j, 2^{j'})), \quad n \to \infty. \tag{A.25}$$

This establishes (A.3).

To show statement (v), first note that

$$o\Big(\frac{a^{2\max\{h_{13}+h_{24}, h_{14}+h_{23}\} - 2(h_{12}+h_{34})}}{n_*} \Big) \Big/ \frac{a^{(h_1+h_2+h_3+h_4) - 2(h_{12}+h_{34})}}{n_*} = o(1),$$

which follows from (3.6) and the fact that

$$2\max\{h_{13}+h_{24}, h_{14}+h_{23}\} \le h_1 + h_2 + h_3 + h_4. \tag{A.26}$$

Thus, by expression (C.56) for $\mathrm{Cov}[W^{(12)}_n(a2^j), W^{(34)}_n(a2^{j'})]$ (established in the proof of Lemma C.5),

$$\begin{aligned} \frac{\sqrt{n_{a,j}}}{a^{\delta_{12}}} \frac{\sqrt{n_{a,j'}}}{a^{\delta_{34}}} \mathrm{Cov}\Big[W^{(12)}_n(a2^j), W^{(34)}_n(a2^{j'}) \Big] &= \frac{2^{\frac{j+j'}{2}} n_*}{a^{\delta_{12}+\delta_{34}}} \mathrm{Cov}\Big[W^{(12)}_n(a2^j), W^{(34)}_n(a2^{j'}) \Big] \\ &= \frac{2^{-\frac{(j+j')}{2}}}{[\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)(1+O(a^{-\varpi_0}))^2]} \\ &\quad \cdot \Big\{ \frac{a^{2(h_{13}+h_{24})}}{a^{h_1+h_2+h_3+h_4}} \frac{1}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \Phi^{jj'}_{13}(2^j k - 2^{j'} k') \Phi^{jj'}_{24}(2^j k - 2^{j'} k') \\ &\quad + \frac{a^{2(h_{14}+h_{23})}}{a^{h_1+h_2+h_3+h_4}} \frac{1}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \Phi^{jj'}_{14}(2^j k - 2^{j'} k') \Phi^{jj'}_{23}(2^j k - 2^{j'} k') \Big\} + o(1). \end{aligned} \tag{A.27}$$

By (A.26) and (A.27), statement (v) holds.

The argument for showing (vi) is an adaptation of the proof of Theorem 3.1 in [53] (see also [58], pp. 510-513; [47], p. 997; and [59], Lemma 2). For notational simplicity, we only write the proof for $m = 2$, where the entries are indexed 1 and 2, and $q_1, q_2 = 1, 2$ denote generic entries. The proof is by means of the Cramér–Wold device. Under (.2), form the vector of wavelet coefficients

$$\mathbf{Y}_n := \Big(d_1(a2^{j_1}, 1), d_2(a2^{j_1}, 1), \ldots, d_1(a2^{j_1}, n_{a,j_1}), d_2(a2^{j_1}, n_{a,j_1}); \ldots;$$
$$d_1(a2^{j_2}, 1), d_2(a2^{j_2}, 1), \ldots, d_1(a2^{j_2}, n_{a,j_2}), d_2(a2^{j_2}, n_{a,j_2})\Big) \in \mathbb{R}^{\Upsilon(n)}, \tag{A.28}$$

where

$$\Upsilon(n) := 2 \sum_{j=j_1}^{j_2} n_{a,j}. \tag{A.29}$$

Let

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_{j_1}, \ldots, \boldsymbol{\theta}_{j_2}) \in \mathbb{R}^{3J}, \tag{A.30}$$

where $J = j_2 - j_1 + 1$ and $\boldsymbol{\theta}_j = (\theta_{j,1}, \theta_{j,12}, \theta_{j,2})^* \in \mathbb{R}^3$, $j = j_1, \ldots, j_2$. Now form the block-diagonal matrix D_n defined by

$$\mathrm{diag}\Big(\underbrace{\frac{1}{n_{a,j_1}}\sqrt{\frac{1}{2^{j_1}}}\Omega_{n,j_1}, \ldots, \frac{1}{n_{a,j_1}}\sqrt{\frac{1}{2^{j_1}}}\Omega_{n,j_1}}_{n_{a,j_1}}; \ldots; \underbrace{\frac{1}{n_{a,j_2}}\sqrt{\frac{1}{2^{j_2}}}\Omega_{n,j_2}, \ldots, \frac{1}{n_{a,j_2}}\sqrt{\frac{1}{2^{j_2}}}\Omega_{n,j_2}}_{n_{a,j_2}}\Big), \tag{A.31}$$

where

$$\Omega_{n,j} = \begin{pmatrix} \dfrac{\theta_{j,1}}{\mathbb{E}\big[d_1^2(a2^j, 0)\big]} & \dfrac{1}{a^{\delta_{12}}}\dfrac{\theta_{j,12}}{2\mathbb{E}\big[d_1(a2^j, 0)d_2(a2^j, 0)\big]} \\ \dfrac{1}{a^{\delta_{12}}}\dfrac{\theta_{j,12}}{2\mathbb{E}\big[d_1(a2^j, 0)d_2(a2^j, 0)\big]} & \dfrac{\theta_{j,2}}{\mathbb{E}\big[d_2^2(a2^j, 0)\big]} \end{pmatrix}, \quad j = j_1, \ldots, j_2. \tag{A.32}$$

In (A.32), it can be understood that the slow growth factors for the main diagonal terms are $\frac{1}{a^{2\delta_{11}}} = \frac{1}{a^{2\delta_{22}}} \equiv 1$. We would like to establish the limiting distribution of the statistic

$$T_n = \sum_{j=j_1}^{j_2} \frac{\boldsymbol{\theta}_j^*}{\sqrt{2^j}} \mathrm{vec}_{\mathcal{S}}\Big[\Big(\frac{1}{a^{\delta_{q_1 q_2}}}\Big)_{q_1,q_2=1,2} \circ W_n(a2^j)\Big]$$
$$= \sum_{j=j_1}^{j_2} \frac{\boldsymbol{\theta}_j^*}{\sqrt{2^j}} \frac{1}{n_{a,j}} \Big(\sum_{k=1}^{n_{a,j}} \frac{d_1^2(a2^j, k)}{\mathbb{E}\big[d_1^2(a2^j, 0)\big]}, \frac{1}{a^{\delta_{12}}} \sum_{k=1}^{n_{a,j}} \frac{d_1(a2^j, k)d_2(a2^j, k)}{\mathbb{E}\big[d_1(a2^j, 0)d_2(a2^j, 0)\big]}, \sum_{k=1}^{n_{a,j}} \frac{d_2^2(a2^j, k)}{\mathbb{E}\big[d_2^2(a2^j, 0)\big]}\Big)^*$$
$$= \mathbf{Y}_n^* D_n \mathbf{Y}_n,$$

where it suffices to consider $\boldsymbol{\theta}$ in (A.30) such that

$$\boldsymbol{\theta}^* \Sigma(H) \boldsymbol{\theta} > 0 \tag{A.33}$$

(see [40], pp. 211 and 214). The matrix $\Sigma(H)$ in (A.33) is obtained from (3.17) and can be written in block form as $\Sigma(H) =: (G^{jj'})_{j,j'=j_1,\ldots,j_2}$, corresponding to block entries of the vector $\boldsymbol{\theta} = (\theta_{j_1}, \ldots, \theta_{j_2})^*$. Let $\Gamma_{\mathbf{Y}_n} = \mathrm{Cov}\,[\mathbf{Y}_n, \mathbf{Y}_n]$, and consider the spectral decomposition $\Gamma_{\mathbf{Y}_n}^{1/2} D_n \Gamma_{\mathbf{Y}_n}^{1/2} = O \Lambda O^*$, where Λ is diagonal with real, and not necessarily positive, eigenvalues

$$\xi_i(a2^j), \quad i = 1, \ldots, \Upsilon(n), \tag{A.34}$$

and O is an orthogonal matrix. Now let $\mathbf{Z} \sim N(0, I_{\Upsilon(n)})$. Then,

$$T_n \stackrel{d}{=} \mathbf{Z}^* \Gamma_{\mathbf{Y}_n}^{1/2} D_n \Gamma_{\mathbf{Y}_n}^{1/2} \mathbf{Z} = \mathbf{Z}^* O \Lambda O^* \mathbf{Z} \stackrel{d}{=} \mathbf{Z}^* \Lambda \mathbf{Z} =: \sum_{i=1}^{\Upsilon(n)} \xi_i(a2^j) Z_i^2.$$

Assume for the moment that

$$\max_{i=1,\ldots,\Upsilon(n)} |\xi_i(a2^j)| = o\Big(\Big(\frac{a}{n}\Big)^{1/2}\Big). \tag{A.35}$$

By (A.33) and (3.17),

$$\frac{n}{a} \mathrm{Var}\,[T_n] = \sum_{j=j_1}^{j_2} \sum_{j'=j_1}^{j_2} \boldsymbol{\theta}_j^* \Big\{ \sqrt{\frac{n}{a2^j}} \sqrt{\frac{n}{a2^{j'}}} \mathrm{Cov}\Big[\ \mathrm{vec}_{\mathcal{S}}\Big[\Big(\frac{1}{a^{\delta_{q_1 q_2}}}\Big)_{q_1,q_2=1,2} \circ W_n(a2^j)\Big],$$
$$\mathrm{vec}_{\mathcal{S}}\Big[\Big(\frac{1}{a^{\delta_{q_1 q_2}}}\Big)_{q_1,q_2=1,2} \circ W_n(a2^{j'})\Big]\ \Big]\Big\} \boldsymbol{\theta}_{j'} \to \sum_{j=j_1}^{j_2} \sum_{j'=j_1}^{j_2} \boldsymbol{\theta}_j^*\, G^{jj'}\, \boldsymbol{\theta}_{j'} > 0.$$

Therefore, there exists a constant $C > 0$ such that, for large enough n, $\frac{n}{a}\mathrm{Var}\,[T_n] \geq C > 0$. In view of condition (A.35),

$$\frac{\max_{i=1,\dots,\Upsilon(n)}|\xi_i(a2^j)|}{\sqrt{\mathrm{Var}\,[T_n]}} \leq C'\left(\frac{n}{a}\right)^{1/2} \max_{i=1,\dots,\Upsilon(n)}|\xi_i(a2^j)| \to 0, \quad n \to \infty.$$

The claim (3.19) is now a consequence of Lemma B.4 in [53].

So, we need to show (A.35). The first step is to establish the bound

$$\sup_{\mathbf{u}\in S^{\Upsilon(n)-1}} |\mathbf{u}^* \Gamma_{\mathbf{Y}_n}^{1/2} D_n \Gamma_{\mathbf{Y}_n}^{1/2}\mathbf{u}| \leq C \max_{j=j_1,\dots,j_2} \frac{1}{n_{a,j}}\|\Omega_{n,j}\| \sup_{\mathbf{u}\in S^{\Upsilon(n)-1}} \mathbf{u}^*\Gamma_{\mathbf{Y}_n}\mathbf{u}. \tag{A.36}$$

Let $\mathbf{u} \in S^{\Upsilon(n)-1}$ and let $\mathbf{v} = \Gamma^{1/2}\mathbf{u}$. We can break up the vector $\mathbf{v}$ into two-dimensional subvectors $v_{\cdot,\cdot}$ to reexpress $\mathbf{v} = (v_{j_1,1},\dots,v_{j_1,n_{a,j_1}};\dots;v_{j_2,1},\dots,v_{j_2,n_{a,j_2}})^*$. Based on the block-diagonal structure of D_n expressed in (A.31),

$$|\mathbf{u}^*\Gamma_{\mathbf{Y}_n}^{1/2} D_n \Gamma_{\mathbf{Y}_n}^{1/2}\mathbf{u}| = |\mathbf{v}^* D_n \mathbf{v}| = \left|\sum_{j=j_1}^{j_2}\sum_{l=1}^{n_{a,j}} v_{j,l}^* \frac{\Omega_{n,j}}{n_{a,j}\sqrt{2^j}} v_{j,l}\right| \leq C \sum_{j=j_1}^{j_2}\sum_{l=1}^{n_{a,j}} \frac{1}{n_{a,j}\sqrt{2^j}}\|\Omega_{n,j}\|\ \|v_{j,l}\|^2$$

$$\leq C\Big(\max_{j=j_1,\dots,j_2}\frac{1}{n_{a,j}\sqrt{2^j}}\|\Omega_{n,j}\|\Big)\sum_{j=j_1}^{j_2}\sum_{l=1}^{n_{a,j}}\|v_{j,l}\|^2 = C\Big(\max_{j=j_1,\dots,j_2}\frac{1}{n_{a,j}\sqrt{2^j}}\|\Omega_{n,j}\|\Big)\mathbf{u}^*\Gamma_{\mathbf{Y}_n}\mathbf{u}, \tag{A.37}$$

where the constant C comes from a change of matrix norms and only depends on the fixed dimension $m = 2$. By taking $\sup_{\mathbf{u}\in S^{\Upsilon(n)-1}}$ on both sides of (A.37), we arrive at (A.36).

The second step towards showing (A.35) consists of analyzing the asymptotic behavior of the right-hand side of (A.36), as $n \to \infty$. For this, we will assume the result of Lemma C.1. So, note that

$$\max_{j=j_1,\dots,j_2}\frac{1}{n_{a,j}}\|\Omega_{n,j}\| \leq C\,\frac{a}{n}\,\frac{1}{a^{2\min\{h_1,h_2,h_{12}\}}}\,\frac{1}{a^{\delta_{12}}}, \quad n \to \infty. \tag{A.38}$$

Moreover, by relation (C.14), the maximum eigenvalue of $\Gamma_{\mathbf{Y}_n}$ is bounded by $\|\Gamma_{\mathbf{Y}_n}\| \leq C a^{2\max\{h_1,h_2\}}$, where $\|\cdot\|$ is the matrix Euclidean norm. Therefore, in view of (A.38) and (C.14), the right-hand side of (A.36) is bounded by $C\frac{a^{2(h_{\max}-h_{\min})+1}}{n}\frac{1}{a^{\delta_{12}}}$. In turn, the latter expression divided by $\sqrt{\frac{a}{n}}$ is bounded by $C\left(\frac{a^{4(h_{\max}-h_{\min})+1}}{n}\right)^{1/2}\frac{1}{a^{\delta_{12}}}$. By condition (3.6), this implies (A.35), and as a result, also (3.19). $\square$

Appendix B. Section 3.3

Proof of Theorem 3.1. Fix $q_1, q_2 = 1,\dots,m$. Based on (3.22), rewrite

$$\frac{1}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}(\widehat{\alpha}_{q_1q_2}-\alpha_{q_1q_2})$$

$$= \sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\log|W_n^{(q_1q_2)}(a2^j)| + \sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\left\{\log\frac{\left|\mathbb{E}\left[S_n^{(q_1q_2)}(a2^j)\right]\right|}{a^{\alpha_{q_1q_2}}} - \log|\Phi_{q_1q_2}^{jj}(0)|\right\}$$

$$+ \frac{1}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\left(\sum_{j=j_1}^{j_2} w_j\,\log|\Phi_{q_1q_2}^{jj}(0)| - \alpha_{q_1q_2}\right). \tag{B.1}$$

By Lemma C.1(ii), and an application of the mean value theorem, for some $\theta(n) > 0$ between $\left|\mathbb{E}\left[S_n^{(q_1q_2)}(a2^j)\right]\right|/a^{\alpha_{q_1q_2}}$ and $|\Phi_{q_1q_2}^{jj}(0)|$ the second term in the sum (B.1) can be bounded in absolute value by

$$\sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\left\{\frac{1}{\theta(n)}\left|\frac{\left|\mathbb{E}\left[S_n^{(q_1q_2)}(a2^j)\right]\right|}{a^{\alpha_{q_1q_2}}} - |\Phi_{q_1q_2}^{jj}(0)|\right|\right\}$$

$$= \sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\frac{1}{|\Phi_{q_1q_2}^{jj}(0)|+o(1)}\frac{C_j}{a^{\varpi_0}} \leq C'\sqrt{\frac{n}{a^{2\delta_{q_1q_2}+1+2\varpi_0}}} \to 0,$$

as $n \to \infty$, where the limit is a consequence of condition (3.6). Also note that, after a change of variable $2^j x = y$ in the expression for $\Phi_{q_1q_2}^{jj}(0)$ (see (3.14)),

$$\Phi_{q_1q_2}^{jj}(0) = 2^{j\alpha_{q_1q_2}}\rho_{q_1q_2}\int_{\mathbb{R}}|y|^{-(\alpha_{q_1q_2}+1)}|\widehat{\psi}(y)|^2 dy =: 2^{j\alpha_{q_1q_2}}c_{q_1q_2} \in \mathbb{R}.$$

Therefore, by (3.22), the third term in the sum (B.1) can be written as

$$\frac{1}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\left(\sum_{j=j_1}^{j_2} w_j\,(j\alpha_{q_1q_2}+\log|c_{q_1q_2}|) - \alpha_{q_1q_2}\right) = 0.$$

So, in regard to the first term in the sum (B.1), consider the weight matrix $M \in M(\frac{m(m+1)}{2}, \frac{m(m+1)}{2}J, \mathbb{R})$ defined by

$$\Big(2^{j_1/2} w_{j_1} I_{\frac{m(m+1)}{2}} \; ; 2^{(j_1+1)/2} w_{j_1+1} I_{\frac{m(m+1)}{2}} \; ; \quad \ldots \quad ; 2^{j_2/2} w_{j_2} I_{\frac{m(m+1)}{2}}\Big),$$

where $I_{\frac{m(m+1)}{2}}$ is a $\frac{m(m+1)}{2} \times \frac{m(m+1)}{2}$ identity matrix and J is given by (3.20). We would like to show that

$$M\left(\mathrm{vec}_{\mathcal{S}}\Big[\Big(\frac{\sqrt{n_{a,j}}}{a^{\delta_{q_1 q_2}}}\Big)_{q_1,q_2=1,\ldots,m} \circ \quad \log \circ |W_n(a2^j)|\Big]\right)_{j=j_1,\ldots,j_2} \stackrel{d}{\to} \mathcal{N}_{\frac{m(m+1)}{2}}(0, MGM^*), \tag{B.2}$$

where $\log \circ |A| := \Big(\log |A_{q_1 q_2}|\Big)_{q_1,q_2=1,\ldots,m}$ for any $m \times m$ real matrix A, and the term post-multiplying the matrix M in (B.2) is a $\frac{m(m+1)}{2}J$-dimensional random vector.

For any $0 < r < 1$, fix a pair q_1, q_2 and an octave j, which specifies one of the entries of the random vector on the left-hand side of (B.2). Define the set $A_n = \{\omega : \min_{q_1,q_2} W_n^{(q_1 q_2)}(a2^j) > r\}$. Under (.2), Proposition 3.1, (vi), implies that $P(A_n) \to 1$ as $n \to \infty$. Thus, for large enough n, in the set A_n the mean value theorem gives the almost sure expression

$$\mathbb{R} \ni 2^{j/2} w_j \frac{\sqrt{n_{a,j}}}{a^{\delta_{q_1 q_2}}} \log|W_n^{(q_1 q_2)}(a2^j)| = 2^{j/2} w_j \frac{\sqrt{n_{a,j}}}{a^{\delta_{q_1 q_2}}} \frac{(W_n^{(q_1 q_2)}(a2^j) - 1)}{\theta_+(W_n^{(q_1 q_2)}(a2^j))} \tag{B.3}$$

for a random variable $\theta_+(W_n^{(q_1 q_2)}(a2^j))$ between $W_n^{(q_1 q_2)}(a2^j)$ and 1. Since $W_n^{(q_1 q_2)}(a2^j) \stackrel{P}{\to} 1$, then $\theta_+(W_n^{(q_1 q_2)}(a2^j)) \stackrel{P}{\to} 1$. By considering (B.3) for all $1 \le q_1 \le q_2 \le m$ and $j = j_1, \ldots, j_2$, relation (B.2) is now a consequence of Proposition 3.1, (vi), and Slutsky's theorem. □

Appendix C. Section 4.1

C.1. General results

Fix $q_1, q_2 = 1, \ldots, m$. For $j = j_1, \ldots, j_2$ and $n \in \mathbb{N}$, consider the jointly Gaussian vector

$$\mathcal{Y}_n = \Big(\frac{d_{q_1}(a2^{j_1}, 1)}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_1})}}, \ldots, \frac{d_{q_1}(a2^{j_1}, n_{a,j_1})}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_1})}}, \ldots, \frac{d_{q_1}(a2^{j_2}, 1)}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_2})}}, \ldots, \frac{d_{q_1}(a2^{j_2}, n_{a,j_2})}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_2})}},$$

$$\frac{d_{q_2}(a2^{j_1}, 1)}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_1})}}, \ldots, \frac{d_{q_2}(a2^{j_1}, n_{a,j_1})}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_1})}}, \ldots, \frac{d_{q_2}(a2^{j_2}, 1)}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_2})}}, \ldots, \frac{d_{q_2}(a2^{j_2}, n_{a,j_2})}{\sqrt{\varrho^{(q_1 q_2)}(a2^{j_2})}}\Big)^* \in \mathbb{R}^{\Upsilon(n)} \tag{C.1}$$

(see (A.29) for the definition of $\Upsilon(n)$). Let

$$\Gamma_{\mathcal{Y}_n} = \mathbb{E}\big[\mathcal{Y}_n \mathcal{Y}_n^*\big] = O \Lambda_{\mathcal{Y}} O^* = O \;\mathrm{diag}(\lambda_{1,\mathcal{Y}}, \ldots, \lambda_{\Upsilon(n),\mathcal{Y}})\; O^*, \quad O \in O(\Upsilon(n)), \tag{C.2}$$

be the associated covariance matrix and its matrix spectral decomposition. The following lemma provides the finite-sample and asymptotic properties of the covariance structure of wavelet coefficients, both from $\Xi^{jj'}$ and $\mathcal{Y}_n$. Note that such covariance structure does not in general correspond to a multivariate stationary stochastic process when multiple octaves j are considered.

Lemma C.1. *For $j = j_1, \ldots, j_2$, $q_1, q_2 = 1, \ldots, m$, and $n \in \mathbb{N}$, the following statements hold.*

(i) *Consider $\Xi^{jj',a}(a(n)(2^j k - 2^{j'} k')) = \mathbb{E}\Big[D(a(n)2^j, k) D(a(n)2^{j'}, k')^*\Big]$ (see (3.12)). For $j = j'$ and $n \in \mathbb{N}$, there is a continuous spectral density $f_{\psi,n}(x)_{q_1 q_2}$ such that we can write*

$$\frac{\Xi^{jj,a}_{q_1 q_2}(a(n)2^j(k-k'))}{(a(n)2^j)^{2h_{q_1 q_2}}} = \int_{-\pi}^{\pi} e^{i(k-k')x} f_{\psi,n}(x)_{q_1 q_2}\, dx, \quad k, k' \in \mathbb{Z}. \tag{C.3}$$

Moreover,

$$|f_{\psi,n}(x)_{q_1 q_2}| \le C, \quad x \in (-\pi, \pi], \tag{C.4}$$

for a constant $C > 0$ that does not depend on n.

(ii) *Let $\Phi^{jj'}_{\cdot\cdot}(z)$ be as in (3.14), $z \in \mathbb{Z}$, and fix any $0 < \xi < 1$. Then,*

$$\Big|\frac{\Xi^{jj',a}_{q_1 q_2}(za(n))}{a(n)^{2h_{q_1 q_2}}} - \Phi^{jj'}_{q_1 q_2}(z)\Big| \le \frac{C}{a(n)^{\varpi_0}}, \quad n \to \infty, \tag{C.5}$$

where ϖ_0 and β are defined in expressions (2.6) and (3.3), respectively.

(iii) *Let $\mathcal{Y}_n$ be as in (C.1), and let $\lambda_{i,\mathcal{Y}}$, $i = 1, \ldots, \Upsilon(n)$, be the eigenvalues of the covariance matrix $\Gamma_{\mathcal{Y}_n}$ (see (A.29) and (C.2)). Then, for some $C > 0$,*

$$\max_{i=1,\ldots,\Upsilon(n)} \lambda_{i,\mathcal{Y}} \le C a(n)^{2(\max\{h_{q_1}, h_{q_2}\} - h_{q_1 q_2})}. \tag{C.6}$$

Proof. We will use the shorthand $q_1 = 1$ and $q_2 = 2$ throughout the proof.

We first show (i). By making the change of variable $y = a2^j x$ in (A.1), we obtain (C.3) with

$$f_{\psi,n}(x)_{12} := \sum_{l=-\infty}^{\infty} \frac{|\widehat{\psi}(x+2\pi l)|^2}{|x+2\pi l|^{2h_{12}+1}}\ \rho_{12} g_{12}\Big(\frac{x+2\pi l}{a2^j}\Big). \tag{C.7}$$

Moreover, by (3.1)–(3.3), $\widehat{\psi}$ is continuous, and hence, by (2.5), (3.3), (3.4) and the dominated convergence theorem, the periodic function $f_{\psi,n}(x)_{12}$ in (C.7) is also continuous on $[-\pi,\pi]$, and hence, bounded. This shows (C.3) and (C.4).

We now turn to (ii). It suffices to show that, for any $0 < \xi < 1$,

$$\Big|\frac{\Xi_{12}^{jj',a}(za)}{a^{2h_{12}}} - \Phi_{12}^{jj'}(z)\Big| \le \frac{C}{a^{\min\{\varpi_0,(1-\xi)(2h_{12}+2\beta)\}}}, \quad n \to \infty, \tag{C.8}$$

since for small enough ξ, conditions (2.7) and (3.3) imply that

$$(1-\xi)(2h_{12}+2\beta) > 2h_{\min} + 2 \ge \varpi_0.$$

In fact, by (A.1), (3.14) and a change of variable $y = ax$,

$$\begin{aligned}
&\frac{\Xi_{12}^{jj',a}(za)}{a^{2h_{12}}} - \Phi_{12}^{jj'}(z)\\
&= \frac{1}{a^{2h_{12}}}\int_{\mathbb{R}} e^{izax}|x|^{-(2h_{12}+1)}\rho_{12}\{g_{12}(x)-1\}\widehat{\psi}(a2^j x)\overline{\widehat{\psi}(a2^{j'}x)}dx\\
&= \Big\{\int_{|y|\le a^{1-\xi}} + \int_{|y|>a^{1-\xi}}\Big\} e^{izy}|y|^{-(2h_{12}+1)}\rho_{12}\Big\{g_{12}\Big(\frac{y}{a}\Big)-1\Big\}\widehat{\psi}(2^j y)\overline{\widehat{\psi}(2^{j'}y)}dy
\end{aligned} \tag{C.9}$$

for any $0 < \xi < 1$. For large enough n, by (2.6), (2.7) and (3.1)–(3.3), the absolute value of the first integral in the sum (C.9) can be bounded by

$$\frac{C}{a^{\varpi_0}}\int_{|y|\le a^{1-\xi}} |y|^{-(2h_{12}+1)+\varpi_0}|\widehat{\psi}(2^j y)\overline{\widehat{\psi}(2^{j'}y)}|dy \le \frac{C'}{a^{\varpi_0}}. \tag{C.10}$$

On the other hand, by (3.2) the absolute value of the second term in the sum (C.9) can be bounded by

$$C\int_{|y|>a^{1-\xi}} |y|^{-(2h_{12}+1)-2\beta}dy \le \frac{C'}{a^{(1-\xi)(2h_{12}+2\beta)}}. \tag{C.11}$$

Expressions (C.10) and (C.11) yield (C.5).

We turn to (iii). For notational simplicity, consider only two octaves j, j', whence $J = j_2 - j_1 + 1 = 2$. Then, from (A.29),

$$\Upsilon(n) = 2(n_{a,j_1} + n_{a,j_2}).$$

Fix $\mathbf{v} \in \mathbb{C}^{\Upsilon(n)}$. For notational simplicity, divide the summation range $k = 1, \dots, \Upsilon(n)$ into the subranges

$$K_1 = \{1,\dots,n_{a,j}\}, \quad K_2 = \{n_{a,j}+1,\dots,n_{a,j}+n_{a,j'}\},$$
$$K_3 = \{n_{a,j}+n_{a,j'}+1,\dots,2n_{a,j}+n_{a,j'}\}, \quad K_4 = \{2n_{a,j}+n_{a,j'}+1,\dots,2(n_{a,j}+n_{a,j'})\}.$$

Define the octave and index functions

$$j(k) = j1_{\{K_1\cup K_3\}}(k) + j'1_{\{K_2\cup K_4\}}(k), \quad q(k) = 1\ 1_{\{K_1\cup K_2\}}(k) + 2\ 1_{\{K_3\cup K_4\}}(k),$$

respectively, which reflects the order of appearance of different octave and index values in the vector (C.1). By (A.1),

$$\begin{aligned}
\mathbf{v}^*\Gamma_{\mathcal{Y}_n}\mathbf{v} &= \sum_{k_1=1}^{\Upsilon(n)}\sum_{k_2=1}^{\Upsilon(n)} v_{k_1}\overline{v}_{k_2}\Big(\Gamma_{\mathcal{Y}_n}\Big)_{k_1,k_2}\\
&= \int_{\mathbb{R}}\sum_{k_1=1}^{\Upsilon(n)}\sum_{k_2=1}^{\Upsilon(n)} \frac{v_{k_1}\overline{v}_{k_2}e^{ia(2^{j(k_1)}k_1-2^{j(k_2)}k_2)x}\widehat{\psi}(a2^{j(k_1)}x)\overline{\widehat{\psi}(a2^{j(k_2)}x)}}{\sqrt{\varrho^{(12)}(a2^{j(k_1)})\varrho^{(12)}(a2^{j(k_2)})}} f_{q(k_1)q(k_2)}(x)dx.
\end{aligned} \tag{C.12}$$

By expanding the double summation in (C.12) into each pair in the Cartesian product $\{K_1, K_2, K_3, K_4\}^2$, we end up with 16 double summation terms under the sign of the integral, i.e., 8 pairs of conjugates. To the cross terms, namely, those involving distinct summation ranges in the index k, we can apply the elementary inequality $|x\overline{y}+\overline{x}y| \le |x|^2+|y|^2$. One such pair, associated with the summation regions $K_2 \times K_3$ and $K_3 \times K_2$, is

$$\begin{aligned}
&\Big|\int_{\mathbb{R}}\Big(\sum_{k_1\in K_3}\frac{v_{k_1}e^{ia2^j k_1 x}\widehat{\psi}(a2^j x)}{\sqrt{\varrho^{(12)}(a2^j)}}\sum_{k_2\in K_2}\frac{\overline{v}_{k_2}e^{-ia2^{j'}k_2 x}\overline{\widehat{\psi}(a2^{j'}x)}}{\sqrt{\varrho^{(12)}(a2^{j'})}}\\
&\quad + \sum_{k_1\in K_2}\frac{v_{k_1}e^{ia2^{j'}k_1 x}\widehat{\psi}(a2^{j'}x)}{\sqrt{\varrho^{(12)}(a2^{j'})}}\sum_{k_2\in K_3}\frac{\overline{v}_{k_2}e^{-ia2^j k_2 x}\overline{\widehat{\psi}(a2^j x)}}{\sqrt{\varrho^{(12)}(a2^j)}}\Big)f_{12}(x)dx\Big|\\
&\le \int_{\mathbb{R}}\Big\{\Big|\sum_{k\in K_3}\frac{v_k e^{ia2^j kx}\widehat{\psi}(a2^j x)}{\sqrt{\varrho^{(12)}(a2^j)}}\Big|^2 + \Big|\sum_{k\in K_2}\frac{\overline{v}_k e^{-ia2^{j'}kx}\overline{\widehat{\psi}(a2^{j'}x)}}{\sqrt{\varrho^{(12)}(a2^{j'})}}\Big|^2\Big\}|f_{12}(x)|dx,
\end{aligned}$$

since $f_{12}(x) = f_{21}(x)$, and analogous bounds hold for the remaining terms. Therefore, (C.12) is bounded by

$$\int_{\mathbb{R}} \Big| \sum_{k\in K_1} v_k e^{ia2^j kx} \Big|^2 \frac{|\widehat{\psi}(a2^j x)|^2}{\varrho^{(12)}(a2^j)} (f_{11}(x) + 3|f_{12}(x)|)dx + \int_{\mathbb{R}} \Big| \sum_{k\in K_2} v_k e^{ia2^{j'} kx} \Big|^2 \frac{|\widehat{\psi}(a2^{j'} x)|^2}{\varrho^{(12)}(a2^{j'})} (f_{11}(x) + 3|f_{12}(x)|)dx$$
$$+ \int_{\mathbb{R}} \Big| \sum_{k\in K_3} v_k e^{ia2^j kx} \Big|^2 \frac{|\widehat{\psi}(a2^j x)|^2}{\varrho^{(12)}(a2^j)} (f_{22}(x) + 3|f_{12}(x)|)dx + \int_{\mathbb{R}} \Big| \sum_{k\in K_4} v_k e^{ia2^{j'} kx} \Big|^2 \frac{|\widehat{\psi}(a2^{j'} x)|^2}{\varrho^{(12)}(a2^{j'})} (f_{22}(x) + 3|f_{12}(x)|)dx.$$

By the change of variable $y = a2^{j(k)}x$ in each integral, breaking them up (in $\mathbb{R}$) into subregions of length 2π and using the periodicity of Fourier sums, we obtain

$$\int_{-\pi}^{\pi} \Big| \sum_{k\in K_1} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_1} |y + 2\pi l|^{-(2h_1+1)} \rho_1 g_1\Big(\frac{y+2\pi l}{a2^j}\Big)$$
$$+ 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^j}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^j)} dy$$
$$+ \int_{-\pi}^{\pi} \Big| \sum_{k\in K_2} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_1} |y + 2\pi l|^{-(2h_1+1)} \rho_1 g_1\Big(\frac{y+2\pi l}{a2^{j'}}\Big)$$
$$+ 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^{j'}}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^{j'})} dy$$
$$+ \int_{-\pi}^{\pi} \Big| \sum_{k\in K_3} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_2} |y + 2\pi l|^{-(2h_2+1)} \rho_2 g_2\Big(\frac{y+2\pi l}{a2^j}\Big)$$
$$+ 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^j}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^j)} dy$$
$$+ \int_{-\pi}^{\pi} \Big| \sum_{k\in K_4} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_2} |y + 2\pi l|^{-(2h_2+1)} \rho_2 g_2\Big(\frac{y+2\pi l}{a2^{j'}}\Big)$$
$$+ 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^{j'}}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^{j'})} dy$$
$$\leq C a^{2(\max\{h_1,h_2\}-h_{12})} \int_{-\pi}^{\pi} \Big\{ \Big| \sum_{k\in K_1} v_k e^{iky} \Big|^2 + \Big| \sum_{k\in K_2} v_k e^{iky} \Big|^2 + \Big| \sum_{k\in K_3} v_k e^{iky} \Big|^2 + \Big| \sum_{k\in K_4} v_k e^{iky} \Big|^2 \Big\} dy$$
$$= C a^{2(\max\{h_1,h_2\}-h_{12})} \mathbf{v}^*\mathbf{v} \tag{C.13}$$

for some $C > 0$. The inequality (C.13) is a consequence of (C.3) (from Lemma C.1(i)) and Proposition 3.1(iii), as applied to $\varrho^{(12)}(a2^{j(k)})$. Thus, the claim (C.6) holds. □

Remark C.1. Fix $q_1, q_2 = 1, \ldots, m$, and recall that $\rho^{(q_1q_2)}(a(n)2^j) = \Xi^{jj,a}_{q_1q_2}(0)$ (see (3.13)). Condition (2.8), the expression for the constant $\Phi^{jj}_{q_1q_2}(0)$ (see the right-hand side of (3.14)) and Lemma C.1(ii), imply that $\rho^{(q_1q_2)}(a(n)2^j)$ is bounded away from zero for large n.

Remark C.2. For $q_1 = 1$ and $q_2 = 2$, let $\mathbf{Y}_n$ and $\mathcal{Y}_n$ be as in (A.28) and (C.1), respectively. Then,

$$\mathbf{Y}_n = \mathcal{P}_n \text{diag}\Big(\underbrace{\sqrt{\varrho^{(12)}(a2^{j_1})}, \ldots, \sqrt{\varrho^{(12)}(a2^{j_1})}}_{n_{a,j_1}}, \ldots, \underbrace{\sqrt{\varrho^{(12)}(a2^{j_2})}, \ldots, \sqrt{\varrho^{(12)}(a2^{j_2})}}_{n_{a,j_2}},$$
$$\underbrace{\sqrt{\varrho^{(12)}(a2^{j_1})}, \ldots, \sqrt{\varrho^{(12)}(a2^{j_1})}}_{n_{a,j_1}}, \ldots, \underbrace{\sqrt{\varrho^{(12)}(a2^{j_2})}, \ldots, \sqrt{\varrho^{(12)}(a2^{j_2})}}_{n_{a,j_2}} \Big) \mathcal{Y}_n$$
$$=: \mathcal{P}_n N_n \mathcal{Y}_n$$

for some permutation matrix $\mathcal{P}_n$. Moreover, since $\Gamma_{\mathbf{Y}_n} = \mathcal{P}_n N_n \Gamma_{\mathcal{Y}_n} N_n \mathcal{P}_n^*$ is a real symmetric matrix, by Lemma C.1(ii) and (iii), we obtain the bound

$$\|\Gamma_{\mathbf{Y}_n}\| = \|\mathcal{P}_n N_n \Gamma_{\mathcal{Y}_n} N_n \mathcal{P}_n^*\| \leq C a(n)^{2\max\{h_1,h_2\}} \tag{C.14}$$

for the maximum eigenvalue of $\Gamma_{\mathbf{Y}_n}$, where $\|\cdot\|$ is the matrix Euclidean norm.

For any $n \in \mathbb{N}$, consider the Gaussian vector $\mathcal{Y}_n$ as in (C.1) but with only one octave j. It will be convenient to reexpress the sum $W^{(12)}_n(a2^j)$ as in (3.10) (with $q_1 = 1$ and $q_2 = 2$) based on a quadratic form. Define the permutation matrix

$$\Pi_n = \begin{pmatrix} \mathbf{0} & I_{n_{a,j}} \\ I_{n_{a,j}} & \mathbf{0} \end{pmatrix} \in O(2n_{a,j}). \tag{C.15}$$

Consider the spectral decomposition (C.2). Then,

$$W_n^{(12)}(a2^j) = \frac{1}{2n_{a,j}}\mathcal{Y}_n^* \Pi_n \mathcal{Y}_n \quad \text{(for one octave } j)$$

$$\stackrel{d}{=} \frac{1}{n_{a,j}} \mathbf{Z}^* \frac{(O\Lambda_{\mathcal{Y}}^{1/2}O^*)\Pi_n(O\Lambda_{\mathcal{Y}}^{1/2}O^*)}{2}\mathbf{Z} \stackrel{d}{=} \frac{1}{n_{a,j}}\mathbf{Z}^* \frac{\Lambda_{\mathcal{Y}}^{1/2}O^*\Pi_n O\Lambda_{\mathcal{Y}}^{1/2}}{2}\mathbf{Z}, \tag{C.16}$$

where $\mathbf{Z} \sim \mathcal{N}_{2n_{a,j}}(0, I_{2n_{a,j}})$. Let

$$S_n = \frac{\Lambda_{\mathcal{Y}}^{1/2}O^*\Pi_n O\Lambda_{\mathcal{Y}}^{1/2}}{2}, \tag{C.17}$$

which is a real symmetric matrix. Write out its spectral decomposition

$$S_n = O_S\Lambda_S O_S^*, \quad O \in O(2n_{a,j}), \quad \Lambda_S = \operatorname{diag}(\lambda_{1,S}, \dots, \lambda_{2n_{a,j},S}). \tag{C.18}$$

Expression (C.16) becomes

$$W_n^{(12)}(a2^j) = \frac{1}{n_{a,j}}\mathbf{Z}^* O_S\Lambda_S O_S^* \mathbf{Z} \stackrel{d}{=} \frac{1}{n_{a,j}}\mathbf{Z}^*\Lambda_S\mathbf{Z} = \sum_{i\in i_+(n)} \frac{\lambda_{i,S}}{n_{a,j}}Z_i^2 + \sum_{i\in i_-(n)} \frac{\lambda_{i,S}}{n_{a,j}}Z_i^2 =: \sum_{i\in i_+(n)} \eta_{i,n}Z_i^2 - \sum_{i\in i_-(n)} \eta_{i,n}Z_i^2, \tag{C.19}$$

where $i_+(n)$ and $i_-(n)$ are the indices for which $\lambda_{i,S}$ is nonnegative or negative, respectively. In particular, note that

$$\operatorname{Var}\left[W_n^{(12)}(a(n)2^j)\right] = \|\boldsymbol{\eta}_n\|_2^2,$$

where

$$\boldsymbol{\eta}_n := (\eta_{1,n}, \dots, \eta_{2n_{a,j},n})^* \tag{C.20}$$

and $\eta_{i,n}$, $i = 1, \dots, 2n_{a,j}$, are as in expression (C.19).

Remark C.3. The following bounds on $\|\boldsymbol{\eta}_n\|_2$ and $\|\boldsymbol{\eta}_n\|_\infty$ will be useful throughout this section.

As a consequence of expression (C.56) and the fact that $\frac{a^{\delta_{12}}}{\sqrt{n_{a,j}}} \to 0$ under (3.6),

$$\frac{\|\boldsymbol{\eta}_n\|_2}{a(n)^{\delta_{12}}} = \sqrt{\mathbb{E}\left[\left(\frac{W_n^{(12)}(a(n)2^j) - 1}{a(n)^{\delta_{12}}}\right)^2\right]} \sim \sqrt{\frac{a(n)}{n}}\, C, \quad n \to \infty, \tag{C.21}$$

for some $C \geq 0$. In particular,

$$W_n^{(12)}(a2^j) \stackrel{L^2(P)}{\longrightarrow} 1, \quad n \to \infty. \tag{C.22}$$

Moreover, expression (C.6) in Lemma C.1(iii), implies that

$$\|\boldsymbol{\eta}_n\|_\infty = \frac{\|\Lambda_S\|}{n_{a,j}} = \frac{1}{n_{a,j}}\|\Lambda_{\mathcal{Y}}^{1/2}O^*\Pi_n O\Lambda_{\mathcal{Y}}^{1/2}\| \leq \frac{\|\Lambda_{\mathcal{Y}}^{1/2}\|^2}{n_{a,j}} = \max_{i=1,\dots,\Upsilon(n)} \frac{\lambda_{i,\mathcal{Y}}}{n_{a,j}} \leq C' \frac{a(n)^{2(\max\{h_1,h_2\}-h_{12})+1}}{n} \tag{C.23}$$

for some $C' \in \mathbb{R}$.

Remark C.4. Note that any moment of $\frac{W_n^{(12)}(a(n)2^j)-1}{a^{\delta_{12}}}$ is bounded in n, i.e.,

$$\left|\mathbb{E}\left[\frac{W_n^{(12)}(a(n)2^j) - 1}{a(n)^{\delta_{12}}}\right]^\kappa\right| = O(1), \quad \kappa \in \mathbb{N}. \tag{C.24}$$

In fact, for even $\kappa \in \mathbb{N}$, expressions (C.19) and (C.23) imply that the left-hand side of (C.24) is bounded by

$$\sum_{i_1,\dots,i_\kappa \in i_+(n)\cup i_-(n)} \frac{\eta_{i_1,n}\dots\eta_{i_\kappa,n}}{a(n)^{\kappa\delta_{12}}}\left|\mathbb{E}\left[(Z_{i_1}^2 - 1)\dots(Z_{i_\kappa}^2 - 1)\right]\right| \leq \frac{C(\mathbb{E}\left[(Z_1^2-1)^2\right])^{\kappa/2}}{a(n)^{\kappa\delta_{12}}} \sum_{i_1,\dots,i_{\kappa/2}\in i_+(n)\cup i_-(n)} \eta_{i_1,n}^2\dots\eta_{i_{\kappa/2},n}^2$$

$$\leq \frac{C(\mathbb{E}\left[(Z_1^2-1)^2\right])^{\kappa/2}}{a(n)^{\kappa\delta_{12}}} n_{a,j}^{\kappa/2}\|\boldsymbol{\eta}_n\|_\infty^\kappa \leq \frac{C'}{a(n)^{\kappa\delta_{12}}}\left(\frac{n}{a}\right)^{\kappa/2}\left(\frac{a^{2(\max\{h_1,h_2\}-h_{12})+1}}{n}\right)^\kappa = \frac{C'}{a(n)^{\kappa\delta_{12}}}\left(\frac{a^{4(\max\{h_1,h_2\}-h_{12})+1}}{n}\right)^{\kappa/2} = O(1),$$

where the last equality is a consequence of (3.6) (a similar derivation holds for odd κ). Expression (C.24) will be used in the proof of Lemma C.4.

The following lemma provides a concentration inequality for centered quadratic forms, and corresponds to Lemma 1 in [60] (see also [61], Lemma 8, and [62], p. 39). It will be used in the ensuing Lemma C.3 to establish a bound on the rate of convergence to zero of the probabilities $P(W_n^{(12)}(a2^j) \leq r)$ and $P(W_n^{(12)}(a(n)2^j) \geq r')$, where $r < 1/2 < 3/2 < r'$ (under (.2)).

Lemma C.2 ([60]). *Let $Z_1, \ldots, Z_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $\eta_1, \ldots, \eta_n \geq 0$, not all zero. Let $\|\boldsymbol{\eta}\|_2$ and $\|\boldsymbol{\eta}\|_\infty$ be the Euclidean square and sup norms of the vector $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)^*$. Also, define the random variable $X = \sum_{i=1}^n \eta_{i,n}(Z_i^2 - 1)$. Then, for every $x > 0$,*

$$P(X \geq 2\|\boldsymbol{\eta}\|_2 \sqrt{x} + 2\|\boldsymbol{\eta}\|_\infty x) \leq \exp(-x), \tag{C.25}$$

$$P(X \leq -2\|\boldsymbol{\eta}\|_2 \sqrt{x}) \leq \exp(-x). \tag{C.26}$$

Lemma C.3. *Let $W_n^{(12)}(a2^j)$ be as in (C.19), and fix $r < 1/2 < 3/2 < r'$. Then, for any $0 < \xi < 1$,*

$$P(W_n^{(12)}(a(n)2^j) \leq r) \leq \exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}, \tag{C.27}$$

$$P(W_n^{(12)}(a(n)2^j) \geq r') \leq \exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}, \tag{C.28}$$

for large enough n.

Proof. Expression (C.19) implies that

$$P(W_n^{(12)}(a2^j) \leq r) = P\Big(\sum_{i \in i_+(n)} \eta_{i,n}(Z_i^2 - 1) \leq -1 + r + \sum_{i \in i_-(n)} \eta_{i,n}(Z_i^2 - 1)\Big).$$

For notational simplicity, let $X_{n_{a,j}} = \sum_{i \in i_+(n)} \eta_{i,n}(Z_i^2 - 1)$ and $Y_{n_{a,j}} = \sum_{i \in i_-(n)} \eta_{i,n}(Z_i^2 - 1)$, which are zero mean random variables. Let

$$\boldsymbol{\eta}_{1,n} = (\eta_{i,n})_{i \in i_+(n)}, \quad \boldsymbol{\eta}_{2,n} = (\eta_{i,n})_{i \in i_-(n)}.$$

By (C.21),

$$\begin{aligned}&\max\{\|\boldsymbol{\eta}_{1,n}\|_2^2, \|\boldsymbol{\eta}_{2,n}\|_2^2\} \leq \|\boldsymbol{\eta}_n\|_2^2 \sim C\frac{a^{2(h_1+h_2)-4h_{12}+1}}{n},\\ &\max\{\|\boldsymbol{\eta}_{1,n}\|_\infty, \|\boldsymbol{\eta}_{2,n}\|_\infty\} \leq \|\boldsymbol{\eta}_n\|_\infty \leq \|\boldsymbol{\eta}_n\|_2,\end{aligned} \tag{C.29}$$

for a constant $C \geq 0$. Moreover, in view of (.2), $\|\boldsymbol{\eta}_{1,n}\|_2 > 0$, $n \in \mathbb{N}$. Suppose, without loss of generality, that

$$\|\boldsymbol{\eta}_{2,n}\|_2 > 0, \quad n \in \mathbb{N}$$

(otherwise, $Y_{n_{a,j}} = 0$ a.s. for n such that $\|\boldsymbol{\eta}_{2,n}\|_2 = 0$). Fix a constant $r < \zeta < 1$. By the independence of $X_{n_{a,j}}$ and $Y_{n_{a,j}}$,

$$P(X_{n_{a,j}} \leq -1 + r + Y_{n_{a,j}}) = \Big\{\int_{-\infty}^{1-\zeta} + \int_{1-\zeta}^{\infty}\Big\} P(X_{n_{a,j}} \leq -1 + r + y) f_{Y_{n_{a,j}}}(y)dy, \tag{C.30}$$

where $f_{Y_{n_{a,j}}}$ is the density function of $Y_{n_{a,j}}$. The first integral in (C.30) is bounded from above by $P(X_{n_{a,j}} \leq -\zeta + r)P(Y_{n_{a,j}} \leq 1 - \zeta)$. Moreover, since $-\zeta + r < 0$, then

$$\begin{aligned}P(X_{n_{a,j}} \leq -\zeta + r) &= P\Big(\sum_{i \in i_+(n)} \eta_{i,n}(Z_i^2 - 1) \leq -\zeta + r\Big)\\ &= P\Big(\sum_{i \in i_+(n)} \eta_{i,n}(Z_i^2 - 1) \leq -2\|\boldsymbol{\eta}_{1,n}\|_2\sqrt{\frac{(\zeta - r)^2}{4\|\boldsymbol{\eta}_{1,n}\|_2^2}}\Big) \leq \exp\Big\{-\frac{(\zeta - 1/2)^2}{4\|\boldsymbol{\eta}_{1,n}\|_2^2}\Big\}\\ &\leq \exp\Big\{-C\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big\}.\end{aligned} \tag{C.31}$$

In (C.31), C does not depend on r and the last two inequalities are a consequence of (C.26) and (C.29), respectively, where

$$2\max\{h_{q_1} + h_{q_2}, 2h_{q_1q_2}\} = 2(h_{q_1} + h_{q_2}), \quad q_1, q_2 = 1, \ldots, m, \tag{C.32}$$

stems from (2.10). On the other hand, for any $0 < \xi < 1$ and large enough n the second integral in (C.30) is bounded from above by

$$\int_{1-\zeta}^{\infty} f_{Y_{n_{a,j}}}(y)dy = P(Y_{n_{a,j}} > 1 - \zeta) = P\Big(\sum_{i \in i_-(n)} \eta_{i,n}(Z_i^2 - 1) > 1 - \zeta\Big). \tag{C.33}$$

Note that, by (C.29),

$$\|\boldsymbol{\eta}_{2,n}\|_2\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{\frac{1-\xi}{2}} + \|\boldsymbol{\eta}_{2,n}\|_\infty\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi} \to 0,$$

for any $0 < \xi < 1$, as $n \to \infty$. Consequently, for large enough n, (C.33) is bounded by

$$\begin{aligned}&P\Big(\sum_{i \in i_-(n)} \eta_{i,n}(Z_i^2 - 1) > 2\|\boldsymbol{\eta}_{2,n}\|_2\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{\frac{1-\xi}{2}} + 2\|\boldsymbol{\eta}_{2,n}\|_\infty\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big)\\ &\quad \leq \exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\},\end{aligned} \tag{C.34}$$

where the last inequality follows from (C.25). From (C.31) and (C.34), since C does not depend on r, we obtain (C.28).

We now turn to (C.28). Fix $\frac{1}{2} < \zeta' < 1 < r'$. Then,

$$P(W_n^{(12)}(a2^j) \geq r') = P(X_{n_{a,j}} \geq r' - 1 + Y_{n_{a,j}})$$
$$= \left\{ \int_{-\infty}^{\zeta'-1} + \int_{\zeta'-1}^{\infty} \right\} P(X_{n_{a,j}} \geq r' - 1 + y) f_{Y_{n_{a,j}}}(y) dy. \quad (C.35)$$

The first and second integral terms in (C.35) are bounded by, respectively,

$$\int_{-\infty}^{\zeta'-1} f_{Y_{n_{a,j}}}(y) dy = P(Y_{n_{a,j}} \leq \zeta' - 1)$$

and

$$P(X_{n_{a,j}} \geq r' + \zeta' - 2),$$

where $\zeta' - 1 < 0 < r' + \zeta' - 2$. Therefore, an argument similar to that for (C.27) can be applied to conclude that (C.28) also holds. □

C.2. Asymptotic covariances

We will establish Theorem 4.1 at the end of this section, after proving Lemmas C.4–C.7. The latter establish the asymptotic behavior of the first four moments of the $\mathbb{R}$-valued random variables $W_n^{(q_1 q_2)}(a(n)2^j)$, $W_n^{(q_3 q_4)}(a(n)2^j)$. So, consider the function

$$\log |S_n^{(q_1 q_2)}(a(n)2^j)| \, 1_{\{|W_n^{(q_1 q_2)}(a(n)2^j)| > r\}}, \qquad 1 \leq q_1 \leq q_2 \leq m, \quad 0 < r < \frac{1}{2}, \quad (C.36)$$

where the truncation works as a regularization of the log function around the origin. In the event $|W_n^{(q_1 q_2)}(a(n)2^j)| \leq r$, we interpret that

$$\log |S_n^{(q_1 q_2)}(a(n)2^j)| \, 1_{\{|W_n^{(q_1 q_2)}(a(n)2^j)| > r\}} = 0 \quad \text{a.s.}$$

Throughout this section, we will make use of the Isserlis theorem (e.g., [63]). For a zero mean, Gaussian random vector $\mathbf{Z} = (Z_1, \ldots, Z_m)^* \in \mathbb{R}^m$, the theorem states that

$$\mathbb{E}[Z_1 \ldots Z_{2k}] = \sum \prod \mathbb{E}\left[Z_i Z_j\right], \quad \mathbb{E}[Z_1 \ldots Z_{2k+1}] = 0, \quad k = 1, \ldots, \lfloor m/2 \rfloor. \quad (C.37)$$

The notation $\sum \prod$ stands for adding over all possible k-fold products of pairs $\mathbb{E}\left[Z_i Z_j\right]$, where the indices partition the set $1, \ldots, 2k$.

The following lemma shows that the high order centered cross moments of $W_n^{(12)}(a(n)2^j)$, $W_n^{(34)}(a(n)2^j)$ are negligible with respect to their low order counterparts.

Lemma C.4. *Let $\kappa_1, \kappa_2 \in \mathbb{N} \cup \{0\}$, $\kappa_1 + \kappa_2 \geq 3$, and fix $0 < r < 1/2$. Then, as $n \to \infty$,*

$$\mathbb{E}\left[\left(\frac{W_n^{(12)}(a(n)2^j) - 1}{a(n)^{\delta_{12}}} \right)^{\kappa_1} \left(\frac{W_n^{(34)}(a(n)2^{j'}) - 1}{a(n)^{\delta_{34}}} \right)^{\kappa_2} 1_{\{\min\{|W_n^{(12)}(a(n)2^{j'})|, |W_n^{(34)}(a(n)2^{j'})|\} > r\}} \right] = o\left[\left(\frac{a(n)}{n} \right)^2 \right]. \quad (C.38)$$

Proof. For each case $\kappa_1, \kappa_2 \in \mathbb{N}$, $\kappa_1 + \kappa_2 \geq 3$, we first show that

$$\mathbb{E}\left[\left(\frac{W_n^{(12)}(a2^j) - 1}{a^{\delta_{12}}} \right)^{\kappa_1} \left(\frac{W_n^{(34)}(a2^{j'}) - 1}{a^{\delta_{34}}} \right)^{\kappa_2} \right] = o\left[\left(\frac{a}{n} \right)^2 \right], \quad (C.39)$$

i.e., without indicators, and then extend the claim to the full expression (C.38).

Consider the case where $\kappa_1 = 1$ and $\kappa_2 = 2$. Then, the left-hand side of (C.39) can be rewritten as

$$\frac{1}{a^{\delta_{12} + 2\delta_{34}}} \frac{1}{n_{a,j} n_{a,j'}^2} \mathbb{E}\left[\sum_{k=1}^{n_{a,j}} \sum_{k_1'=1}^{n_{a,j'}} \sum_{k_2'=1}^{n_{a,j'}} \left(\frac{d_1(a2^j, k) d_2(a2^j, k)}{\varrho^{(12)}(a2^j)} - 1 \right) \left(\frac{d_3(a2^{j'}, k_1') d_4(a2^{j'}, k_1')}{\varrho^{(34)}(a2^{j'})} - 1 \right) \left(\frac{d_3(a2^{j'}, k_2') d_4(a2^{j'}, k_2')}{\varrho^{(34)}(a2^{j'})} - 1 \right) \right]. \quad (C.40)$$

Starting from assumption (.2), for notational simplicity relabel the generic terms under the summation sign in (C.49) as $X_1 = d_1(a2^j, k)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_2 = d_2(a2^j, k)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_3 = d_3(a2^{j'}, k_1')/\sqrt{\varrho^{(34)}(a2^{j'})}$, $X_4 = d_4(a2^{j'}, k_1')/\sqrt{\varrho^{(34)}(a2^{j'})}$, $X_5 = d_3(a2^{j'}, k_2')/\sqrt{\varrho^{(34)}(a2^{j'})}$, $X_6 = d_4(a2^{j'}, k_2')/\sqrt{\varrho^{(34)}(a2^{j'})}$. Then,

$$\mathbb{E}[(X_1 X_2 - 1)(X_3 X_4 - 1)(X_5 X_6 - 1)] = \mathbb{E}[X_1 X_2 X_3 X_4 X_5 X_6] - \{\mathbb{E}[X_1 X_2 X_3 X_4] + \mathbb{E}[X_1 X_2 X_5 X_6] + \mathbb{E}[X_3 X_4 X_5 X_6]\} + 2. \quad (C.41)$$

By applying the Isserlis relation (C.37) to the six-fold and four-fold products in (C.41), the latter expression becomes

$$\mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_5]\,\mathbb{E}[X_4X_6] + \mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_6]\,\mathbb{E}[X_4X_5] + \mathbb{E}[X_1X_4]\,\mathbb{E}[X_2X_5]\,\mathbb{E}[X_3X_6] + \mathbb{E}[X_1X_4]\,\mathbb{E}[X_2X_6]\,\mathbb{E}[X_3X_5]$$
$$+ \mathbb{E}[X_1X_5]\,\mathbb{E}[X_2X_3]\,\mathbb{E}[X_4X_6] + \mathbb{E}[X_1X_5]\,\mathbb{E}[X_2X_4]\,\mathbb{E}[X_3X_6] + \mathbb{E}[X_1X_6]\,\mathbb{E}[X_2X_3]\,\mathbb{E}[X_4X_5] + \mathbb{E}[X_1X_6]\,\mathbb{E}[X_2X_4]\,\mathbb{E}[X_3X_5]\,. \quad (C.42)$$

The asymptotic behavior of the summation of each term in the seven-fold sum (C.42) can be established in the same way, so we only study the first one. Up to a constant, the latter is asymptotically equivalent to

$$\frac{a^{(2h_{13}+2h_{23}+2h_4)-(2h_{12}+4h_{34})}}{a^{\delta_{12}+2\delta_{34}}\ n_{a,j}}\frac{1}{n^2_{a,j'}}\sum_{k=1}^{n_{a,j}}\sum_{k'_1=1}^{n_{a,j'}}\sum_{k'_2=1}^{n_{a,j'}}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k'_1)\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]}{a^{2h_{23}}}\frac{\mathbb{E}\Big[d_4(a2^{j'},k'_1)d_4(a2^{j'},k'_2)\Big]}{a^{2h_4}}$$

$$=\frac{a^{2(h_{13}+h_{23}+h_4)}}{a^{h_1+h_2+2(h_3+h_4)}\ n_{a,j}}\frac{1}{n^2_{a,j'}}\sum_{k'_1=1}^{n_{a,j'}}\sum_{k'_2=1}^{n_{a,j'}}\frac{\mathbb{E}\Big[d_4(a2^{j'},k'_1)d_4(a2^{j'},k'_2)\Big]}{a^{2h_4}}\cdot\Big\{\sum_{k=1}^{n_{a,j}}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k'_1)\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]}{a^{2h_{23}}}\Big\}. \tag{C.43}$$

However, the summation in k in expression (C.43) is bounded by

$$\Big|\sum_{k=1}^{n_{a,j}}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k'_1)\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]}{a^{2h_{23}}}\Big|$$

$$\le\sum_{k=1}^{n_{a,j}}\Big(\Big|\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k'_1)\Big]-\Phi^{jj'}_{13}(a(2^jk-2^{j'}k'_1))}{a^{2h_{13}}}\Big|+\Big|\frac{\Phi^{jj'}_{13}(a(2^jk-2^{j'}k'_1))}{a^{2h_{13}}}\Big|\Big)$$

$$\Big(\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]-\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|+\Big|\frac{\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|\Big)$$

$$\le\sum_{k=1}^{n_{a,j}}\Big\{\Big|\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k'_1)\Big]-\Phi^{jj'}_{13}(a(2^jk-2^{j'}k'_1))}{a^{2h_{13}}}\Big|\cdot\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]-\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|$$

$$+\Big|\frac{\Phi^{jj'}_{13}(a(2^jk-2^{j'}k'_1))}{a^{2h_{13}}}\Big|\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]-\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|$$

$$+\Big|\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k'_1)\Big]-\Phi^{jj'}_{13}(a(2^jk-2^{j'}k'_1))}{a^{2h_{13}}}\Big|\Big|\frac{\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|$$

$$+\Big|\frac{\Phi^{jj'}_{13}(a(2^jk-2^{j'}k'_1))}{a^{2h_{13}}}\Big|\Big|\frac{\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|\Big\}\le C, \tag{C.44}$$

where C does not depend on k. To justify the last inequality, we only look at the second term in (C.44), since the remaining terms can be analyzed in a similar way. It suffices to proceed as in [53], in particular around expression (B.31) in the latter reference. Indeed, starting from (3.2), suppose without loss of generality that

$$\text{supp}(\psi)=[0,1].$$

For $0<\varepsilon<1/2$, decompose

$$\sum_{k=1}^{n_{a,j}}\Big|\Phi^{jj'}_{13}(2^jk-2^{j'}k'_1)\Big|\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]-\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|$$

$$=\sum_{k=1}^{n_{a,j}}\Big(1_{\Big\{\frac{\max\{2^j,2^{j'}\}}{|2^jk-2^{j'}k'_1|}>\varepsilon\Big\}}+1_{\Big\{\frac{\max\{2^j,2^{j'}\}}{|2^jk-2^{j'}k'_1|}\le\varepsilon\Big\}}\Big)\Big|\Phi^{jj'}_{13}(2^jk-2^{j'}k'_1)\Big|\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]-\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|. \tag{C.45}$$

The first sum term in (C.45) only contains a finite number of terms, where such number does not depend on k or k'_1. Moreover,

$$\max\Big\{|\Phi^{jj'}_{13}(2^jk-2^{j'}k'_1)|,\ \Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k'_2)\Big]-\Phi^{jj'}_{23}(a(2^jk-2^{j'}k'_2))}{a^{2h_{23}}}\Big|\Big\}\le C,$$

where C does not depend k, k'_1 or k''_2. The latter statement follows from the fact that $\Phi^{jj'}_{13}$ is, up to a change of sign, the wavelet variance of a fBm and from (A.24). Moreover, by Proposition II.2 in [47] and again by (A.24), the second sum term in (C.45) is bounded by $C\sum_{z\in\mathbb{Z}\backslash\{0\}}z^{-4}$. This shows that (C.45) is bounded by a constant not depending on k, k'_1 or k'_2. Hence, (C.44) holds.

Consequently, up to a constant the absolute value of (C.43) is bounded from above by

$$C'\frac{a^{2(h_{13}+h_{23}+h_4)+2}}{a^{h_1+h_2+2(h_3+h_4)}\ n^2}\frac{1}{n_{a,j'}}\sum_{k'_1=1}^{n_{a,j'}}\sum_{k'_2=1}^{n_{a,j'}}\Big|\frac{\mathbb{E}\Big[d_4(a2^{j'},k'_1)d_4(a2^{j'},k'_2)\Big]}{a^{2h_4}}\Big|\le C''\Big(\frac{a}{n}\Big)^2, \tag{C.46}$$

where we used the fact that $\frac{1}{a^{\delta_{13}+\delta_{23}}}\le 1$. This establishes (C.39).

So, rewrite

$$\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2\Big]-\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)1_{\{W_n^{(12)}(a2^j)>r\}}\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2 1_{\{W_n^{(34)}(a2^{j'})>r\}}\Big]$$

$$=\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2\Big\{1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})>r\}}$$

$$+\ 1_{\{W_n^{(12)}(a2^j)>r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}+1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}\Big\}\Big]. \tag{C.47}$$

The asymptotic behavior of every term on the right-hand side of (C.47) can be established in the same way, so we only look at the first one. For any $0<\xi<1$, the Cauchy–Schwarz inequality, Lemma C.3 and expression (C.39) yield

$$\Big|\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2 1_{\{W_n^{(34)}(a2^{j'})>r\}}\Big]\Big|$$

$$\le\sqrt{\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^2\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^4\Big]}\sqrt{P(W_n^{(12)}(a2^j)\le r)}$$

$$\le C\Big(\frac{a}{n}\Big)\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big\}, \tag{C.48}$$

where C does not depend on r. Moreover, under (3.6),

$$\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big\}=o\Big(\frac{a}{n}\Big).$$

Therefore, (C.38) holds. The remaining cases where $\kappa_1+\kappa_2=3$ can be handled similarly.

Now consider the case where $\kappa_1=4$ and $\kappa_2=0$. Then, the left-hand side of (C.39) can be rewritten as

$$\frac{1}{a^{4\delta_{12}}}\frac{1}{n_{a,j}^4}\mathbb{E}\Big[\sum_{k_1=1}^{n_{a,j}}\sum_{k_2=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\Big(\frac{d_1(a2^j,k_1)d_2(a2^j,k_1)}{\varrho^{(12)}(a2^j)}-1\Big)\Big(\frac{d_1(a2^j,k_2')d_2(a2^j,k_2)}{\varrho^{(12)}(a2^j)}-1\Big)$$

$$\Big(\frac{d_1(a2^j,k_3)d_2(a2^j,k_3)}{\varrho^{(12)}(a2^j)}-1\Big)\Big(\frac{d_1(a2^j,k_4)d_2(a2^j,k_4)}{\varrho^{(12)}(a2^j)}-1\Big)\Big]. \tag{C.49}$$

Starting from the assumption (.2), for notational simplicity relabel the generic terms under the summation sign in (C.49) as $X_1=d_1(a2^j,k_1)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_2=d_2(a2^j,k_1)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_3=d_1(a2^j,k_2)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_4=d_2(a2^j,k_2)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_5=d_1(a2^j,k_3)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_6=d_2(a2^j,k_3)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_7=d_1(a2^j,k_4)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_8=d_2(a2^j,k_4)/\sqrt{\varrho^{(12)}(a2^j)}$. Then,

$$\mathbb{E}[(X_1X_2-1)(X_3X_4-1)(X_5X_6-1)(X_7X_8-1)]$$

$$=\Big\{\mathbb{E}[X_1X_2X_3X_4X_5X_6X_7X_8]\Big\}$$

$$-\Big\{\mathbb{E}[X_3X_4X_5X_6X_7X_8]+\mathbb{E}[X_1X_2X_5X_6X_7X_8]+\mathbb{E}[X_1X_2X_3X_4X_7X_8]+\mathbb{E}[X_1X_2X_3X_4X_5X_6]\Big\}$$

$$+\Big\{\mathbb{E}[X_5X_6X_7X_8]+\mathbb{E}[X_3X_4X_7X_8]+\mathbb{E}[X_1X_2X_7X_8]$$

$$+\mathbb{E}[X_1X_2X_3X_4]+\mathbb{E}[X_1X_2X_5X_6]+\mathbb{E}[X_3X_4X_5X_6]\Big\}$$

$$-\Big\{\mathbb{E}[X_1X_2]+\mathbb{E}[X_3X_4]+\mathbb{E}[X_5X_6]+\mathbb{E}[X_7X_8]-1\Big\}. \tag{C.50}$$

Namely, we arrive at an expression with four terms between braces. By relabeling them A, B, C and D, respectively, we can write

$$\mathbb{E}[(X_1X_2-1)(X_3X_4-1)(X_5X_6-1)(X_7X_8-1)]=A-B+C-D. \tag{C.51}$$

We claim that, by an application of the Isserlis theorem, the expression (C.51) can be written as a sum of products of the form

$$\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\,\mathbb{E}\big[X_{l_5}X_{l_6}\big]\,\mathbb{E}\big[X_{l_7}X_{l_8}\big], \tag{C.52}$$

where $l_1<l_3<l_5<l_7$ and no pair $\mathbb{E}[X_\cdot X_\cdot]$ has consecutive indices (the latter condition implies that no pair is identically 1 with respect to summation in k_1, k_2, k_3 and k_4 in expression (C.49)).

In fact, recall that $\mathbb{E}[X_1X_2]=\mathbb{E}[X_3X_4]=\mathbb{E}[X_5X_6]=\mathbb{E}[X_7X_8]=1$. Note that the Isserlis theorem breaks up any term in the original sum (C.50) into a product of expectations of the form $\mathbb{E}[X_\cdot X_\cdot]$, where each product can only be identically 1 (with respect to summation in k_1, k_2, k_3 and k_4) if each odd index l is paired with $l+1$ (e.g., after decomposing A by Isserlis, the only term which is identically 1 is $\mathbb{E}[X_1X_2]\,\mathbb{E}[X_3X_4]\,\mathbb{E}[X_5X_6]\,\mathbb{E}[X_7X_8]$). Hence, after applying the Isserlis theorem to A, B and C, each if the 11 terms contained in the sum $A-B+C$ ends up with exactly one term identically equal to 1. Therefore, we obtain $(1-4+6-3)\times 1=0$ the right-hand side of (C.51). Hence, the full expression (C.51) contains no term identically 1, and, in addition, we no longer need to account for D. Next, note that, after applying the Isserlis theorem to (C.50), no resulting term can be of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\times 1^3$, where 1 stands for "identically 1" with respect to summation in some index $k_\cdot$ (and $\mathbb{E}\big[X_{l_1}X_{l_2}\big]$ is not identically 1). So, consider terms of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\times 1^2$, where, again, 1 stands for "identically 1" (and $\mathbb{E}\big[X_{l_1}X_{l_2}\big]$ and $\mathbb{E}\big[X_{l_3}X_{l_4}\big]$ are not identically 1). For the sake of clarity, consider the particular term $\mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_4]\times 1^2$. After applying the Isserlis theorem, the latter ends up appearing (once) in the expansions of A, of $\mathbb{E}[X_1X_2X_3X_4X_7X_8]$ and $\mathbb{E}[X_1X_2X_3X_4X_5X_6]$ in B, and of $\mathbb{E}[X_1X_2X_3X_4]$ in C. Thus, on the right-hand side of (C.51), we obtain $(1-2+1)\times(\mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_4]\times 1^2)=0$. Since the same is true for every other term of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\times 1^2$, the full expression (C.51) contains no such terms and, in addition, we no longer need to account for C. Now consider terms of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\,\mathbb{E}\big[X_{l_5}X_{l_6}\big]\times 1$, where 1 stands for "identically 1" (and $\mathbb{E}\big[X_{l_1}X_{l_2}\big]$, $\mathbb{E}\big[X_{l_3}X_{l_4}\big]$ and $\mathbb{E}\big[X_{l_5}X_{l_6}\big]$ are not identically 1). Note that, after applying the Isserlis theorem, B ends up containing the sum of all possible terms of this form, and each one appears only once. Since they also all appear only once in the expansion of A, the terms stemming from A and B cancel. Therefore, (C.50) is made up of the sum of terms of the form (C.52), as claimed.

We are now in a position to establish the asymptotic behavior of (C.49). For notational simplicity, we focus on the particular term of the form $\mathbb{E}[X_1X_5]\,\mathbb{E}[X_2X_6]\,\mathbb{E}[X_3X_7]\,\mathbb{E}[X_4X_8]$. After taking summations, the resulting expression is asymptotically equivalent to

$$C\,\frac{a^{4h_1+4h_2}}{a^{4\delta_{12}+8h_{12}}}\,\frac{1}{n_{a,j}^4}\sum_{k_1=1}^{n_{a,j}}\sum_{k_2=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\frac{\mathbb{E}\big[d_1(a2^j,k_1)d_1(a2^j,k_3)\big]}{a^{2h_1}}\,\frac{\mathbb{E}\big[d_2(a2^j,k_1)d_2(a2^j,k_3)\big]}{a^{2h_2}}$$
$$\frac{\mathbb{E}\big[d_1(a2^j,k_2)d_1(a2^j,k_4)\big]}{a^{2h_1}}\,\frac{\mathbb{E}\big[d_2(a2^j,k_2)d_2(a2^j,k_4)\big]}{a^{2h_2}}$$
$$=\frac{C}{n_{a,j}^2}\Big\{\frac{1}{n_{a,j}}\sum_{k_1=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\frac{\mathbb{E}\big[d_1(a2^j,k_1)d_1(a2^j,k_3)\big]}{a^{2h_1}}\,\frac{\mathbb{E}\big[d_2(a2^j,k_1)d_2(a2^j,k_3)\big]}{a^{2h_2}}$$
$$\times\frac{1}{n_{a,j}}\sum_{k_2=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\frac{\mathbb{E}\big[d_1(a2^j,k_2)d_1(a2^j,k_4)\big]}{a^{2h_1}}\,\frac{\mathbb{E}\big[d_2(a2^j,k_2)d_2(a2^j,k_4)\big]}{a^{2h_2}}\Big\}.$$

The terms $|\mathbb{E}\big[d_2(a2^j,k_1)d_2(a2^j,k_3)\big]/a^{2h_2}|$, $|\mathbb{E}\big[d_2(a2^j,k_2)d_2(a2^j,k_4)\big]/a^{2h_2}|$ are bounded in k_1, k_2, k_3 and k_4. Moreover, by a reasoning analogous to the one leading to (C.46), the averages $n_{a,j}^{-1}\sum_{k_1=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\mathbb{E}\big[d_1(a2^j,k_1)d_1(a2^j,k_3)\big]/a^{2h_1}$ and $n_{a,j}^{-1}\sum_{k_2=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\mathbb{E}\big[d_1(a2^j,k_2)d_1(a2^j,k_4)\big]/a^{2h_1}$ converge to constants as $n\to\infty$. This shows (C.39) for $\kappa_1=4$ and $\kappa_2=0$.

A simple adaptation of the argument for the case $\kappa_1=1$ and $\kappa_2=2$ (see (C.47)) shows that (C.38) also holds when $\kappa_1=4$ and $\kappa_2=0$, as claimed.

We now turn to the case where $\kappa_1>2$ and $\kappa_2>2$. Let $\zeta_0>1/2$ be a fixed constant. Then,

$$\Big|\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^{\kappa_1}\Big(\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big)^{\kappa_2}\Big]\Big|$$
$$\le\mathbb{E}\Big[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^{\kappa_1}\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}\Big(1_{\{|W_n^{(12)}(a2^j)-1|\ge\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}+1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}$$
$$+1_{\{|W_n^{(12)}(a2^j)-1|\ge\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|<\zeta_0\}}+1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|<\zeta_0\}}\Big)\Big].\tag{C.53}$$

However, by the Cauchy–Schwarz inequality,

$$\mathbb{E}\Big[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^{\kappa_1}\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}\Big]$$
$$\le\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1}\mathbb{E}\Big[\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}1_{\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}\Big]$$
$$\le\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1}\sqrt{\mathbb{E}\Big(\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big)^{2\kappa_2}P(|W_n^{(34)}(a2^j)-1|\ge\zeta_0)}$$
$$\le C\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1}\sqrt{P(|W_n^{(34)}(a2^j)-1|\ge\zeta_0)}=O\Big[\Big(\frac{a}{n}\Big)^2\Big],$$

where the last inequality and the equality are consequences of (C.24) and Lemma C.3, respectively. The same bound applies to the first and third terms in the sum (C.53). Therefore, the latter is bounded by

$$O\Big[\Big(\frac{a}{n}\Big)^2\Big]+\mathbb{E}\Big[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^{\kappa_1}\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|<\zeta_0\}}\Big]$$
$$\le O\Big[\Big(\frac{a}{n}\Big)^2\Big]+\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1-2}\Big(\frac{\zeta_0}{a^{\delta_{34}}}\Big)^{\kappa_2-2}\mathbb{E}\Big[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^2\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^2\Big]$$
$$\le O\Big[\Big(\frac{a}{n}\Big)^2\Big]+\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1-2}\Big(\frac{\zeta_0}{a^{\delta_{34}}}\Big)^{\kappa_2-2}\sqrt{\mathbb{E}\Big[\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big]^4}\sqrt{\mathbb{E}\Big[\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big]^4}$$
$$=O\Big[\Big(\frac{a}{n}\Big)^2\Big].\tag{C.54}$$

In (C.54), the first inequality is a consequence of the Cauchy–Schwarz inequality, and the second inequality follows from (C.38) for $\kappa_1 = 4$ and $\kappa_2 = 0$ or $\kappa_1 = 0$ and $\kappa_2 = 4$. In addition, by a simple adaptation of the argument for the case $\kappa_1 = 1$ and $\kappa_2 = 2$ (see (C.47)), (C.38) also holds for $\kappa_1 > 2$ and $\kappa_2 > 2$.

The cases where $\kappa_1 = 1$ and $\kappa_2 \geq 4$ or $\kappa_1 \geq 4$ and $\kappa_2 = 1$ can be tackled by a similar procedure. □

The following lemma expresses, up to a residual term, the cross-covariance (first cross-moment) between sample wavelet variances in terms of the functions (3.14).

Lemma C.5. *Let $\Phi^{jj'}_{\cdot\cdot}(z)$ and n_* be as in* (3.14) and (4.1)*, respectively. For $0 < r < 1/2$,*

$$\mathbb{E}\Big[(W_n^{(12)}(a(n)2^j) - 1)1_{\{|W_n^{(12)}(a(n)2^j)|>r\}}(W_n^{(34)}(a(n)2^{j'}) - 1)1_{\{|W_n^{(34)}(a(n)2^{j'})|>r\}}\Big]$$
$$= \frac{2^{-(j+j')}}{\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)\{1 + O(a(n)^{-\varpi_0})\}^2}\Big\{\frac{a(n)^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\Phi^{jj'}_{13}(2^jk - 2^{j'}k')\Phi^{jj'}_{24}(2^jk - 2^{j'}k')\Big]$$
$$+ \frac{a(n)^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\Phi^{jj'}_{14}(2^jk - 2^{j'}k')\Phi^{jj'}_{23}(2^jk - 2^{j'}k')\Big] + o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{12}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big)\Big\}, \tag{C.55}$$

as $n \to \infty$.

Proof. As in the proof of Lemma C.4, we first drop the indicator functions on the left-hand side of (C.55) and investigate the limit. We will show that

$$\mathrm{Cov}\Big[W_n^{(12)}(a2^j), W_n^{(34)}(a2^{j'})\Big] = \frac{2^{-(j+j')}}{[\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)(1 + O(a^{-\varpi_0}))^2]}$$
$$\cdot\Big\{\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\Phi^{jj'}_{13}(2^jk - 2^{j'}k')\Phi^{jj'}_{24}(2^jk - 2^{j'}k')$$
$$+ \frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\Phi^{jj'}_{14}(2^jk - 2^{j'}k')\Phi^{jj'}_{23}(2^jk - 2^{j'}k')$$
$$+ o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{14}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big)\Big\}. \tag{C.56}$$

In fact, the left-hand side of (C.56) can be written as

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j) - 1)(W_n^{(34)}(a2^{j'}) - 1)\Big] = \frac{1}{n_{a,j}n_{a,j'}}\sum_{k=1}^{n_{a,j}}\sum_{k'=1}^{n_{a,j'}}\Big\{\mathbb{E}\Big[\frac{d_1(a2^j,k)d_2(a2^j,k)}{\varrho^{(12)}(a2^j)}\frac{d_3(a2^{j'},k')d_4(a2^{j'},k')}{\varrho^{(34)}(a2^{j'})}\Big] - 1\Big\}. \tag{C.57}$$

By the Isserlis relation (C.37), the first term in the argument of the sum (C.57) can be reexpressed as

$$\frac{\mathbb{E}\Big[d_1(a2^j,k)d_2(a2^j,k)d_3(a2^{j'},k')d_4(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}$$
$$= \Big\{1 + \frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})} + \frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}\Big\}. \tag{C.58}$$

By Lemma C.1(ii), and (C.58), we can rewrite (C.57) as

$$\frac{1}{n_{a,j}n_{a,j'}}\sum_{k=1}^{n_{a,j}}\sum_{k'=1}^{n_{a,j'}}\Big\{\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})} + \frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}\Big\}$$
$$= \frac{2^{-(j+j')}}{\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)(1 + O(a^{-\varpi_0}))^2}$$
$$\Big\{\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}}\Big]$$
$$+ \frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{14}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{23}}}\Big]\Big\}. \tag{C.59}$$

By adding and subtracting the counterparts $\Phi^{jj'}_{\cdot\cdot}(2^j k - 2^{j'} k')$ for each term, up to the factor $2^{-(j+j')}/\{\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)(1+O(a^{-\varpi_0}))^2\}$ the expression (C.59) can be written as

$$\begin{aligned}
&\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}} - \Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Big)\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}} - \Phi^{jj'}_{24}(2^j k - 2^{j'}k')\Big)\\
&+\Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}} - \Phi^{jj'}_{24}(2^j k - 2^{j'}k')\Big)\\
&+\Phi^{jj'}_{24}(2^j k - 2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}} - \Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Big) + \{\Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Phi^{jj'}_{24}(2^j k - 2^{j'}k')\}\Big]\\
&+\frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Big\{\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{14}}} - \Phi^{jj'}_{14}(2^j k - 2^{j'}k')\Big)\\
&\cdot\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{23}}} - \Phi^{j'j}_{23}(2^j k - 2^{j'}k')\Big)\\
&+\Phi^{jj'}_{14}(2^j k - 2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{14}}} - \Phi^{jj'}_{23}(2^j k - 2^{j'}k')\Big)\\
&+\Phi^{jj'}_{23}(2^j k - 2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{23}}} - \Phi^{jj'}_{14}(2^j k - 2^{j'}k')\Big) + \Phi^{jj'}_{14}(2^j k - 2^{j'}k')\Phi^{jj'}_{23}(2^j k - 2^{j'}k')\Big\}\Big]\\
&= \frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Phi^{jj'}_{24}(2^j k - 2^{j'}k')\\
&+\frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Phi^{jj'}_{14}(2^j k - 2^{j'}k')\Phi^{jj'}_{23}(2^j k - 2^{j'}k') + o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{14}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big).
\end{aligned} \tag{C.60}$$

It remains to justify the order of the error term in (C.60). So, by adapting the proof of (A.16), we obtain

$$\begin{aligned}
&\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\Big|\sum_{k=1}^{2^j n_*}\sum_{k'=1}^{2^{j'}n_*}\Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}} - \Phi^{jj'}_{24}(2^j k - 2^{j'}k')\Big)\Big|\\
&= \frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}o(1) = o\Big(\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big).
\end{aligned}$$

The same bound holds for

$$\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\Big|\sum_{k=1}^{2^j n_*}\sum_{k'=1}^{2^{j'}n_*}\Phi^{jj'}_{24}(2^j k - 2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}} - \Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Big)\Big|$$

and

$$\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\Big|\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}} - \Phi^{jj'}_{13}(2^j k - 2^{j'}k')\Big)\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}} - \Phi^{jj'}_{24}(2^j k - 2^{j'}k')\Big)\Big|.$$

By extending this analysis to the remaining terms of (C.60), we obtain analogous bounds and the error term $o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{14}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big)$, as claimed.

In view of (C.59) and (C.60), it suffices to show that the indicator functions on the left-hand side of (C.55) do not affect the approximation order. In fact,

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)(W_n^{(34)}(a2^{j'})-1)\Big]-\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}(W_n^{(34)}(a2^{j'})-1)1_{\{W_n^{(34)}(a2^{j'})>r\}}\Big]$$
$$=\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)(W_n^{(34)}(a2^{j'})-1)\cdot\Big(1_{\{W_n^{(12)}(a2^j)>r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}$$
$$+1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})>r\}}+1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}\Big)\Big]. \tag{C.61}$$

Define

$$h^*=\max_{q_1,q_2=1,2,3,4}h_{q_1q_2},\quad h_*=\min_{q_1,q_2=1,2,3,4}h_{q_1q_2}. \tag{C.62}$$

For $0<\xi'<1$, by the Cauchy–Schwarz inequality, expression (C.39) (from Lemma C.4) and Lemma C.3, the first term on the right-hand side of (C.61) is bounded by

$$\Big|\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}(W_n^{(34)}(a2^{j'})-1)1_{\{W_n^{(34)}(a2^{j'})\le r\}}\Big]\Big|$$
$$\le a^{\delta_{12}+\delta_{34}}\sqrt{\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^2\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2\Big]}\sqrt{P(W_n^{(34)}(a2^j)\le r)}$$
$$\le C\Big(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\Big)\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi'}\Big\}$$
$$=o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{12}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big),$$

where the constant C does not depend on r and the last equality is a consequence of (3.6). Similar bounds hold for the remaining terms on the right-hand side of (C.61). Therefore, the expression (C.55) follows. □

The following lemma describes the decay rate of the first individual truncated moment of the wavelet variance.

Lemma C.6. *For any* $0<\xi<1$ *and* $0<r<1/2$,

$$\max\Big\{\Big|\mathbb{E}\Big[\{W_n^{(12)}(a(n)2^j)-1\}1_{\{W_n^{(12)}(a(n)2^j)\le r\}}\Big]\Big|;\ \Big|\mathbb{E}\Big[\{W_n^{(12)}(a(n)2^j)-1\}1_{\{W_n^{(12)}(a(n)2^j)>r\}}\Big]\Big|\Big\}$$
$$=O\Big(\frac{a(n)^{h_1+h_2-2h_{12}+1/2}}{\sqrt{n}}\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}\Big). \tag{C.63}$$

Proof. Notice that

$$0=\mathbb{E}\big[W_n^{(12)}(a2^j)-1\big]=\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}\Big]+\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big].$$

Hence,

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}\Big]=-\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big].$$

However, by the Cauchy–Schwarz inequality,

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big]\Big|\le\sqrt{\mathrm{Var}\Big[W_n^{(12)}(a2^j)\Big]}\sqrt{P(W_n^{(12)}(a2^j)\le r)}. \tag{C.64}$$

By expressions (C.64), (C.22) and Lemma C.3, the expression (C.63) follows. □

The following lemma establishes the decay of the covariances between truncated terms (C.36) and indicators involving wavelet variances, or between indicators only.

Lemma C.7. *For any* $0<\xi<1$ *and* $0<r<1/2$,

(i)

$$\mathrm{Cov}\Big[1_{\{|W_n^{(12)}(a2^{j'})|>r\}},1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\Big]\le\exp\Big\{-\frac{1}{2}\Big[\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}+\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big]\Big\}; \tag{C.65}$$

(ii)

$$\mathrm{Var}\Big[\log|W_n^{(12)}(a(n)2^j)|1_{\{W_n^{(12)}(a(n)2^j)<-r\}}\Big]\le(C+\log^2(r))\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a(n)^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\} \tag{C.66}$$

and

$$\mathrm{Var}\Big[\log W_n^{(12)}(a(n)2^j)1_{\{W_n^{(12)}(a(n)2^j)>r\}}\Big]\le\log^2(r)\exp\Big\{-\Big(\frac{n}{a(n)^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}+o(1); \tag{C.67}$$

(iii)

$$\mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|\,1_{\{|W_n^{(12)}(a2^j)|>r\}},\,1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\right]$$
$$\leq\sqrt{\log^2(r)\exp\Big\{-\Big(\frac{n}{a(n)^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}+o(1)\Big\}}\cdot\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big\}. \tag{C.68}$$

In (C.66), $C>0$ *does not depend on* r.

Proof. We first show (C.65). By the Cauchy–Schwarz inequality, the left-hand side of (C.65) is bounded from above by

$$\sqrt{\mathrm{Var}\left[1_{\{|W_n^{(12)}(a2^j)|>r\}}\right]}\sqrt{\mathrm{Var}\left[1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\right]}. \tag{C.69}$$

Moreover, for $0<\xi<1$ and the octave j, Lemma C.3 implies that

$$\mathrm{Var}\left[1_{\{|W_n^{(12)}(a2^j)|>r\}}\right]=P(|W_n^{(12)}(a2^j)|>r)P(|W_n^{(12)}(a2^j)|\leq r)\leq\exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}. \tag{C.70}$$

The same bound holds for the octave j' in (C.69). Thus, (C.65) holds.

To prove (C.66), note that $\mathrm{Var}\left[\log|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)<-r\}}\right]$ is bounded by

$$\mathbb{E}\Big[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)<-r\}}\Big]$$
$$=\mathbb{E}\Big[\log^2|W_n^{(12)}(a2^j)|\Big(1_{\{W_n^{(12)}(a2^j)\leq-1/2\}}+1_{\{-1/2<W_n^{(12)}(a2^j)<-r\}}\Big)\Big]$$
$$\leq C\mathbb{E}\Big[|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\leq-1/2\}}\Big]+\log^2(r)P(-1/2<W_n^{(12)}(a2^j)<-r)$$
$$\leq C\sqrt{\mathbb{E}\Big[W_n^{(12)}(a2^j)^2\Big]P(W_n^{(12)}(a2^j)\leq-1/2)}+\log^2(r)P(-1/2<W_n^{(12)}(a2^j)<-r)$$
$$\leq C'\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}+\log^2(r)\exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}.$$

This establishes (C.66).

To prove (C.67), note that $\mathrm{Var}\left[\log W_n^{(12)}(a2^j)1_{\{W_n^{(12)}(a2^j)>r\}}\right]$ is bounded by

$$\mathbb{E}\Big[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)>r\}}\Big]$$
$$=\mathbb{E}\Big[\log^2|W_n^{(12)}(a2^j)|\Big(1_{\{r<W_n^{(12)}(a2^j)<1/2\}}+1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\Big)\Big]$$
$$\leq\log^2(r)P(r<W_n^{(12)}(a2^j)<1/2)+\mathbb{E}\Big[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\Big].$$

However, for some $C>0$,

$$\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\leq CW_n^{(12)}(a2^j)^2,$$

where, by (C.22), $W_n^{(12)}(a2^j)\overset{P}{\to}1$ and

$$\mathbb{E}\big[W_n^{(12)}(a2^j)^2\big]=\mathrm{Var}W_n^{(12)}(a2^j)+1\to1,\quad n\to\infty.$$

Therefore, by the dominated convergence theorem for convergence in probability,

$$\lim_{n\to\infty}\mathbb{E}\Big[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\Big]=0.$$

This establishes (C.67).

To show (C.68), again by applying the Cauchy–Schwarz inequality, the left-hand side of (C.68) is bounded from above by

$$\sqrt{\mathrm{Var}\left[\log|W_n^{(12)}(a2^j)|1_{\{|W_n^{(12)}(a2^j)|>r\}}\right]}\sqrt{\mathrm{Var}\left[1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\right]}. \tag{C.71}$$

Hence, the bound follows from (C.65) and (C.67). □

We are now in a position to establish Theorem 4.1.

Proof of Theorem 4.1. Fix $0<\xi<1$ and recall that $n_*=\frac{n}{a2^{j+j'}}$. Then,

$$\mathrm{Cov}\left[\log|S_n^{(12)}(a2^j)|\,1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\,\log|S_n^{(34)}(a2^{j'})|\,1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\right]$$
$$=\mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|\,1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\,\log|W_n^{(34)}(a2^{j'})|\,1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\right]$$

$$
\begin{aligned}
&+\log\left|\mathbb{E}\left[d_3(a2^{j},0)d_4(a2^{j'},0)\right]\right| \mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|\,1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\,1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\right]\\
&+\log\left|\mathbb{E}\left[d_1(a2^{j},0)d_2(a2^{j},0)\right]\right| \mathrm{Cov}\left[1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\,\log|W_n^{(34)}(a2^{j'})|\,1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\right]\\
&+\log\left|\mathbb{E}\left[d_1(a2^{j},0)d_2(a2^{j},0)\right]\right| \log\left|\mathbb{E}\left[d_3(a2^{j'},0)d_4(a2^{j'},0)\right]\right| \mathrm{Cov}\left[1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\,1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\right]\\
&=\mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\,\log|W_n^{(34)}(a(n)2^{j'})|1_{\{|W_n^{(34)}(a(n)2^{j'})|>r_n\}}\right]\\
&\quad+o\Big(\Big(\frac{a(n)^{4h_{\max}-4h_{\min}}}{n_*}\Big)^2\Big).
\end{aligned}
\tag{C.72}
$$

The last two equalities in (C.72) are a consequence of Lemma C.1, (ii), as applied to $\mathbb{E}\left[d_1(a2^j,0)d_2(a2^j,0)\right]$ and $\mathbb{E}\left[d_3(a2^{j'},0)d_4(a2^{j'},0)\right]$, and of the bound (C.68) (from Lemma C.7(iii)) under the condition (4.2).

Therefore, it suffices to show that the main term on the right-hand side of (C.72) is equal to the main term on the right-hand side of (4.3). By accounting for absolute values, the covariance term in the former can be broken up into a sum of four terms, namely,

$$
\mathrm{Cov}\left[\log W_n^{(12)}(a2^j)1_{\{W_n^{(12)}(a2^j)>r_n\}},\,\log W_n^{(34)}(a2^{j'})1_{\{W_n^{(34)}(a2^{j'})>r_n\}}\right]
\tag{C.73}
$$

plus the remainder

$$
\begin{aligned}
&\mathrm{Cov}\left[\log W_n^{(12)}(a2^j)\,1_{\{W_n^{(12)}(a2^j)>r_n\}},\,\log|W_n^{(34)}(a2^j)|\,1_{\{W_n^{(34)}(a2^j)<-r_n\}}\right]\\
&+\mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|\,1_{\{W_n^{(12)}(a2^j)<-r_n\}},\,\log W_n^{(34)}(a2^j)\,1_{\{W_n^{(34)}(a2^j)>r_n\}}\right]\\
&+\mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|\,1_{\{W_n^{(12)}(a2^j)<-r_n\}},\,\log|W_n^{(34)}(a2^j)|\,1_{\{W_n^{(34)}(a2^j)<-r_n\}}\right].
\end{aligned}
\tag{C.74}
$$

By the Cauchy–Schwarz inequality, the bounds (C.66) and (C.67) (from Lemma C.7, (ii)) and condition (4.2), the absolute value of the second term in the sum (C.74) is bounded by

$$
\sqrt{\mathrm{Var}\left[\log|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)<-r_n\}}\right]\mathrm{Var}\left[\log W_n^{(34)}(a2^j)1_{\{W_n^{(34)}(a2^j)>r_n\}}\right]}=o\Big(\Big(\frac{a(n)^{4h_{\max}-4h_{\min}}}{n_*}\Big)^2\Big).
$$

By a similar argument, the same bound holds for the remaining terms in the sum (C.74). Thus, it suffices to focus on (C.73). In the following derivations, expressions involving individual sample wavelet variance terms will be expressed in terms of $W_n^{(12)}(a2^j)$, but analogous expressions hold when substituting $W_n^{(34)}(a2^{j'})$ for $W_n^{(12)}(a2^j)$.

Fix $0<\xi<1$. For any given j, write out the almost sure Taylor expansion

$$
\log W_n^{(12)}(a2^j)\,1_{\{W_n^{(12)}(a2^j)>r_n\}}=\Big\{(W_n^{(12)}(a2^j)-1)-\frac{1}{2}\Big(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\Big)^2\Big\}1_{\{W_n^{(12)}(a2^j)>r_n\}},
\tag{C.75}
$$

where $\theta_+(W_n^{(12)}(a2^j))\in[\min\{W_n^{(12)}(a2^j),1\},\max\{W_n^{(12)}(a2^j),1\}]$. Then,

$$
\begin{aligned}
&\mathbb{E}\left[\log W_n^{(12)}(a2^j)\ \log W_n^{(34)}(a2^{j'})\ 1_{\min\{\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\right]\\
&=\mathbb{E}\left[(W_n^{(12)}(a2^j)-1)(W_n^{(34)}(a2^{j'})-1)1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\right]\\
&-\frac{1}{2}\mathbb{E}\left[(W_n^{(12)}(a2^j)-1)\Big(\frac{W_n^{(34)}(a2^{j'})-1}{\theta_+(W_n^{(34)}(a2^{j'}))}\Big)^2 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\right]\\
&-\frac{1}{2}\mathbb{E}\left[\Big(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\Big)^2(W_n^{(34)}(a2^{j'})-1)\ 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\right]\\
&+\frac{1}{4}\mathbb{E}\left[\Big(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\Big)^2\Big(\frac{W_n^{(34)}(a2^{j'})-1}{\theta_+(W_n^{(34)}(a2^{j'}))}\Big)^2 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\right].
\end{aligned}
\tag{C.76}
$$

For $0<r_n<1/2$, recast

$$
\begin{aligned}
&\Big(\frac{W_n^{(12)}(a2^j)-1}{\widehat{\theta}_+(W_n^{(12)}(a2^j))}\Big)^2 1_{\{W_n^{(12)}(a2^j)>r_n\}}\\
&=\Big(\frac{W_n^{(12)}(a2^j)-1}{\widehat{\theta}_+(W_n^{(12)}(a2^j))}\Big)^2\Big(1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}+1_{\{W_n^{(12)}(a2^j)\geq 1/2\}}\Big)\\
&\leq\Big(\frac{W_n^{(12)}(a2^j)-1}{r_n}\Big)^2 1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}+\Big(\frac{W_n^{(12)}(a2^j)-1}{1/2}\Big)^2 1_{\{W_n^{(12)}(a2^j)\geq 1/2\}}.
\end{aligned}
\tag{C.77}
$$

Therefore, up to a constant, we can bound the fourth term in (C.76) by

$$
\begin{aligned}
&\mathbb{E}\left[\left|\left(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\right)^2\left(\frac{W_n^{(34)}(a2^j)-1}{\theta_+(W_n^{(34)}(a2^j))}\right)^2 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\right|\right]\\
&\le \frac{1}{r_n^4}\mathbb{E}\left[(W_n^{(12)}(a2^j)-1)^2 1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\right]\\
&\quad+\frac{1}{(r_n/2)^2}\mathbb{E}\left[(W_n^{(12)}(a2^j)-1)^2 1_{\{W_n^{(12)}(a2^j)\ge 1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\right]\\
&\quad+\frac{1}{(r_n/2)^2}\mathbb{E}\left[(W_n^{(12)}(a2^j)-1)^2 1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{W_n^{(34)}(a2^j)\ge 1/2\}}\right]\\
&\quad+\frac{1}{(1/2)^4}\mathbb{E}\left[(W_n^{(12)}(a2^j)-1)^2 1_{\{W_n^{(12)}(a2^j)\ge 1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{W_n^{(34)}(a2^{j'})\ge 1/2\}}\right].
\end{aligned}
\tag{C.78}
$$

By Lemma C.4, the fourth term in the sum on the right-hand side of (C.78) is bounded by

$$
O\left(\left(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\right)^2\right).
\tag{C.79}
$$

By the Cauchy–Schwarz inequality, Lemma C.4 and condition (4.2), the first term in the sum (C.78) is bounded by

$$
\begin{aligned}
&\frac{a^{2(\delta_{12}+\delta_{34})}}{r_n^4}\sqrt{\mathbb{E}\left[\left(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\right)^4\left(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\right)^4\right]}\cdot\sqrt{\mathbb{E}\left[1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\right]}\\
&\le \frac{a^{2(\delta_{12}+\delta_{34})}}{r_n^4}O\left(\frac{a}{n}\right)\sqrt{P(r_n<W_n^{(12)}(a2^j)<1/2)P(r_n<W_n^{(34)}(a2^j)<1/2)}\\
&\le \frac{a^{2(\delta_{12}+\delta_{34})}}{r_n^4}O\left(\frac{a}{n}\right)\exp\left\{-\frac{1}{2}\left[\left(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\right)^{1-\xi}+\left(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\right)^{1-\xi}\right]\right\}\\
&=O\left(\left(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\right)^2\right),
\end{aligned}
\tag{C.80}
$$

since $2(\delta_{12}+\delta_{34})\le 8h_{\max}-8h_{\min}$. The second term in the sum (C.78) is bounded by

$$
\begin{aligned}
&\frac{Ca^{2(\delta_{12}+\delta_{34})}}{r_n^2}\sqrt{\mathbb{E}\left[\left(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\right)^4\left(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\right)^4\right]}\cdot\sqrt{\mathbb{E}\left[1_{\{W_n^{(12)}(a2^j)\ge 1/2\}}1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\right]}\\
&\le \frac{Ca^{2(\delta_{12}+\delta_{34})}}{r_n^2}O\left(\frac{a}{n}\right)\sqrt{P(W_n^{(12)}(a2^j)\ge 1/2)P(r_n<W_n^{(34)}(a2^j)<1/2)}\\
&\le \frac{Ca^{2(\delta_{12}+\delta_{34})}}{r_n^2}O\left(\frac{a}{n}\right)\exp\left\{-\frac{1}{2}\left(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\right)^{1-\xi}\right\}=O\left(\left(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\right)^2\right).
\end{aligned}
\tag{C.81}
$$

An analogous bound holds for the third term in the sum (C.78). Therefore, by (C.79), (C.80) and (C.81), the fourth term in (C.76) is of the order

$$
O\left(\left(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\right)^2\right).
$$

By a similar reasoning, the same conclusion holds for the second and third terms in (C.76). Therefore, by (C.76) and Lemma C.5, we conclude that (C.72) is equal to the right-hand side of (4.3), as claimed. □

Remark C.5. For $q=q_1=q_2$, $W_{n,-}^{(qq)}(a2^j)=0$ a.s. (see (C.19)). Then, the existence of the moment $\mathbb{E}\left[\log^l|W_n^{(qq)}(a2^j)|\right]$, $l\in\mathbb{N}$, can be directly established by applying relation (96) in [54]. Moreover, the analysis of moments in this section can be extended without the truncation based on the sequence (4.2).

References

[1] B. Mandelbrot, Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier, J. Fluid Mech. 62 (1974) 331–358.
[2] R. Lopes, N. Betrouni, Fractal and multifractal analysis: a review, Med. Image Anal. 13 (2009) 634–649.
[3] S. Akselrod, D. Gordon, F.A. Ubel, D.C. Shannon, A.C. Berger, R.J. Cohen, Power spectrum analysis of heart rate fluctuation: a quantitative probe of beat-to-beat cardiovascular control, Science 213 (4504) (1981) 220–222.
[4] K. Kiyono, Z.R. Struzik, N. Aoyagi, Y. Yamamoto, Multiscale probability density function analysis: non-Gaussian and scale-invariant fluctuations of healthy human heart rate, IEEE Trans. Biomed. Eng. 53 (1) (2006) 95–102.
[5] P. Ciuciu, G. Varoquaux, P. Abry, S. Sadaghiani, A. Kleinschmidt, Scale-free and multifractal dynamic properties of fmri signals during rest and task, Front. Physiol. 3 (186) (2012) 1–18.
[6] P. Ciuciu, P. Abry, B.J. He, Interplay between functional connectivity and scale-free dynamics in intrinsic fMRI networks, NeuroImage 95 (186) (2014) 248–263.

[7] B.J. He, J. Zempel, A. Snyder, M. Raichle, The temporal structures and functional significance of scale-free brain activity, Neuron 66 (2010) 353–369.
[8] B.J. He, Scale-free brain activity: past, present, and future, Trends Cogn. Sci. 18 (9) (2014) 480–487.
[9] P. Abry, D. Veitch, Wavelet analysis of long-range dependent traffic, IEEE Trans. Inform. Theory 44 (1) (1998) 2–15.
[10] P. Abry, R. Baraniuk, P. Flandrin, R. Riedi, D. Veitch, Multiscale nature of network traffic, IEEE Signal Process. Mag. 19 (3) (2002) 28–46.
[11] B. Mandelbrot, A multifractal walk down wall street, Sci. Am. 280 (2) (1999) 70–73.
[12] P. Abry, H. Wendt, S. Jaffard, When Van Gogh meets Mandelbrot: multifractal classification of painting's texture, Signal Process. 93 (3) (2013) 554–572.
[13] C.R. Johnson Jr., E. Hendriks, I.J. Berezhnoy, E. Brevdo, S.M. Hughes, I. Daubechies, J. Li, E. Postma, J.Z. Wang, Processing for artist identification: Computerized analysis of Vincent van Gogh's painting brushstrokes, IEEE Signal Process. Mag. 25 (2008) 37–48.
[14] P. Abry, S.G. Roux, H. Wendt, P. Messier, A.G. Klein, N. Tremblay, P. Borgnat, S. Jaffard, B. Vedel, J. Coddington, L. Daffner, Multiscale anisotropic texture analysis and classification of photographic prints: art scholarship meets image processing algorithms, IEEE Signal Process. Mag. 32 (4) (2015) 18–27.
[15] G. Samorodnitsky, M.S. Taqqu, Stable Non-GAussian Random Processes, Chapman and Hall, New York, 1994.
[16] M.S. Taqqu, Fractional Brownian motion and long range dependence, in: P. Doukhan, G. Oppenheim, M.S. Taqqu (Eds.), Theory and Applications of Long-Range Dependence, Birkhäuser, Boston, 2003, pp. 5–38.
[17] J.-M. Bardet, G. Lang, G. Oppenheim, A. Philippe, S. Stoev, M. Taqqu, Semi-parametric estimation of the long-range dependence parameter: A survey, in: P. Doukhan, G. Oppenheim, M.S. Taqqu (Eds.), Theory and Applications of Long-Range Dependence, Birkhäuser, Boston, 2003, pp. 557–577.
[18] P. Flandrin, Wavelet analysis and synthesis of fractional Brownian motion, IEEE Trans. Inform. Theory 38 (1992) 910–917.
[19] D. Veitch, P. Abry, A wavelet-based joint estimator of the parameters of long–range dependence, IEEE Trans. Inform. Theory 45 (3) (1999) 878–897.
[20] P.-O. Amblard, J.-F. Coeurjolly, Identification of the multivariate fractional Brownian motion, IEEE Trans. Signal Process. 59 (11) (2011) 5152–5168.
[21] G. Didier, V. Pipiras, Integral representations and properties of operator fractional Brownian motions, Bernoulli 17 (1) (2011) 1–33.
[22] J.-F. Coeurjolly, P.-O. Amblard, S. Achard, Wavelet analysis of the multivariate fractional Brownian motion, ESAIM Probab. Stat. 17 (2013) 592–604.
[23] G. Didier, H. Helgason, P. Abry, Demixing multivariate-operator self-similar processes, in: Proc. IEEE Int. Conf. Acoust. Speech, and Signal Proc., ICASSP, Australia, Brisbane, 2015, pp. 3671–3675.
[24] P. Abry, G. Didier, Wavelet estimation for operator fractional Brownian motion, Bernoulli (2017) 1–30 in press.
[25] S. Achard, D.S. Bassett, A. Meyer-Lindenberg, E. Bullmore, Fractal connectivity of long-memory networks, Phys. Rev. E 77 (3) (2008) 036104.
[26] H. Wendt, A. Scherrer, P. Abry, S. Achard, Testing fractal connectivity in multivariate long memory processes, in: Proc. IEEE Int. Conf. Acoust. Speech, and Signal Proc., ICASSP, Taipei, Taiwan, 2009, pp. 2913–2916.
[27] B. Podobnik, H.E. Stanley, Detrended cross-correlation analysis: a new method for analyzing two nonstationary time series, Phys. Rev. Lett. 100 (8) (2008) 084102.
[28] W.-X. Zhou, Multifractal detrended cross-correlation analysis for two nonstationary signals, Phys. Rev. E 77 (6) (2008) 066211.
[29] Z.-Q. Jiang, W.-X. Zhou, Multifractal detrending moving-average cross-correlation analysis, Phys. Rev. E 84 (1) (2011) 016106.
[30] B. Podobnik, Z.-Q. Jiang, W.-X. Zhou, H.E. Stanley, Statistical tests for power-law cross-correlated processes, Phys. Rev. E 84 (6) (2011) 066118.
[31] L. Kristoufek, Mixed-correlated ARFIMA processes for power-law cross-correlations, Physica A 392 (24) (2013) 6484–6493.
[32] L. Kristoufek, Spectrum-based estimators of the bivariate Hurst exponent, Phys. Rev. E 90 (6) (2014) 062802.
[33] K. Dong, Y. Gao, L. Jing, Correlation tests of the engine performance parameter by using the detrended cross-correlation coefficient, J. Korean Phys. Soc. 66 (4) (2015) 539–543.
[34] L. Kristoufek, Can the bivariate Hurst exponent be higher than an average of the separate Hurst exponents? Physica A 431 (2015) 124–127.
[35] L. Kristoufek, Finite sample properties of power-law cross-correlations estimators, Physica A 419 (2015) 513–525.
[36] L. Kristoufek, On the interplay between short and long term memory in the power-law cross-correlations setting, Physica A 421 (2015) 218–222.
[37] I. Lobato, Consistency of the averaged cross-periodogram in long memory series, J. Time Series Anal. 18 (2) (1997) 137–155.
[38] D. Wang, B. Podobnik, D. Horvatić, H.E. Stanley, Quantifying and modeling long-range cross correlations in multiple time series with applications to world stock indices, Phys. Rev. E 83 (4) (2011) 046121.
[39] R. Sela, C. Hurvich, The averaged periodogram estimator for a power law in coherency, J. Time Series Anal. 33 (2) (2012) 340–363.
[40] P. Brockwell, R. Davis, Time Series: Theory and Methods, second ed., Springer, 1991.
[41] I.N. Lobato, A semiparametric two-step estimator in a multivariate long memory model, J. Econometrics 90 (1) (1999) 129–153.
[42] P. Robinson, Multivariate Local Whittle estimation in stationary systems, Ann. Statist. 36 (5) (2008) 2508–2530.
[43] F.S. Nielsen, Local Whittle estimation of multi-variate fractionally integrated processes, J. Time Series Anal. 32 (3) (2011) 317–335.
[44] S. Kechagias, V. Pipiras, Definitions and representations of multivariate long-range dependent time series, J. Time Series Anal. 36 (1) (2015) 1–25.
[45] S. Mallat, A Wavelet Tour of Signal Processing, Academic Press, San Diego, CA, 1998.
[46] H. Cramér, M.R. Leadbetter, Stationary and Related Stochastic Processes: Sample Function Properties and their Applications, Courier Dover Publications, 1967.
[47] J.-M. Bardet, Statistical study of the wavelet analysis of fractional Brownian motion, IEEE Trans. Inform. Theory 48 (4) (2002) 991–999.
[48] P. Abry, P. Flandrin, M. Taqqu, D. Veitch, Wavelets for the analysis, estimation and synthesis of scaling data, in: Self-Similar Network Traffic and Performance Evaluation, Wiley, 2000, pp. 39–88.
[49] S. Stoev, V. Pipiras, M. Taqqu, Estimation of the self-similarity parameter in linear fractional stable motion, Signal Process. 82 (12) (2002) 1873–1901.
[50] H. Helgason, V. Pipiras, P. Abry, Synthesis of multivariate stationary series with prescribed marginal distributions and covariance using circulant matrix embedding, Signal Process. 91 (8) (2011) 1741–1758.
[51] H. Helgason, V. Pipiras, P. Abry, Fast and exact synthesis of stationary multivariate gaussian time series using circulant embedding, Signal Process. 91 (5) (2011) 1123–1133.
[52] E. Lehmann, G. Casella, Theory of Point Estimation, Springer, 1998.
[53] P. Abry, G. Didier, Wavelet estimation for operator fractional Brownian motion: supplementary material, Bernoulli (2017) 1–15 in press.
[54] E. Moulines, F. Roueff, M. Taqqu, On the spectral density of the wavelet coefficients of long-memory time series with application to the log-regression estimation of the memory parameter, J. Time Series Anal. 28 (2) (2007) 155–187.
[55] P. Abry, P. Flandrin, On the initialization of the discrete wavelet transform, IEEE Signal Process. Lett. 1 (2) (1994) 32–34.
[56] S. Combrexelle, H. Wendt, G. Didier, P. Abry, Multivariate scale-free dynamics: Testing fractal connectivity, in: IEEE Int. Conf. Acoust. Speech, and Signal Proces., ICASSP, New Orleans, USA, 2017.
[57] G.A. Jones, J.M. Jones, Elementary Number Theory, Springer-Verlag, Berlin, 1998.
[58] J.-M. Bardet, Testing for the presence of self-similarity of Gaussian time series having stationary increments, J. Time Series Anal. 21 (5) (2000) 497–515.
[59] J. Istas, G. Lang, Quadratic variations and estimation of the local Hölder index of a Gaussian process, Ann. Inst. Henri Poincaré 33 (4) (1997) 407–436.
[60] B. Laurent, P. Massart, Adaptive estimation of a quadratic functional by model selection, Ann. Statist. (2000) 1302–1338.
[61] L. Birgé, P. Massart, Minimum contrast estimators on sieves: exponential bounds and rates of convergence, Bernoulli 4 (3) (1998) 329–375.
[62] S. Boucheron, G. Lugosi, P. Massart, Concentration Inequalities: A Nonasymptotic Theory of Independence, Oxford University Press, 2013.
[63] C. Vignat, A generalized Isserlis theorem for location mixtures of Gaussian random vectors, Statist. Probab. Lett. 82 (1) (2012) 67–71.