Symbolic possibilistic logic: completeness and inference methods

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Abstract

This paper studies the extension of possibilistic logic to the case when weights attached to formulas are symbolic. These weights then stand for variables that lie in a totally ordered scale, and only partial knowledge is available on the relative strength of these weights in the form of inequality constraints. Reasoning in symbolic possibilistic logic means solving two problems. One is to compute symbolic expressions representing the weights of conclusions of a possibilistic knowledge base. The other problem is that of comparing the relative strength of derived weights, so as to find out if one formula is more certain than another one. Regarding the first problem, a proof of the soundness and the completeness of this logic according to the relative certainty semantics in the sense of necessity measures is provided. Based on this result, two syntactic inference methods are suggested. The first one shows how to use the notion of minimal inconsistent subsets and known techniques that compute them, so as to obtain the symbolic expression representing the necessity degree of a possibilistic formula. A second family of methods computes prime implicates and takes inspiration from the concept of assumption-based theory. It enables symbolic weights attached to consequences to be simplified in the course of their computation, taking inequality constraints into account. Finally, an algorithm is proposed to find if a consequence is more certain than another one. A comparison with the original version of symbolic possibilistic logic introduced by Benferhat and Prade in 2005 is provided.

Keywords: Possibilistic logic, partial order, hitting sets, consequence finding algorithms, minimal inconsistent subsets.

1 Introduction

Possibilistic logic [16] is an approach to reasoning under uncertainty using totally ordered propositional bases. In this logic, each formula is assigned a degree, often encoded by a weight belonging to (0, 1], seen as a lower bound on the certainty level of the formula. Such degrees of certainty obey graded versions of the principles that found the notions of belief or knowledge in epistemic logic, namely the conjunction of two formulas is not believed less than the least believed of the conjuncts. This is the basic axiom of degrees of necessity in possibility theory [17]. See [19] for a recent survey of possibilistic logic. Deduction in possibilistic logic follows the rule of the weakest link, as postulated by Rescher [30]: the strength of an inference chain is that of the least certain formula involved in this chain. The weight of a formula in the deductive closure is then the weight of the strongest path leading from the base to the formula. Possibilistic logic has developed techniques for knowledge representation and reasoning in various areas, such as non-monotonic reasoning, belief revision and belief merging (see references in [19]).

More than 10 years ago, a natural extension of possibilistic logic was proposed using partially ordered symbolic weights attached to formulas [4, 7], we call here *symbolic possibilistic logic* (SPL). In SPL, weights represent ill-known certainty values on a totally ordered scale. Only partial

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knowledge on the relative strength of weights is supposed to be available. The basic motivation for this variant of possibilistic logic is that, in some practical situations, it is difficult to assume that the weights attached to formulas are known precisely enough to be rank-ordered.

We give two examples of such a situation. The first one is when we assume that pieces of information come from several sources and the weight attached to a formula reflects the reliability of the source supplying it [15]. Some sources are known to be more reliable than other ones, but there may be a lack of knowledge about the relative reliability of some other sources.

EXAMPLE 1

Assume that different agents exchange information about potential participants in a forthcoming meeting.

- Agent A1 says: Albert, Betty and Chris will not come together; if Albert, Betty and David come then the meeting will not be quiet; however if Albert, Betty and Eva come, the meeting will be productive.
- Agent A2 says: if the meeting begins late, then it will not be quiet; if David comes, then Chris comes too.
- Agent A3 says: if Betty comes, then the meeting will begin late; Chris cannot attend the meeting if it begins late.

Moreover, we assume that Agent A1 and Agent A2 are known to be more reliable than Agent A3, but it is not clear whether A1 is more reliable than A2 or not. It can be represented by attaching a symbolic weight to each agent. These weights are then assigned to the assertions given by the agents. The three weights cannot be rank-ordered.

Another example inspired by [1] concerns a setting where users can access only part of the available information in a knowledge base, according to the access rights that they have been granted. Each assertion in the knowledge base is attached a symbolic label representing a confidentiality level of the assertion. Confidentiality labels are symbolic and are only partially ordered. The higher the label in the hierarchy, the more publicly available is the corresponding information item. Given a consequence ϕ of the knowledge base, the composite weight p attached to it after deduction reflects its degree of public availability. Suppose that the user is allowed information items with labels of degree not less than q and that $q > p$. It means that some confidential information to which the user has no right to access is necessary to derive ϕ. So the user will be denied access to ϕ.

The idea of a symbolic possibilistic logic (SPL) was first introduced by Benferhat *et al*. [4], later more thoroughly described in [7]. In the latter paper, only weak inequality constraints between symbolic weights are assumed. A symbolic possibilistic knowledge base along with knowledge pertaining to weights is encoded in propositional logic, augmenting the atomic formulas with other ones pertaining to weights. In [7], the authors give a deduction method for plausible inference in this logic using the idea of forgetting variables, and strict inequalities between symbolic weights attached to consequences are obtained by default.

This generalisation of possibilistic logic differs from other approaches that represent sets of formulas equipped with a partial order in the setting of conditional logics [24]. It also contrasts with another line of research consisting in viewing a partial order on weights as a family of total orders, thus viewing a symbolic possibilistic base as a set of usual possibilistic bases [5].

In a previous publication [31], we had introduced a variant of SPL where constraints between symbolic weights are strict, and we compared this logic with a logic of relative certainty that uses partially ordered knowledge bases, and where atomic formulas directly express that a propositional formula is more certain than another. In the present work, which extends a conference paper [8], we propose a complete account of SPL, where we assume both strict and weak inequality constraints between weights and we define the weighted completion of a possibilistic knowledge base. Our approach yields a partial order on the language, while the alternative partially ordered generalisations of possibilistic logic [5, 6, 7] only compute a set of plausible consequences. We provide a full completeness proof, outlined in [31], but absent from [7]. This proof is different from the completeness proof of standard possibilistic logic as, contrary to the latter, we cannot rely on classical inference from sets of formulas having at least a given certainty degree. It relies on the notion of hitting sets.

Inference methods that compute the symbolic weight attached to a conclusion are suggested, especially some inspired by the literature on abductive reasoning initiated by Reiter [29], some rooted in assumption-based truth-maintenance systems [11] and the computation of prime implicates. We also provide a simple-minded algorithm to compare symbolic weight expressions. In this paper, we do not try to precisely determine the computational complexity of our inference methods nor do we try to optimize them. Such complexity results can be borrowed from existing ones in the consequence-finding literature [28], and can be applied to SPL inference.

The paper is organized as follows. In Section 2, we provide the formal background on standard possibilistic logic. Then, in Section 3 we give the main definitions and concepts of SPL, including axioms and semantics. In Section 4 we prove the soundness and completeness of SPL using hitting sets. In the next Section 5 we show how to use existing tools from the literature on abduction, minimal inconsistent subsets, assumption-based theory, variable-forgetting and more general prime implicate computation methods in order to compute the symbolic weights attached to conclusions and we give a method to compare their relative strengths. Finally, Section 6 discusses related works especially [7]. An alternative semantics for symbolic possibilistic bases is discussed in the Appendix.

2 Background on standard possibilistic logic

We consider a propositional language $\mathcal{L}$, based on a finite set of propositional variables denoted by first Greek letters $\alpha, \beta, \gamma, \ldots$, where composite formulas are denoted by other Greek letters ξ, ϕ, ψ, ..., and Ω is the finite set of interpretations of $\mathcal{L}$. The set of models of ϕ is denoted by $[\phi]$, which is a subset of Ω. We denote by $\vdash$ the classical syntactic inference and by $\models$ the classical semantic inference.

Possibilistic logic is an extension of classical logic and it encodes conjunctions of weighted formulas of the form (ϕ_j, p_j) where ϕ_j is a propositional formula and $p_j \in\ ]0, 1]$. The weight p_j is interpreted as a lower bound on the certainty level of ϕ_j, namely $N(\phi_j) \geq p_j > 0$, where N is a necessity measure in the sense of possibility theory [18].

Let us recall basic notions of possibility theory [17]. A possibility distribution is a mapping $\pi : \Omega \to [0, 1]$ expressing to what extent a situation, encoded as an interpretation, is plausible. At least one situation must be fully plausible ($\pi(\omega) = 1$ for some ω), and $\pi(\omega) = 0$ means that this situation is impossible. A possibility measure can be defined on subsets of Ω from a possibility distribution as $\Pi(A) = \max_{\omega \in A} \pi(\omega)$ expressing the plausibility of any event A. Note that $\Pi(\varnothing) = 0$ and $\Pi(\Omega) = 1$. The necessity measure expressing certainty levels is defined by conjugacy: $N(A) = 1 - \Pi(\overline{A})$ where $\overline{A}$ denotes the complement of A. Note that $N(\varnothing) = 0$ and $N(\Omega) = 1$.

In the following, we attach possibility and necessity degrees to propositional formulas, letting $\Pi(\phi) = \Pi([\phi])$. In particular:

$$N(\phi_j) = \min_{\omega \notin [\phi_j]} (1 - \pi(\omega)).$$

The basic axiom of necessity measures can then be written as

$$N(\phi \wedge \psi) = \min(N(\phi), N(\psi)).$$

Note that the most elementary form of possibilistic logic considers neither negations nor disjunctions of weighted formulae (see [20] for generalized possibilistic logic, where these connectives make sense).

2.1 Semantics of possibilistic logic bases using possibility distributions

A possibilistic logic base (Poslog base) is a finite set of weighted formulas $\Sigma = \{(\phi_j, p_j) : j = 1, \cdots, m\}$. The (fuzzy) set of models of a PL base is defined by a possibility distribution π_Σ on Ω defined as follows.

First, each weighted formula (ϕ_j, p_j) can be associated with a possibility distribution π_j on Ω defined by [16]:

$$\pi_j(\omega) = \begin{cases} 1 & \text{if } \omega \in [\phi_j], \\ 1 - p_j & \text{if } \omega \notin [\phi_j], \end{cases} \tag{1}$$

and then π_Σ is obtained by the fuzzy set conjunction of the π_j's:

$$\pi_\Sigma(\omega) = \min_j \pi_j(\omega). \tag{2}$$

The rationale is based on a minimal commitment principle called *the principle of minimal specificity*. It presupposes that any situation ω remains possible unless explicitly ruled out [17]. A possibility distribution π is less specific (less informative) than π' in the wide sense if $\pi \geq \pi'$ (π leaves at least as many possible interpretations as π'). The principle of minimal specificity tends to maximize possibility degrees.

We can define the semantics of possibilistic logic in terms of the satisfaction of a Poslog base Σ by a possibility distribution π on Ω as $\pi \models \Sigma$ if and only if $N(\phi_j) \geq p_j, j = 1, \cdots, m$ where $N(\phi_j)$ is the degree of necessity of ϕ_j w.r.t. π. Then, we can show that $\pi \models \Sigma$ if and only if $\pi \leq \pi_\Sigma$ [16]. It indicates that the possibilistic logic semantics is based on the selection of the least informative possibility distribution that satisfies Σ.

Let N_Σ be the necessity measure induced by π_Σ. It can be checked that $N_\Sigma(\phi_j) = \min_{\omega \notin [\phi_j]}(1 - \pi_\Sigma(\omega)) \geq p_j$. Note that the initial order of the possibilistic base can be modified according to the logical dependencies between formulas. It may occur that $N_\Sigma(\phi_j) > p_j$. It is the case e.g. if $\exists i, \phi_i \models \phi_j$ and $p_i > p_j$.This feature contrasts with conditional logics of relative possibility or certainty, like in [26, 24, 31] where syntactic expressions of the form $\phi > \psi$ in a knowledge base express constraints such as $N(\phi) > N(\psi)$ that will be enforced in the deductive closure, and may lead to a contradiction (if for instance $\phi \models \psi$ is valid).

DEFINITION 1
The deductive (semantic) closure of Σ is defined as follows:

$$\mathcal{C}_\pi(\Sigma) = \{(\phi, N_\Sigma(\phi)) : \phi \in \mathcal{L},\ N_\Sigma(\phi) > 0\}.$$

The semantics of possibilistic logic allows to replace a weighted conjunction $(\wedge_i \phi_i, p)$ by the set of formulas (ϕ_i, p) without altering the underlying possibility distribution, since $N(\phi \wedge \psi) = \min(N(\phi), N(\psi))$. Therefore from the minimum specificity principle, we can associate the same weight to each sub-formula in a conjunction. As a consequence, we can turn any possibilistic base into a semantically equivalent weighted base of clauses.

Since the weights are only lower bounds, they never add inconsistency to the base, contrary to logics of relative certainty pointed out above. The only reason for inconsistency comes from the classical inconsistency of $\Sigma^* = \{\phi_1, \cdots, \phi_m\}$, we call the *skeleton* of Σ. If Σ^* is inconsistent, one may have that both $N_\Sigma(\phi) > 0$ and $N_\Sigma(\neg\phi) > 0, \forall\phi \in \mathcal{L}$. In this case, $Cons(\Sigma) = \max_\omega \pi_\Sigma(\omega) < 1$ represents the *degree of consistency* of the possibilistic base Σ.

PROPOSITION 1 ([16])
$\min_{\phi\in\Sigma^*} N_\Sigma(\phi) = 1 - Cons(\Sigma)$.

Then, we can define the set of non-trivial consequences of Σ as:

$$\mathcal{C}_\pi^{nt}(\Sigma) = \{(\phi, N_\Sigma(\phi)) : \phi \in \mathcal{L},\ N_\Sigma(\phi) > 1 - Cons(\Sigma)\}$$

which coincides with $\mathcal{C}_\pi(\Sigma)$ if $Cons(\Sigma) = 1$. In the latter case, it holds that $\min(N_\Sigma(\phi), N_\Sigma(\neg\phi)) = 0, \forall\phi \in \mathcal{L}$, so that the skeleton of $\mathcal{C}_\pi(\Sigma)$ is consistent.

In the general case, $\phi \in \mathcal{C}_\pi^{nt}(\Sigma)$ is called a *plausible* consequence of Σ. The meaning of plausible consequences is explained by the following property: ϕ is a plausible consequence of Σ if and only if ϕ is satisfied in all preferred models according to π_Σ [3]. If we define the strict cut at level p of Σ as $\Sigma_p^> = \{(\phi_j, p_j) : p_j > p\}$, it is easy to see that $\mathcal{C}_\pi^{nt}(\Sigma) = \mathcal{C}_\pi(\Sigma^>_{1-Cons(\Sigma)})$.

Note that $N_\Sigma(\phi)$ can be expressed directly, without explicitly referring to the possibility distribution π_Σ on models. Let $\Sigma(\omega)$ denote the formulas in Σ^* satisfied by the interpretation ω. Since

$$1 - \pi_\Sigma(\omega) = \max_{j:\phi_j\notin\Sigma(\omega)} p_j,$$

which corresponds to the so-called 'best-out' ordering [2], it follows that

$$N_\Sigma(\phi) = \min_{\omega\not\models\phi} \max_{j:\phi_j\notin\Sigma(\omega)} p_j. \tag{3}$$

2.2 Syntactic inference in possibilistic logic

A sound and complete syntactic inference $\vdash_\pi$ for possibilistic logic can be defined with the following axioms and inference rules [16]:

Axioms

- $(\phi \to (\psi \to \phi), 1)$
- $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)), 1)$
- $(((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi), 1)$

Inference rules:

- Weakening: if $p_i > p_j$ then $(\phi, p_i) \vdash_\pi (\phi, p_j)$
- Modus Ponens: $\{(\phi \to \psi, p), (\phi, p)\} \vdash_\pi (\psi, p)$

The axioms are those of propositional logic with weight 1. The Modus Ponens rule embodies the adjunction law of accepted beliefs at any level, assuming they form a deductively closed set [13]. It is related to axiom K of modal epistemic logic [22]. The soundness and completeness of possibilistic logic for the above proof system can be translated by the following equality [16]:

$$N_\Sigma(\phi) = \max\{p : \Sigma \vdash_\pi (\phi, p)\}.$$

Note that we can also express inference in possibilistic logic by classical inference on cuts [16]:

$$N_\Sigma(\phi) = \max\{p : (\Sigma_p^{\geq})^* \vdash \phi\} \tag{4}$$

where $\Sigma_p^{\geq} = \{(\phi_j, p_j) : p_j \geq p\}$ is the weak cut at level p of Σ, and $(\Sigma_p^{\geq})^*$ its skeleton.

We can compute the degree of inconsistency $Inc(\Sigma)$ of a possibilistic base Σ syntactically as follows:

$$Inc(\Sigma) = \max\{p : \Sigma \vdash_\pi (\bot, p)\}.$$

It can be proved that [16, 3]:

- $Inc(\Sigma) = 1 - Cons(\Sigma) = 1 - \max_{\omega\in\Omega} \pi_\Sigma(\omega)$
- $N_\Sigma(\phi) = Inc(\Sigma \cup (\neg\phi, 1))$

So, in standard possibilistic logic, there are four ways of defining the deductive closure of a totally ordered base: the semantic approach based on a ranking of interpretations, the syntactic approach based on Modus Ponens and Weakening, the classical approach based on cuts and reasoning by refutation. They are equivalent and yield the same deductive closure. However, we will see in the next sections that this is no longer true if weights are symbols that are partially ordered.

3 Symbolic Possibilistic Logic

In SPL, only partial knowledge is available on the relative strength of weights attached to formulas. This approach was proposed in [4, 7]. We revisit this proposal by allowing the use of strict inequality constraints, unlike the original proposal where only weak inequalities were assumed. We also complement the original proposal, giving the proof that, independently of the constraints existing between symbolic weights, SPL is sound and complete.

3.1 Syntax of SPL

A symbolic possibilistic base is again a set of weighted formulas. But here, the weights are symbols, and we consider that we have only partial knowledge of the total order between these weights. Weights can actually be symbolic expressions involving elementary variables taking values on a totally ordered scale (such as]0, 1]), along with a set of constraints over these weights, describing what is known of their relative strength. The name 'symbolic possibilistic logic' indicates that symbolic computations are performed on the weights.

The set $\mathcal{P}$ of symbolic expressions p_j acting as weights is recursively obtained using a finite set of variables (called elementary weights) $H = \{a_1, \ldots, a_k, \ldots\}$ taking value on]0, 1] and max / min expressions built on H:

- $H \subset \mathcal{P}, 0, 1 \in \mathcal{P}$;
- if $p_i, p_j \in \mathcal{P}$ then $\max(p_i, p_j), \min(p_i, p_j) \in \mathcal{P}$.

We also suppose that $1 \geq a_i > 0, \forall i$.

Let $\Sigma = \{(\phi_i, p_i), i = 1, \cdots, n\}$ be a symbolic possibilistic base (SPL base for short) where p_i is from now on a max / min expression built on H.

3.2 *Constraints between weights*

The knowledge about symbolic weights is encoded by a finite set $C = C^{>} \cup C^{\geq}$ of dominance constraints between elementary weights: $C^{>}$ contains strict constraints $a_i > a_j$, where $>$ is asymmetric and transitive, and $C^{\geq}$ contains weak constraints $a_i \geq a_j$, where $\geq$ is transitive and reflexive. Moreover, we assume that $a_i > a_j \geq a_k$ implies $a_i > a_k$ and $a_i \geq a_j > a_k$ implies $a_i > a_k$. This is in agreement with the assumption that the variables a_i take values on a totally ordered scale. Note that in [7], only weak constraints are used ($C^{>} = \varnothing$), while in [31, 8], there are only strict dominance constraints ($C^{\geq} = \varnothing$).

For inference purposes, we shall be especially interested in conclusions of the form $p > q$ where p and q are symbolic expressions respectively attached to formulas ϕ and ψ in SPL. A valuation is a positive mapping $v : H \to (0, 1]$ that instantiates all elementary weights. Its domain is easily extended to all max/min expressions built on H. Let $\mathcal{V}$ denote the set of valuations. We denote by $v \models C$, the fact that the valuation v obeys constraints in C. In this paper, $p > q$ means that this inequality holds for all valuations $v \models C$, i.e. in *all* instantiations of p, q in agreement with the constraints.

The inference $C \vDash p > q$ then formally means:

DEFINITION 2
$C \vDash p > q$ iff $\forall v \in \mathcal{V}, v \models C$ implies $v(p) > v(q)$.

Informally, any valuation of symbols appearing in p, q (on $]0, 1]$) which satisfies the constraints in C also satisfies $p > q$.

Apart from the properties of $>$ and $\geq$, we also must take for granted as axioms the following property that may be useful for inferring strict dominance statements: $\max(p, q) \geq \min(p, r)$, in particular, $\max(p, q) \geq p$ and $p \geq \min(p, r)$, for any symbolic expressions p, q, r.

REMARK 1
We could assume that C contains dominance constraints between max/min expressions. It is clear that by distributivity any symbolic expression can be put in the form $\min_{i=1}^{r} \max_{k=1}^{n} a_{i_k}$ or in the form $\max_{j=1}^{s} \min_{l=1}^{m} a_{j_l}$, where the a_{i_k}'s and a_{j_l}'s are simple variables on [0, 1]. For instance, the symbolic expression $\min(\max(a, \min(b, c)), d)$ can be put in the form $\max(\min(a, d), \min(b, c, d))$ or in the form $\min(\max(a, b), \max(a, c), d)$. So any constraint of the form $p > q$ can take the canonical form

$$\forall i = 1, \ldots r, j = 1, \ldots s, \max_{k=1,\cdots,n} a_{i_k} \quad > \quad \min_{\ell=1,\cdots,m} a_{j_\ell} \text{ where } a_{i_k}, a_{j_\ell} \in H$$

i.e. $\forall i = 1, \ldots r, j = 1, \ldots s, \exists k \in \{1, \cdots, n\}, \exists \ell \in \{1, \cdots, m\}, a_{i_k} > a_{j_\ell}$.

A similar property, replacing $>$ by $\geq$ holds for weak constraints $p \geq q$. This remark points out that the handling of dominance constraints between max/min expressions comes down to considering disjunctions of sets of constraints between elementary weights, which means an increase of complexity for making inferences in SPL. In this paper, we thus restrict to assuming dominance constraints between elementary weights.

3.3 *Syntactic inference in SPL*

To define inference at the syntactic level, we must slightly reformulate the inference rules of possibilistic logic in order to account for the symbolic nature of the weights.

- Fusion: $\{(\phi, p), (\phi, p')\} \vdash_\pi (\phi, \max(p, p'))$
- Weakening: $(\phi, p) \vdash_\pi (\phi, p')$ if $p \geq p'$
- Modus Ponens: $\{(\phi \to \psi, p), (\phi, p)\} \vdash_\pi (\psi, p)$

This proof system is denoted by S_{SPL}. Note that the Weighted Modus Ponens rule follows from the above rules:

Weighted Modus Ponens: $\{(\phi \to \psi, p), (\phi, p')\} \vdash_\pi (\psi, \min(p, p'))$.

EXAMPLE 1 (continued)

Considering again the meeting example, let α, β, γ, δ, ϵ denote the statement that respectively Albert, Betty, Chris, David and Eva come to the meeting. Let κ stand for a quiet meeting, λ stand for a late meeting, ρ for a productive meeting. Let a_1, a_2, a_3 be the reliabilities of reporting agents A1, A2 and A3, respectively. The SPL knowledge base looks as follows:

$A1$: $(\neg(\alpha \wedge \beta \wedge \gamma), a_1), (\neg(\alpha \wedge \beta \wedge \delta) \vee \neg\kappa, a_1), (\neg(\alpha \wedge \beta \wedge \epsilon) \vee \rho, a_1)$
$A2$: $(\neg\lambda \vee \neg\kappa, a_2), (\neg\delta \vee \gamma, a_2)$
$A3$: $(\neg\beta \vee \lambda, a_3), (\neg\lambda \vee \neg\gamma, a_3)$
C : $a_1 > a_3, a_2 > a_3$.

Suppose Albert, Betty and Eva come for sure (with weight 1). Then the reader can check that (ρ, a_1) can be inferred, as well as $(\neg\kappa, \min(a_2, a_3))$. The former statement is more certain than the latter.

If B is a subset of the skeleton Σ^* of Σ, that classically implies ϕ, it is clear that $\Sigma \vdash_\pi (\phi, \min_{\phi_j \in B} p_j)$. Therefore using syntactic inference, we can compute the expression representing the strength of deduction of ϕ from Σ:

$$N_\Sigma^\vdash(\phi) = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} p_j. \tag{5}$$

Note that in the above expression, it is sufficient to maximize on subsets B that are minimal for inclusion among those that imply ϕ.

The proof system S_{SPL} can be used to define the degree of inconsistency of an SPL base denoted by $Inc(\Sigma)$.

$$Inc(\Sigma) = N_\Sigma^\vdash(\bot) = \max\{p : \Sigma \vdash_\pi (\bot, p)\}$$

We can check that, like in standard possibilistic logic:

PROPOSITION 2

$N_\Sigma^\vdash(\phi) = N_{\Sigma \cup \{(\neg\phi, 1)\}}^\vdash(\bot)$.

PROOF. It follows from the fact that $B \subseteq \Sigma^*$ implies ϕ if and only if $B \cup \{\neg\phi\}$ is inconsistent. ∎

To determine if one formula is more certain than another, we must compare the relative strength of these formulas via their composite weights in the weighted closure:

DEFINITION 3

(Σ, C) syntactically implies that ϕ is more certain than ψ (denoted by $(\Sigma, C) \vdash_\pi \phi > \psi$) iff $C \models N_\Sigma^\vdash(\phi) > N_\Sigma^\vdash(\psi)$.

EXAMPLE 2

Suppose the language $\mathcal{L}$ has atomic variables α, β. Let $\Sigma = \{(\alpha, a), (\neg\alpha \vee \beta, b), (\neg\alpha, c), (\neg\beta, d)\}$ and $C = \{a > b, b > c, b > d\}$. Then, $N_\Sigma^\vdash(\beta) = \max(\min(a, b), \min(a, c)) = \max(b, c) = b$ and $N_\Sigma^\vdash(\alpha) = a$. So, $(\Sigma, C) \vdash_\pi \alpha > \beta$.

As in standard possibilistic logic, we define plausible inference in SPL.

DEFINITION 4
ϕ is a plausible consequence of (Σ, C), denoted by $(\Sigma, C) \vdash_{PL} \phi$ iff $(\Sigma, C) \vdash_{\pi} \phi > \bot$, i.e.

$$C \vDash N_{\Sigma}^{\vdash}(\phi) > Inc(\Sigma) \left(= N_{\Sigma}^{\vdash}(\bot)\right).$$

EXAMPLE 3
Suppose the language $\mathcal{L}$ has atomic variables α, β and let $\Sigma = \{(\alpha, a), (\neg\alpha \vee \beta, b), (\neg\beta, c)\}$. We have $\Sigma \vdash_{\pi} (\beta, \min(a, b))$. It can be checked that $N_{\Sigma}^{\vdash}(\beta) = \min(a, b)$, $N_{\Sigma}^{\vdash}(\neg\beta) = c$, and $N_{\Sigma}^{\vdash}(\bot) = \min(a, b, c)$. Let $C = \{a > c, b > c\}$. Since $C \models \min(a, b) > c$, we obtain $N_{\Sigma}^{\vdash}(\beta) > N_{\Sigma}^{\vdash}(\bot)$, and β is a plausible consequence of (Σ, C).

Note that in SPL, comparing the strength degrees of formulas as per Definitions 3 and 4 requires that the set of strict constraints $C^{>}$ is not empty. Otherwise, no strict inequality can be inferred between formula weights.

3.4 Semantics of symbolic possibilistic bases

In SPL, a formula (ϕ_i, p_i) is still interpreted as a constraint $N(\phi_i) \geq p_i$ on a possibility distribution. The semantics of an SPL base can be defined as in standard possibilistic logic by means of a possibility distribution associated with a possibilistic base, as per Equations (1) and (2). This possibility distribution will attach a symbolic expression to each interpretation. However, the presence of terms of the form $1 -.$ prevents $\pi_{\Sigma}(\omega)$ from lying in $\mathcal{P}$. So, in the case of SPL, it is more convenient to express an *impossibility* distribution $\iota_{\Sigma} = 1 - \pi_{\Sigma}$ since $\iota_{\Sigma}(\omega) \in \mathcal{P}$, namely: $\forall\omega \in \Omega$:

$$\iota_{\Sigma}(\omega) = 1 - \pi_{\Sigma}(\omega) = \begin{cases} \max_{j:\phi_j \notin \Sigma(\omega)} p_j \\ 0 \text{ if } \Sigma(\omega) = \Sigma^*. \end{cases}$$

As in standard possibilistic logic, we can turn any SPL base into a semantically equivalent weighted base of clauses, and restrict to such bases.

Let Σ be an SPL base, and $\omega, \omega' \in \Omega$ be two interpretations. As in the numerical setting, we define a partial ordering on Ω, induced by Σ, i.e. the counterpart of the possibilistic ordering:

$$\omega >_{\Sigma} \omega' \text{ iff } C \vDash \iota_{\Sigma}(\omega) < \iota_{\Sigma}(\omega').$$

The weighted completion of the SPL base Σ is defined as follows:

DEFINITION 5
Let Σ be an SPL base of the form $\{(\phi_i, p_i), i = 1, \ldots, n\}$. Its weighted completion $\hat{\Sigma}$ is given by $\hat{\Sigma} = \{(\phi, N_{\Sigma}(\phi)) : \phi \in \Sigma^*\}$ where $\Sigma^* = \{\phi_1, \cdots, \phi_n\}$ and $N_{\Sigma}(\phi)$ is the min/max expression:

$$N_{\Sigma}(\phi) = \min_{\omega \not\vDash \phi} \iota_{\Sigma}(\omega) = \min_{\omega \not\vDash \phi} \max_{j:\phi_j \notin \Sigma(\omega)} p_j. \tag{6}$$

Note that in contrast with standard possibilistic logic, the above expression $N_{\Sigma}(\phi)$ cannot be simplified down to a single weight. However, we can restrict to subsets $\Sigma(\omega)$ which are maximal for inclusion. Moreover we can see that the following equivalence relating N_{Σ} and ι_{Σ} holds:

PROPOSITION 3
$C \models N_{\Sigma}(\phi) > N_{\Sigma}(\psi)$ is equivalent to

$$\forall \nu \models C, \exists \omega' \models \neg\psi, \forall \omega \models \neg\phi, \nu(\iota_{\Sigma}(\omega')) < \nu(\iota_{\Sigma}(\omega)).$$

PROOF. $C \models N_\Sigma(\phi) > N_\Sigma(\psi) \iff \forall v \models C, v(N_\Sigma(\phi)) > v(N_\Sigma(\psi))$
$\iff \forall v \models C, v(\min_{\omega \not\models \phi} \iota_\Sigma(\omega)) > v(\min_{\omega' \not\models \psi} \iota_\Sigma(\omega'))$
$\iff \forall v \models C, \min_{\omega \not\models \phi} v(\iota_\Sigma(\omega))) > \min_{\omega' \not\models \psi} v(\iota_\Sigma(\omega')))$
$\iff \forall v \models C, \exists \omega' \models \neg\psi, \forall \omega \models \neg\phi, v(\iota_\Sigma(\omega')) < v(\iota_\Sigma(\omega)).$ ■

This is clearly less demanding than another definition such as $\exists \omega' \models \neg\psi, \forall \omega \models \neg\phi$, $C \models \iota_\Sigma(\omega') < \iota_\Sigma(\omega)$, which is an extension of the best-out ordering. Indeed in the above proposition, the choice of $\omega' \models \neg\psi$ depends on v, but the inequality $v(\iota_\Sigma(\omega')) < v(\iota_\Sigma(\omega))$ must hold for a single ω' and all v in the latter. See the appendix for a comparison between best-out and usual possibilistic semantics of SPL.

In the following, we adopt the semantics of SPL with constraints by comparing the symbolic necessity expressions (6) in the completion, based on constraints on weights. Namely, we define the semantic inference of dominance statements in SPL by defining a partial order on the language $\mathcal{L}$ as follows:

$$(\Sigma, C) \models \phi > \psi \text{ if and only if } C \models N_\Sigma(\phi) > N_\Sigma(\psi). \tag{7}$$

Then, we can say that ϕ is semantically more certain than ψ.[1] One reason is that, in the symbolic framework, this inference is more productive than the one based on enforcing ordering $\iota_\Sigma(\omega') < \iota_\Sigma(\omega)$ for all valuations, for each pair of countermodels of ϕ and ψ. Another reason is that the syntactic inference leads to also attaching symbolic degrees of strength $N^{\vdash}_\Sigma(\phi)$ to formulas, again leading to a partial order on the language. We shall prove that these two partial orderings are the same.

4 The completeness of SPL

In this section, we prove that, irrespective of the set of constraints between symbolic weights, the inference system S_{SPL} is sound and complete for the semantics of SPL defined from the possibilistic ordering $>_\Sigma$. This is done by proving that the two expressions of the necessity degrees of formulas in the syntactic and the semantic closures of a possibilistic base in SPL are equal.

In standard possibilistic logic, the proof of completeness [16] relies on cuts, due to Equation 4. This method no longer works with symbolic weights, due to the fact that the result we try to prove does not use the existing ordering constraints on symbolic weights. Actually we provide a direct proof that the two expressions of $N_\Sigma(\phi)$ and $N^{\vdash}_\Sigma(\phi)$ coincide independently of constraints in C:

PROPOSITION 4 (Soundness and Completeness)
$N_\Sigma(\phi) = N^{\vdash}_\Sigma(\phi), \forall \phi \in \mathcal{L}.$

No completeness proof appears in [7], where the focus is on plausible inference according to a principle different from the one proposed here (see Section 6 for a discussion). An outline of our proof was published in [31], and a first draft of the full proof in [8]. We give it here in details for the sake of completeness.

In this proof, we need the notion of hitting-set (also known as hypergraph transversal [23]), useful in the formalization of abductive reasoning [29] and other topics in databases or formal concept analysis [10].

[1] We can define $(\Sigma, C) \models \phi \geq \psi$ similarly.

DEFINITION 6 (Hitting-set)
Let $\mathcal{B}$ be a collection of sets. A hitting-set of $\mathcal{B}$ is a set $H \subseteq \cup_{B_i \in \mathcal{B}} B_i$ such that $H \cap B_i \neq \varnothing$ for each $B_i \in \mathcal{B}$. A hitting-set H of $\mathcal{B}$ is minimal (for set-inclusion) if and only if no strict subset of H is a hitting-set of $\mathcal{B}$.

We have to prove that $\min_{\omega \not\models \phi} \max_{j:\phi_j \notin \Sigma(\omega)} p_j = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} p_j$. We separately address cases according to whether Σ^* is consistent or not.

4.1 *Consistent case*

Suppose that Σ^* is consistent. We can simplify the two symbolic expressions:

- For $N^{\vdash}_{\Sigma}(\phi)$, it is sufficient to consider the minimal (for set-inclusion) subsets of Σ^*, say B_i, $i = 1, \ldots, n$, that imply ϕ:

$$N^{\vdash}_{\Sigma}(\phi) = \max_{i=1}^{n} \min_{\phi_j \in B_i} p_j.$$

  This is due to the fact that the values of symbolic weights belong to a totally ordered set, and therefore, if $A \subset B$, then $\min_{\phi_j \in B} p_j \leq \min_{\phi_j \in A} p_j$. This property would be false if C were considered as a mere partial order on H, without such a semantics (e.g., min and max would not be defined).
- For $N_{\Sigma}(\phi)$, it is sufficient to consider the interpretations ω such that $\omega \not\models \phi$ and $\Sigma(\omega)$ is maximal (for set-inclusion):

$$N_{\Sigma}(\phi) = \min_{\omega \not\models \phi, \Sigma(\omega) \text{ maximal}} \max_{j:\phi_j \notin \Sigma(\omega)} p_j.$$

The expression $N_{\Sigma}(\phi)$ can be simplified since the subsets of the form $\Sigma(\omega)$ that are maximal (for set-inclusion) such that $\omega \not\models \phi$ are exactly the maximal (for set-inclusion) subsets of Σ^* consistent with $\neg\phi$. Such a subset will be denoted by $M_{\neg\phi} \in \mathcal{M}_{\neg\phi}$. So, we can write: $N_{\Sigma}(\phi) = \min_{M_{\neg\phi} \in \mathcal{M}_{\neg\phi}} \max_{\phi_j \notin M_{\neg\phi}} p_j$.

We need four lemmas.

LEMMA 1
If Σ^* is a minimal (for set-inclusion) base that implies ϕ, then $N_{\Sigma}(\phi) = N^{\vdash}_{\Sigma}(\phi)$.

PROOF. $N^{\vdash}_{\Sigma}(\phi) = \min_{\phi_j \in \Sigma^*} p_j$. So, $\forall \omega, \max_{j:\phi_j \notin \Sigma(\omega)} p_j \geq N^{\vdash}_{\Sigma}(\phi)$ so $N_{\Sigma}(\phi) \geq N^{\vdash}_{\Sigma}(\phi)$. Conversely, for all $\phi_k \in \Sigma^*$, $\Sigma^* \backslash \{\phi_k\} \not\vdash \phi$. So there is a model ω_k of $\Sigma^* \backslash \{\phi_k\}$ which is not a model of ϕ. So $\Sigma(\omega_k) = \Sigma^* \backslash \{\phi_k\}$. Therefore we have:

$$N_{\Sigma}(\phi) = \min_{\omega \not\models \phi} \max_{j:\phi_j \notin \Sigma(\omega)} p_j \leq \min_{\omega_k} \max_{j:\phi_j \notin \Sigma(\omega_k)} p_j = \min_{\phi_k \in \Sigma^*} p_k = N^{\vdash}_{\Sigma}(\phi).$$

■

As an immediate consequence, we have:

COROLLARY 1
$N_{\Sigma}(\phi) \geq N^{\vdash}_{\Sigma}(\phi)$.

PROOF. $N^{\vdash}_{\Sigma}(\phi) = \max_{i=1}^{n} N^{\vdash}_{B_i}(\phi)$ where B_i, $i = 1, \ldots, n$ are the minimal subsets that imply ϕ. So, following Lemma 1, $N^{\vdash}_{\Sigma}(\phi) = \max_{i=1}^{n} N_{B_i}(\phi)$. For each B_i, it is clear that $\iota_{\Sigma} \leq \iota_{B_i}$, so that from Equation (6), $N_{\Sigma}(\phi) \geq N_{B_i}(\phi)$. So $N_{\Sigma}(\phi) \geq N^{\vdash}_{\Sigma}(\phi)$. ■

Using distributivity, we can rewrite the syntactic necessity degree $N^{\vdash}_{\Sigma}(\phi)$ in terms of the minimal hitting-sets of the collection of sets $\mathcal{B} = \{B_1, \ldots, B_n\}$.

LEMMA 2
$N^{\vdash}_{\Sigma}(\phi) = \min_{H \in \mathcal{S}} \max_{\phi_j \in H} p_j$, where $\mathcal{S}$ denotes the set of all the minimal hitting-sets H of $\mathcal{B} = \{B_1, \ldots, B_n\}$.

PROOF. $N^{\vdash}_{\Sigma}(\phi) = \max_{i=1}^{n} \min_{\phi_j \in B_i} p_j$. Using distributivity of min vs max, we can rewrite $\max_{i=1}^{n} \min_{\phi_j \in B_i} p_j$ as $\min_{k=1}^{r} \max_{\phi_l \in E_k} p_l$, where the E_k are obtained by picking one formula in each B_i, in every possible way. So the sets E_k are all the hitting-sets of $\mathcal{B} = \{B_1, \ldots, B_n\}$. It is easy to see that it is sufficient to consider the sets E_k which are minimal (for set-inclusion) when computing $\min_{k=1}^{r} \max_{\phi_l \in E_k} p_l$. These minimal hitting-sets are exactly the elements H of $\mathcal{S}$. ∎

Let $\overline{\Sigma(\omega)}$ denote the set of formulas in Σ^* falsified by ω.

LEMMA 3
$\forall \omega \not\models \phi, \overline{\Sigma(\omega)}$ contains a hitting-set of $\{B_1, \ldots B_n\}$ (i.e. $\forall i, B_i \cap \overline{\Sigma(\omega)} \neq \varnothing$).

PROOF. Let $\omega \not\models \phi$ such that $\exists B_i, B_i \cap \overline{\Sigma(\omega)} = \varnothing$. So $B_i \subseteq \Sigma(\omega)$. However, as $\omega \not\models \phi$ we have $\Sigma(\omega) \nvdash \phi$ and then $B_i \nvdash \phi$. This contradicts the fact that $B_i \vdash \phi$. ∎

Note that the above result holds in particular when $\overline{\Sigma(\omega)}$ is minimal. The subsets $\overline{\Sigma(\omega)}$ such that $\omega \not\models \phi$ that are minimal (for set-inclusion) are the complements of the maximal subsets $M_{\neg\phi}$ of Σ^* consistent with $\neg\phi$.

So, we have:

LEMMA 4
The complement of each minimal hitting-set H of $\{B_1, \ldots B_n\}$ is a maximal subset of Σ^* consistent with $\neg\phi$ (called $M_{\neg\phi}$ above).

PROOF. Let $H = \{\phi_1, \ldots, \phi_n\}$ be a minimal hitting-set of $\{B_1, \ldots, B_n\}$ with $\phi_i \in B_i$. The set $\overline{H}$ is consistent, and it is consistent with $\neg\phi$. Otherwise, $\overline{H} \vdash \phi$ and so $\exists B_i \subseteq \overline{H}$ such that $B_i \vdash \phi$. This is impossible because by definition of H, $H \cap B_i \neq \varnothing$. So $\overline{H}$ is consistent with $\neg\phi$. In addition $\overline{H}$ is maximal consistent with $\neg\phi$. Indeed, if we add $\phi_i \in H$ to $\overline{H}$, $H \setminus \{\phi_i\}$ is no longer a hitting-set. Therefore $\exists B_j$ such that $(H \setminus \{\phi_i\}) \cap B_j = \varnothing$. Then $B_j \subseteq \overline{H} \cup \{\phi_i\}$ and so $\overline{H} \cup \{\phi_i\} \vdash \phi$ which proves that $\overline{H} \cup \{\phi_i\}$ is not consistent with $\neg\phi$. So $\exists M_{\neg\phi} = \overline{H}$. ∎

As an immediate consequence, we have:

COROLLARY 2
$N_{\Sigma}(\phi) \leq N^{\vdash}_{\Sigma}(\phi)$

PROOF. Using Lemma 2:

$$N^{\vdash}_{\Sigma}(\phi) = \min_{H \in \mathcal{S}} \max_{\phi_j \in H} p_j = \min_{M_{\neg\phi} = \overline{H}, H \in \mathcal{S}} \max_{\phi_j \notin M_{\neg\phi}} p_j \text{ (Lemma 4)}$$
$$\geq \min_{M_{\neg\phi} \in \mathcal{M}_{\neg\phi}} \max_{\phi_j \notin M_{\neg\phi}} p_j = N_{\Sigma}(\phi).$$

∎

This result along with Corollary 1 proves soundness and completeness when the SPL base is consistent.

In fact, there is an exact correspondence between the set of the maximal subsets of Σ^* consistent with $\neg\phi$ and the set of the minimal hitting-sets of the set $\{B_1, \dots, B_n\}$ of minimal consistent subsets that imply ϕ, namely $\mathcal{M}_{\neg\phi} = \{\overline{H}, H \in \mathcal{S}\}$.

COROLLARY 3
For each maximal subset of Σ^* consistent with $\neg\phi$, $M_{\neg\phi}$, there exists a minimal hitting-set H of $\{B_1, \dots B_n\}$ such that $M_{\neg\phi} = \overline{H}$.

PROOF. $\overline{M_{\neg\phi}}$ is a minimal subset of the form $\overline{\Sigma(\omega)}$ with $\omega \models \neg\phi$. By Lemma 3, $\overline{\Sigma(\omega)}$ contains a minimal hitting-set H. By Lemma 4, its complement is a maximal subset of Σ^* consistent with $\neg\phi$. It can therefore only be $M_{\neg\phi}$. ∎

4.2 Inconsistent case

Suppose that Σ^* is inconsistent. Now, some of its minimal subsets that imply ϕ may be inconsistent. We have the following results:

- Let $I_1, \dots, I_p$ be the minimal inconsistent subsets (MIS) of Σ^*. The inconsistency degree of Σ is $Inc(\Sigma) = N^{\vdash}_{\Sigma}(\bot) = \max_{k=1}^{p} \min_{\phi_j \in I_k} p_j$, and
$$N^{\vdash}_{\Sigma}(\phi) = \max(Inc(\Sigma), \max_{i=1}^{n} \min_{\phi_j \in B_i} p_j),$$
the B_is being the minimal subsets (consistent or not) which imply ϕ.
- $N^{\vdash}_{\Sigma}(\phi) \geq Inc(\Sigma)$.
- The definition of $N_{\Sigma}(\phi)$ is the same as in the consistent case. However, observe that $\forall\omega$, $\Sigma(\omega) \subset \Sigma^*$ (since $\Sigma(\omega)$ is consistent).

Now we are able to prove Proposition 4 for this case, noticing that we just have to augment the set $\{B_1, \dots, B_n\}$ with the minimal inconsistent subsets of Σ^* that contain none of the B_i's:

- Lemma 1 can be used. Now, in the Lemma, we assume that Σ^* is an inconsistent base (hence it implies ϕ), all subsets of which are consistent but none of them implies ϕ.
- Corollary 1 holds, where we use both the $B_i's$ and the I_k's (note that minimality does not exclude inconsistency).
- The minimal sets implying ϕ are all the $B_i's$ plus some of the I_k's, and we take their hitting sets.
- For Lemma 3, $\Sigma(\omega)$ is always consistent. So, we may notice that $I_i \not\subset \Sigma(\omega)$ trivially, in the proof.
- The proof of Lemma 4 still holds, since the sets $\overline{H}$ are consistent, as the $M_{\neg\phi}$. So, we have again the validity of Corollary 2.

So soundness and completeness hold even if the base Σ^* is inconsistent.

REMARK 2
It may happen that some minimal inconsistent subset I_i of Σ^* is not minimal implying ϕ. For instance, if $\Sigma = \{(\phi, a), (\neg\phi, b)\}$, then the unique minimal subset implying ϕ is $\{\phi\}$. In that case,

$$N^{\vdash}_{\Sigma}(\phi) = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} p_j = \max(\min(a, b), a) = a = N_{\Sigma}(\phi).$$

Similarly, $N^{\vdash}_{\Sigma}(\neg\phi) = b$. So we have $N^{\vdash}_{\Sigma}(\bot) = \min(a, b) \leq N^{\vdash}_{\Sigma}(\phi)$ and $N^{\vdash}_{\Sigma}(\bot) \leq N^{\vdash}_{\Sigma}(\neg\phi)$. We have $\{a\} \subset \{a, b\}$ but it cannot be concluded that $N^{\vdash}_{\Sigma}(\bot) < N^{\vdash}_{\Sigma}(\neg\phi)$.

As a consequence of Propositions 2 and 4, reasoning by refutation is valid: $N_{\Sigma\cup\{(\neg\phi,1)\}}(\bot) = N_{\Sigma}(\phi)$.

5 Toward inference methods in SPL

In this section, we will suggest two syntactic inference methodologies that calculate the necessity degree $N^{\vdash}_{\Sigma}(\phi)$ of a possibilistic formula. The first approach is based on the calculation of minimal inconsistent subsets. The second one is based on a propositional logical encoding of an SPL base, where symbolic weights are represented by propositional variables, and the use of prime implicates, which deals with the simplification of complex weights attached to consequences during the derivation process. We assume for simplicity that the weights bearing on formulas of the original SPL base are elementary, with possibility of assigning the same weight to different formulas. However, as we shall see, this assumption is inessential, and can easily be dropped.

5.1 Syntactic inference based on minimal inconsistent subsets

Given a formula ϕ, computing the expression in equation (5) requires the determination of all minimal subsets B_i such that $B_i \vdash \phi$. Some of the minimal subsets that imply ϕ may be inconsistent. In that case, they are minimal inconsistent in Σ^*.

LEMMA 5
Let $B \subseteq \Sigma^*$ be a minimal subset that implies ϕ. If B is inconsistent, then it is minimal inconsistent in Σ^*.

PROOF. Assume B is inconsistent and that $B' \subset B$ and B' is minimal inconsistent. Then $B' \vdash \phi$ trivially, which contradicts the hypothesis on B. ■

So, if $B \subseteq \Sigma^*$ is a minimal subset implying ϕ, either B is consistent or B is a minimal inconsistent subset of Σ^*. It follows easily that:

PROPOSITION 5
Let $B_1, \cdots, B_k$ be the minimal consistent subsets of Σ^* implying ϕ. Let $I_1, \cdots, I_l$ be the minimal inconsistent subsets in Σ^* *which do not contain any* B_j, $j = 1 \cdots k$, then $N^{\vdash}_{\Sigma}(\phi) = \max(\max_{i=1}^{k} \min_{\phi_j \in B_i} p_j, \max_{i=1}^{l} \min_{\phi_j \in I_i} p_j)$.

Besides, we know that $B \subseteq \Sigma^*$ is minimal implying ϕ if and only if B is minimal such that $B \cup \{\neg\phi\}$ is inconsistent. We can prove even more:

PROPOSITION 6
Let Σ be an SPL base.

1. If K is a minimal inconsistent subset of $\Sigma^* \cup \{\neg\phi\}$ containing $\neg\phi$, then $K \setminus \{\neg\phi\}$ is consistent, minimal implying ϕ.
2. If B is consistent and minimal implying ϕ then $B \cup \{\neg\phi\}$ is a minimal inconsistent subset of $\Sigma^* \cup \{\neg\phi\}$.

PROOF.

1. Let $K \subseteq \Sigma^* \cup \{\neg\phi\}$ such that $\neg\phi \in K$ and K is minimal inconsistent in $\Sigma^* \cup \{\neg\phi\}$. Let $B = K \setminus \{\neg\phi\}$. Clearly, $B \subseteq \Sigma^*$ is consistent, and $B \vdash \phi$ as $B \cup \{\neg\phi\}$ is inconsistent. Assume that

B is not minimal implying ϕ; then, there exists $B' \subset B$ such that $B' \vdash \phi$. So, $B' \cup \{\neg\phi\} \subset K$ is inconsistent, which is in contradiction with the hypothesis that K is minimal inconsistent in $\Sigma^* \cup \{\neg\phi\}$.

2. Let $B \subseteq \Sigma^*$ be minimal implying ϕ and assume that B is consistent. The set $K = B \cup \{\neg\phi\}$ is inconsistent in $\Sigma^* \cup \{\neg\phi\}$ and contains $\neg\phi$. Assume that K is not minimal inconsistent in $\Sigma^* \cup \{\neg\phi\}$. There exists $K' \subset K$ such that K' is inconsistent, and we can even assume that K' is minimal inconsistent.

    - Either $\neg\phi \in K'$, so by the first point we have that $K' \setminus \{\neg\phi\}$ is minimal implying ϕ and $K' \setminus \{\neg\phi\} \subset K \setminus \{\neg\phi\} = B$, in contradiction with the hypothesis on B.
    - Either $\neg\phi \notin K'$, so $K' \subset B$. As K' is inconsistent, B is also inconsistent, in contradiction with the hypothesis on B.

    So K is minimal inconsistent in $\Sigma^* \cup \{\neg\phi\}$. ■

Due to Proposition 5 and Proposition 6, computing $N_\Sigma^\vdash(\phi)$ amounts to determining:

- the set of minimal inconsistent subsets K_i of $\Sigma^* \cup \{\neg\phi\}$ containing $\neg\phi$;
- the minimal inconsistent subsets of Σ^* which do not contain any of the $B_i = K_i \setminus \{\neg\phi\}$'s obtained in the previous step.

The above computation comes down to the well-known problem of determining the minimal inconsistent subsets, forming a set $MIS(S)$, of a given set of formulas S. Let $\mathcal{B}^\vdash(\phi) = \{B \subseteq \Sigma^* | B \cup \{\neg\phi\} \in MIS(\Sigma^* \cup \{\neg\phi\})\}$ and $\mathcal{I}(\phi) = \{B \in MIS(\Sigma^*) | B$ does not contain any base from $\mathcal{B}^\vdash(\phi)\}$. Then let $\mathcal{B}(\phi) = \mathcal{B}^\vdash(\phi) \bigcup \mathcal{I}(\phi)$. The necessity degree of a formula ϕ can be computed as follows:

$$N_\Sigma^\vdash(\phi) = \max_{B \in \mathcal{B}(\phi)} \min_{\phi_j \in B} p_j. \tag{8}$$

The most efficient methods for solving the *MIS* problem exploit the duality that exists between minimal inconsistent subsets $MIS(S)$, and maximal consistent subsets $MCS(S)$, and the fact that checking the consistency of a base is less time-consuming than checking its inconsistency [25]. Given a propositional base S, $MIS(S)$ is obtained from $MCS(S)$ using hitting-sets [25, 27]. It is clear that we can use these efficient methods to compute the symbolic weights of consequences of SPL bases. However, these composite weights can be simplified due to the knowledge of constraints between elementary weights in C. The next section tries to operate such simplifications during the computation of symbolic weight expressions.

5.2 Inference methods based on the derivation of prime implicates

In this section, we present another syntactic method for SPL inference, inspired by an approach to abductive reasoning, whereby we consider elementary symbolic weights as additional propositional variables. Namely, the elementary weights involved in the computation of $N_\Sigma^\vdash(\phi)$ are viewed as assumptions that explain the certainty of ϕ. It suggests using assumption-based reasoning, as in the framework of Assumption-based Truth Maintenance Systems (ATMS) [11] designed to manage dependencies in a knowledge base.

An assumption-based theory is a pair $(\mathcal{K}, \mathcal{A})$, where $\mathcal{K}$ is a consistent base of propositional formulas, and $\mathcal{A}$ is a set of propositional variables called assumptions. An interesting form of reasoning in assumption-based theories is inference from assumptions to their consequences. In the general useful case, all assumptions cannot be true simultaneously in the context of $\mathcal{K}$. So it is necessary to determine incompatible subsets of assumptions. In the ATMS framework, minimal incompatible subsets of assumptions are called *nogoods*.

DEFINITION 7
Let $(\mathcal{K}, \mathcal{A})$ be an assumption-based theory.

- Any subset E of $\mathcal{A}$ is called an environment
- An environment E is $\mathcal{K}$-incoherent if and only if $E \cup \mathcal{K}$ is inconsistent
- A nogood is a minimal (for set-inclusion) $\mathcal{K}$-incoherent environment

In the following, we show how to encode an SPL base in order to use assumption-based reasoning for computing the necessity degree of a formula. More precisely, we show how the inconsistency degree of an SPL base can be computed from the nogoods of an appropriate assumption-based theory. That will enable us to compute the necessity degree of a formula, owing to the following result (Section 3.3, Proposition 2):

$$N^{\vdash}_{\Sigma}(\phi) = N^{\vdash}_{\Sigma \cup \{(\neg\phi,1)\}}(\bot) = Inc(\Sigma \cup \{(\neg\phi, 1)\}).$$

An SPL base Σ can be expressed as a standard propositional base, by turning each elementary weight a_i into a propositional variable (for simplicity still denoted by a_i) interpreted as an assumption, and replacing each SPL formula (ϕ_i, a_i) by the propositional formula $\neg a_i \vee \phi_i$.

DEFINITION 8
Let Σ be an SPL base. The associated assumption-based theory $(\mathcal{K}_\Sigma, \mathcal{A}_\Sigma)$ is defined by:

- $\mathcal{K}_\Sigma = \{\neg a_i \vee \phi_i | (\phi_i, a_i) \in \Sigma\}$
- $\mathcal{A}_\Sigma = \{a_i | (\phi_i, a_i) \in \Sigma\}$

This idea goes back to [14] where it was noticed that the set of models of a clause of the form $\neg\psi \vee \phi$ is the same as the set of models of the possibilistic logic formula (ϕ, ψ) where, in the latter expression, ψ is viewed as a Boolean symbolic necessity weight whose value depends on the considered interpretation. Let χ_ψ be the characteristic function of the set of models of ψ ($\chi_\psi(\omega) = 1$ if $\omega \models \psi$, and 0 otherwise). Then it holds

$$\pi_{(\phi,\psi)}(\omega) = \chi_{\neg\psi\vee\phi}(\omega) = \begin{cases} 1 \text{ if } \omega \models \phi \\ 1 - \chi_\psi(\omega) \text{ otherwise.} \end{cases}$$

As shown in the previous subsection, in order to compute $Inc(\Sigma)$, we have to consider the minimal inconsistent subsets $I_1, \ldots, I_p$ of Σ^*, so that

$$Inc(\Sigma) = N^{\vdash}_{\Sigma}(\bot) = \max_{k=1}^{p} \min_{\phi_j \in I_k} a_j.$$

Moreover, we only need the weights associated with the formulas belonging to these subsets. With the encoding of Definition 8, it is easy to see that each minimal inconsistent subset I_k of Σ^* exactly corresponds to a nogood $\mathcal{N}_k$ with respect to the assumption-based theory $(\mathcal{K}_\Sigma, \mathcal{A}_\Sigma)$.

So, it follows that:

PROPOSITION 7
Given an SPL base Σ and the associated assumption-based theory $(\mathcal{K}_\Sigma, \mathcal{A}_\Sigma)$, let $\{\mathcal{N}_1, \cdots, \mathcal{N}_p\}$ be the set of *nogoods* of $(\mathcal{K}_\Sigma, \mathcal{A}_\Sigma)$. We have: $Inc(\Sigma) = N^{\vdash}_{\Sigma}(\bot) = \max_{k=1}^{p} \min_{a \in \mathcal{N}_k} a$.

As we are rather interested in $N^{\vdash}_{\Sigma\cup\{(\neg\phi,1)\}}(\bot)$, we have to determine the minimal inconsistent subsets of $(\Sigma\cup\{(\neg\phi, 1)\})^*$. Note that adding $(\neg\phi, 1)$ to the base Σ can be encoded by adding the formula $\neg\phi$ to the propositional base $\mathcal{K}_\Sigma$. Indeed, no assumption is needed to explain the certainty of $\neg\phi$, as $\neg\phi$ is assumed to be certainly true. So the assumption-based theory associated with $\Sigma \cup \{(\neg\phi, 1)\}$ is $(\mathcal{K}_\Sigma \cup \{\neg\phi\}, \mathcal{A}_\Sigma)$.

As a consequence, we have:

PROPOSITION 8
Consider an SPL base Σ, and its associated assumption-based theory $(\mathcal{K}_\Sigma, \mathcal{A}_\Sigma)$ as well as a formula ϕ. Let $\{\mathcal{N}_1, \cdots, \mathcal{N}_k\}$ be the *nogoods* of $(\mathcal{K}_\Sigma \cup \{\neg\phi\}, \mathcal{A}_\Sigma)$. It holds that: $N^{\vdash}_{\Sigma}(\phi) = \max_{k=1}^{p} \min_{a\in\mathcal{N}_k} a$.

EXAMPLE 4
Suppose the language $\mathcal{L}$ has atomic variables α, β. Let $\Sigma = \{(\neg\alpha \vee \beta, a), (\alpha, b), (\neg\beta, c), (\beta, d)(\neg\alpha, e)\}$. Σ is encoded by:

- $\mathcal{K}_\Sigma = \{\neg a \vee \neg\alpha \vee \beta, \neg b \vee \alpha, \neg c \vee \neg\beta, \neg d \vee \beta, \neg e \vee \neg\alpha\}$
- $\mathcal{A}_\Sigma = \{a, b, c, d, e\}$

The set of nogoods of $(\mathcal{K}_\Sigma \cup \{\neg\beta\}, \mathcal{A}_\Sigma)$ is $\{\{d\}, \{a, b\}, \{b, e\}\}$. So $N^{\vdash}_{\Sigma}(\beta) = \max(d, \min(a, b), \min(b, e))$.

Nogoods can be computed using consequence finding algorithms (see [28] for an extensive survey of the state of the art in the XXth century). In particular, nogoods can be characterized in terms of special prime implicates, as explained below:

DEFINITION 9
Let Γ be a set of literals of the propositional language, and $\mathcal{K}$ a set of formulas.

- A clause κ is a Γ-implicate of $\mathcal{K}$ iff $\mathcal{K} \models \kappa$ holds and all the literals of κ belong to Γ.
- A clause κ is a Γ-prime implicate of $\mathcal{K}$ iff
  1. κ is a Γ-implicate of $\mathcal{K}$, and
  2. for every Γ-implicate κ' of $\mathcal{K}$, if $\kappa' \models \kappa$ holds, then $\kappa \models \kappa'$ holds.

The set of Γ-prime implicates of $\mathcal{K}$ is denoted by $PI_\Gamma(\mathcal{K})$.

PROPOSITION 9 ([28])
Let $(\mathcal{K}, \mathcal{A})$ be an assumption-based theory. Let $\neg\mathcal{A}$ denote the set of negative literals associated with the variables of $\mathcal{A}$. The set of nogoods of $(\mathcal{K}, \mathcal{A})$ is $\{Lit(\delta) | \neg\delta \in PI_{\neg\mathcal{A}}(\mathcal{K})\}$, where $Lit(\delta)$ denotes the set of literals of δ.

Note that in the above proposition, all elements of $PI_{\neg\mathcal{A}}(\mathcal{K})$ are clauses built on $\neg\mathcal{A}$, $\neg\delta$ is a clause, so δ is a conjunction of variables of $\mathcal{A}$.

As a consequence, the set of nogoods used in Proposition 8 for obtaining $N^{\vdash}_{\Sigma}(\phi)$ can be computed as $\{Lit(\delta) | \neg\delta \in PI_{\neg\mathcal{A}_\Sigma}(\mathcal{K}_\Sigma \cup \{\neg\phi\})\}$.

Many algorithms have been designed for computing prime implicates and Γ-prime implicates of a base $\mathcal{K}$ (see [28] for a survey). The computation of the Γ-prime implicates of $\mathcal{K}$ from the prime implicates of $\mathcal{K}$ can be achieved in an efficient way:

- One of the available methods is based on the technique of ‘variable forgetting’. In words, the Γ-prime implicates of $\mathcal{K}$ are obtained as the prime implicates of a new base, denoted by

Forget$(\mathcal{K}, Lit(\mathcal{K}) \setminus \Gamma)$, in which all the literals outside of Γ are eliminated from $\mathcal{K}$. It remains to compute *Forget*$(\mathcal{K}, Lit(\mathcal{K}) \setminus \Gamma)$, which can be computationally expensive. Forgetting a literal ℓ from $\mathcal{K}$ comes down to computing $\mathcal{K}_{\ell \leftarrow \top} \vee (\neg\ell \wedge \mathcal{K})$ where $\mathcal{K}_{\ell \leftarrow \top}$ is what remains of $\mathcal{K}$ after making ℓ true. The variable-forgetting method is used by Benferhat and Prade [7] to compute $N^{\vdash}_{\Sigma}(\phi)$.
- Resolution-based approaches to the direct generation of Γ-prime implicates also exist, inspired by resolution-based approaches to the computation of prime implicates.
- Other algorithms take advantage of the structure of the base $\mathcal{K}$, especially when $\mathcal{K}$ is given in Disjunctive Normal Form (DNF). For instance, forgetting a literal from a formula in DNF comes down to removing each contradictory term in it, then removing the literal from each remaining term.

Note that computing nogoods is exactly the same problem as calculating the MIS of $\Sigma^* \cup \{\neg\phi\}$. However, one of the benefits of the ATMS method is the fact that everything is computed in terms of weights (encoded by assumptions). The next step is to simplify the expression of $N^{\vdash}_{\Sigma}(\phi)$ prior to comparing the degrees of two formulas, using the constraints on weights. Using ATMS methods, simplifications can be carried out in the course of calculations. This is the topic of the next section.

5.3 Simplifying complex symbolic weights

Due to Proposition 8, $N^{\vdash}_{\Sigma}(\phi) = \max_{k=1}^{p} \min_{a \in \mathcal{N}_k} a$ where $\{\mathcal{N}_1, \cdots, \mathcal{N}_p\}$ are the nogoods of $(\mathcal{K}_\Sigma \cup \{\neg\phi\}, \mathcal{A}_\Sigma)$. So, $N^{\vdash}_{\Sigma}(\phi)$ can be simplified:

- first by replacing each set of weights $\mathcal{N}_i$ by the reduced set of weights $W = \min_C(\mathcal{N}_i)$ consisting of the least elementary weights in $\mathcal{N}_i$ according to the partial order defined by the constraints in C,
- then by deleting the dominated sets W_i in the following sense: W_i is dominated by W_j iff $C \vDash \forall a \in W_j, \exists b \in W_i, a \geq b$.

Note that when performing the above simplifications, all constraints in C may be interpreted as weak constraints. Actually, the fact that a constraint is strict is not exploited when simplifying a max/min expression. Strict constraints will be instrumental later for deriving strict comparisons between two complex weights $N^{\vdash}_{\Sigma}(\phi)$ and $N^{\vdash}_{\Sigma}(\psi)$. It is natural to simplify their expressions prior to comparing them.

In the assumption-based reasoning method, one can think of exploiting constraints on weights at the moment we are producing them and simplify the sets of weights. Namely, these simplifications can be made in the course of the calculations, with an appropriate logical encoding of the constraints, interpreted as weak ones. The idea, already exploited in [7], is to complete the assumption-based theory with formulas encoding the constraints, so that the nogoods obtained from the new base are exactly the reduced sets of weights W_i that are non-dominated.

More precisely, let $\mathcal{N}$ be a nogood of the assumption-based theory $(\mathcal{K}, \mathcal{A})$. Knowing that $a_1 \geq a_2$ with $\{a_1, a_2\} \subseteq \mathcal{N}$, we have $\min_{a \in \mathcal{N}} a = \min_{a \in \mathcal{N} \setminus \{a_1\}} a$. From Definition 7, it follows easily that, given $\{a_1, a_2\} \subseteq \mathcal{N}$, $\mathcal{N} \setminus \{a_1\}$ is a nogood of $(\mathcal{K} \cup \{\neg a_2 \vee a_1\}, \mathcal{A})$ iff $\mathcal{N}$ is a nogood of $(\mathcal{K}, \mathcal{A})$.

Moreover, let $\mathcal{N}_i$ be dominated by $\mathcal{N}_j$, meaning $C \vDash \forall a \in \mathcal{N}_j, \exists b \in \mathcal{N}_i, a \geq b$. If we replace each $a \geq b$ by the formula $\neg b \vee a$, it is easy to see that the formula $\bigvee_{b \in \mathcal{N}_i} \neg b$ is no longer a $\neg\mathcal{A}-$prime implicate of $(\mathcal{K} \cup \{a \to b : a \geq b \in C\}, \mathcal{A})$.

The discussion above leads to the following logical encoding of the constraints, similar to the one in [7]:

DEFINITION 10
Let (Σ, C) be an SPL base with constraints. The logical encoding of C is defined by the base $\mathcal{K}_C = \{\neg b \vee a : (a > b) \in C \text{ or } (a \geq b) \in C\}$.

It follows that:

PROPOSITION 10
Given an SPL base (Σ, C) with constraints, the assumption-based theory $(\mathcal{K}_\Sigma, \mathcal{A}_\Sigma)$ and the base $\mathcal{K}_C$. The reduced sets of weights that are non-dominated $\{W_1, \cdots, W_k\}$ are the nogoods of $(\mathcal{K}_\Sigma \cup \mathcal{K}_C \cup \{\neg\phi\}, \mathcal{A}_\Sigma)$.

In other words, from the nogoods of $(\mathcal{K}_\Sigma \cup \mathcal{K}_C \cup \{\neg\phi\}, \mathcal{A}_\Sigma)$, we obtain directly a simplified form of the degree $N^{\vdash}_\Sigma(\phi)$ under the form $\max_{k=1}^{p} \min_{a \in W_k} a$, where the W_k are reduced non-dominated sets of weights.

Example 4 (continued) *Let* $\Sigma = \{(\neg\alpha \vee \beta, a), (\alpha, b), (\neg\beta, c), (\beta, d)(\neg\alpha, e)\}$ *and* $C = \{a > d, d > e, d > c, b > e, c > e\}$.

- $\mathcal{K}_\Sigma = \{\neg a \vee \neg\alpha \vee \beta, \neg b \vee \alpha, \neg c \vee \neg\beta, \neg d \vee \beta, \neg e \vee \neg\alpha\}$
- $\mathcal{A}_\Sigma = \{a, b, c, d, e\}$
- $\mathcal{K}_C = \{\neg d \vee a, \neg e \vee d, \neg c \vee d, \neg e \vee b, \neg e \vee c\}$

The set of nogoods of $(\mathcal{K}_\Sigma \cup \{\neg\beta\}, \mathcal{A}_\Sigma)$ is $\{\{d\}, \{a, b\}, \{b, e\}\}$. Using C we can reduce $\{b, e\}$ to $\{e\}$ and $N^{\vdash}_\Sigma(\beta)$ to $\max(d, \min(a, b))$. But note that

- The set of nogoods of $(\mathcal{K}_\Sigma \cup \mathcal{K}_C \cup \{\neg\beta\}, \mathcal{A}_\Sigma)$ is exactly $\{\{d\}, \{a, b\}\}$.
- The set of nogoods of $(\mathcal{K}_\Sigma \cup \mathcal{K}_C \cup \{\alpha\}, \mathcal{A}_\Sigma)$ is $\{\{c\}\}$. So $N^{\vdash}_\Sigma(\neg\alpha) = c$.

5.4 Comparing complex symbolic weights

Once we have obtained their simplified expressions, the last step is to compare the certainty degrees $N^{\vdash}_\Sigma(\phi)$ and $N^{\vdash}_\Sigma(\psi)$ of two formulas ϕ and ψ, which are max/min expressions, using the constraints in C. We have to check whether $C \vDash N^{\vdash}_\Sigma(\phi) > N^{\vdash}_\Sigma(\psi)$ in order to decide if ϕ is more certain than ψ. More precisely, we must prove that

$$\max_{k=1}^{p} \min_{a \in W_k} a > \max_{j=1}^{q} \min_{b \in V_j} b$$

where the W_k's (resp. V_j's) are reduced non-dominated sets of weights obtained for ϕ (resp. ψ). By construction, elementary weights within W_k and V_j are not comparable using constraints in C.

It amounts to finding an expression $\min(a_{i1}, \cdots a_{in})$ in $N^{\vdash}_\Sigma(\phi)$ which dominates all expressions $\min(b_{j1}, \cdots, b_{jm})$ in $N^{\vdash}_\Sigma(\psi)$. Then, for each $j = 1 \cdots q$, for the pair of sets (W_k, V_j) we have to check whether there exists $b_j \in V_j$ such that $C \models a_k > b_j$, for all $a_k \in W_k$. All in all, we have to check whether

$$C \models \exists k, \forall j, \exists b_j \in V_j, \forall a_k \in W_k, a_k > b_j.$$

This is obtained by Algorithm 2 (for max–max comparison) that calls Algorithm 1 (for the inner min–min comparisons). It is clear that this step of the computation exploits strict constraints between elementary weights in C, possibly combined with weak ones via transitivity. As said earlier, if there were no strict constraints in C, there would be no way of producing strict ones here.

Algorithm 1: *Comp_Min*

Data: F and G two sets of weights, C a set of constraints.
Result: $\min F > \min G$?
———————————————

Dec:=false;
while *Dec=false and* $b_i \in G$ **do**
    Dec:=true;
    while *Dec=true and* $a_i \in F$ **do**
        Dec:=Dec $\wedge\ a_i > b_i \in C$;
return *Dec;*

Algorithm 2: *Comp_Max*

Data: $\mathcal{F}$ and $\mathcal{G}$ two families of sets of weights, C a set of constraints
Result:
$\max_{F \in \mathcal{F}} \min F > \max_{G \in \mathcal{G}} \min G$?
—————————————————

Dec:=false;
while *Dec=false and* $E_j \in \mathcal{F}$ **do**
    Dec:=true;
    while *Dec=true and* $E_i \in \mathcal{G}$ **do**
        Dec:=Dec $\wedge$ *Comp_Min*(E_i, E_j, C);
return *Dec;*

Example 4 (continued) $\Sigma = \{(\neg\alpha \vee \beta, a), (\alpha, b), (\neg\beta, c), (\beta, d)(\neg\alpha, e)\}$ and $C = \{a > d, d > e, d > c, b > e, c > e\}$. We want to compare $N_\Sigma^\vdash(\beta)$ and $N_\Sigma^\vdash(\neg\alpha)$. We have: $N_\Sigma^\vdash(\beta) = \max(d, \min(a, b))$ and $N_\Sigma^\vdash(\neg\alpha) = c$. Now, as $d > c \in C$, we have $\max(d, \min(a, b)) \geq d > c$, so we can conclude that $C \models N_\Sigma^\vdash(\beta) > N_\Sigma^\vdash(\neg\alpha)$, i.e. $(\Sigma, C) \vdash_\pi \beta > \neg\alpha$.

REMARK 3
In the above considerations, we have assumed that weights attached to formulas in $\mathcal{K}$ are elementary. However, this assumption can be dropped provided that the constraints in C only compare elementary weights. Indeed, weighted formulas of the form $(\phi, \min(a, b))$ or $(\phi, \max(a, b))$ can be handled without any problem when computing the weight of conclusions. In the assumption-based approach, $(\phi, \max(a, b))$ can be replaced by two formulas $\neg a \vee \phi$ and $\neg b \vee \phi$, and $(\phi, \min(a, b))$ by the formula $\neg a \vee \neg b \vee \phi$ without altering the semantic content of the SPL base. Moreover, in the assumption-based approach, any dominance constraint between max/min expressions can be replaced by a disjunction of formulas of the form $\neg a \vee b$ in $\mathcal{K}_C$. However when comparing the weights of conclusion and checking a possible strict dominance, the presence of strict constraints between max/min expressions in C will be difficult to handle and the two above algorithms are insufficient to that effect.

6 Related works

The question of encoding a partially ordered knowledge base using a symbolic counterpart of possibilistic logic has been addressed previously in [7] (a preliminary draft being [4]). These papers have introduced SPL, and especially proposed to encode symbolic possibilistic pairs in propositional logic like in subsections 5.2 and 5.3. However there are several differences with the present approach, that we discuss below.

1. For the computation of symbolic weights attached to consequences, each possibilistic formula (ϕ, a) in SPL is encoded in [7] as a formula $A \vee \phi$ where A is a variable supposed to mean '$\geq a$', i.e. $[a, 1]$ (while we use $\neg a \vee \phi \in \mathcal{K}_\Sigma$). Encoding an SPL possibilistic formula in one way or another is immaterial, for the purpose of deriving and simplifying symbolic weights.
2. Constraints between weights in [7] are weak, of the form $p \geq q$ with complex max–min weights, and lead to a partial preorder on C. In contrast, in our paper we can express both weak and strict inequality constraints (while C contained only strict inequalities in [8]). Note that the propositional encoding of C using material implications between reified elementary weights can only express a weak inequality between them (e.g. $[a, 1] \subseteq [b, 1]$).
3. In [7], the necessity of drawing plausible conclusions from an SPL base while only using weak constraints led them to propose a definition of semantic inference on weights, we denote by $C \mid\!\sim p > q$, very different from the one in our approach. They define $C \mid\!\sim p > q$ as $C \vDash p \geq q$ and $C \nvDash q \geq p$. This is somewhat analogous to strict Pareto order between vectors. However, following this view, we could infer $N_\Sigma(\alpha \vee \beta) > N_\Sigma(\alpha)$ from $\Sigma = \{(\alpha, a), (\beta, b)\}$ and $C = \varnothing$. Indeed, one has $N_\Sigma(\alpha) = a, N_\Sigma(\alpha \vee \beta) = \max(a, b)$, $C \vDash \max(a, b) \geq a$ but not $C \vDash a \geq \max(a, b)$, hence $C \mid\!\sim \max(a, b) > a$. This definition sounds questionable, unless we assume that distinct variables always take different values as in [4]. It is problematic because it amounts to interpreting strict inequality as the impossibility of proving a weak one, which is non-monotonic in essence. In our method, $p > q$ holds provided that it holds for all valuations of p, q in accordance with the constraints, which is a more standard interpretation of strict inequality between two symbolic expressions. Under our semantics, the absence of strict constraints forbids the derivation of strict relative certainty between formulas.
4. Finally in [7], the focus is on deducing plausible conclusions only, i.e. conclusions whose weight is strictly greater than the level of inconsistency, in other words, formulas ϕ such that $C \mid\!\sim N_\Sigma(\phi) > N_\Sigma(\bot)$, while in our paper we try to build a partial order on the language, where $\phi > \psi$ whenever $C \vDash N_\Sigma(\phi) > N_\Sigma(\psi)$.

SPL is different from conditional logics due to Lewis [26] or Halpern [24], or yet the logic of relative certainty in [31]. Such logics syntactically encode the domination constraints between propositional formulas, using higher order atomic formulas of the form $\psi > \phi$ that model statements expressing that ψ is at least as certain as ϕ. Conjunctions, negations and disjunctions of such atomic formulas are allowed.

A major difference with conditional logics is that in SPL, the SPL base $\Sigma = \{(\phi, a), (\psi, b)\}$ with $C = \{a > b\}$ does not imply that $(\Sigma, C) \models \phi > \psi$. For instance, if $\phi \models \psi$, we shall only conclude that $(\Sigma, C) \models \psi \geq \phi$, since $\Sigma \vdash_\pi (\psi, \max(a, b))$. In contrast, assuming that $\psi > \phi$ holds in a conditional logic, while $\phi \models \psi$, will lead to a contradiction as $\psi > \psi$ can then be derived from the axioms of such conditional logic (e.g. in [31], from the orderliness axiom stating that from $\psi > \phi$ we can derive $\psi \vee \chi > \phi \wedge \xi$, and the irreflexivity of the relation $>$). On the contrary, SPL restores an ordering of formulas compatible with classical entailment, while a knowledge base in a conditional logic is a fragment of the final partial ordering on the language.

Another important difference is that SPL embeds a minimal commitment assumption (through the definition of the symbolic impossibility distribution ι_Σ on the interpretations), while conditional logics do not. For instance, in the conditional logic of [31], whose semantics is in terms of partial orderings between sets of interpretations, we conclude neither $\phi > \psi$ nor $\phi > \xi$ from $\phi > \psi \wedge \xi$, where $\psi \wedge \xi$ is consistent, and, say, $\psi \wedge \xi \models \phi$. In SPL, from $\Sigma = \{(\phi, a), (\psi \wedge \xi, b)\}$ and $C = \{a > b\}$, we can compute $N_\Sigma(\psi) = N_\Sigma(\xi) = b$ (minimal commitment), and check that $C \models N_\Sigma(\phi) > N_\Sigma(\psi) = N_\Sigma(\xi)$. So, in SPL, we conclude that *both* $\phi > \psi$ and $\phi > \xi$ hold. In other words, SPL derives a conclusion valid only for the minimally specific necessity measure in agreement with the constraints.

In contrast, the conditional logic of [31] does not assume minimal commitment. Its conclusions are valid for all necessity measures in agreement with the partial certainty order (for this semantics of the conditional logic of [31], see [9]). Concluding $\phi > \psi$ or $\phi > \xi$ from $\phi > \psi \wedge \xi$ in the conditional logic of [31] would mean to deduce that one of the statements '$N(\phi) > N(\psi)$, $\forall N$ such that $N(\phi) > N(\psi \wedge \xi)$' or '$N(\phi) > N(\xi)$, $\forall N$ such that $N(\phi) > N(\psi \wedge \xi)$' is valid, which is clearly not the case. So the conditional logic does not lend itself to a semantics in terms of an unknown underlying necessity ordering constrained by the comparative knowledge base. Indeed, under this epistemic view, a constraint such as $\phi > \psi \wedge \xi$ means $\exists N$, $N(\phi) > N(\psi \wedge \xi)$. One should conclude from it that $\exists N$, $N(\phi) > N(\psi)$ or $\exists N$, $N(\phi) > N(\xi)$, since $N(\psi \wedge \xi) = \min(N(\psi), N(\xi))$, i.e. one of $N(\phi) > N(\psi)$ or $N(\phi) > N(\xi)$ holds for the actual N. In fact, SPL chooses the least specific among them, for which $N(\phi) > N(\psi)$ and $N(\phi) > N(\xi)$ hold at the same time.

Finally, inferring $\phi > \psi$ from an SPL base with constraints (Σ, C) comes down to inferring $N_\Sigma(\phi) > N_\Sigma(\psi) \in (0, 1]$ for all standard possibilistic bases Σ_v obtained by replacing symbolic weights a_i by numbers $v(a_i) \in (0, 1]$ in agreement with C. In contrast, a knowledge base in the conditional logic of relative certainty [31] cannot be clearly interpreted as a partially known total ordering of formulas.

7 Conclusion

This paper is an extensive presentation of Symbolic Possibilistic Logic (SPL), with partially ordered symbolic weights. It is one possible approach to the study of inference from a partially ordered propositional base. We provide a proof of the soundness and completeness of this logic, that generalizes the completeness proof of standard possibilistic logic, based on maximal consistent subsets. Two syntactic inference methodologies are outlined, which allow us to infer new formulas with complex symbolic weights (necessity degrees of formulas): one that requires the enumeration of minimal inconsistent subsets to calculate necessity degrees; the other one uses results from the consequence finding literature (assumption-based approaches). It enables weighted formulas and constraints over weights to be encoded in propositional logic, so as to simplify the max/min expressions attached to deduced formulas during the inference process. Then such simplified max/min expressions need to be compared in view of the existing domination constraints between elementary weights.

This paper clearly shows the qualitative nature of possibilistic logic, even if in its original form, it handles numbers attached to propositional formulas. It is shown that possibilistic logic can be viewed as a variant of assumption-based reasoning. The use of symbolic weights may be instrumental when only partial knowledge of the strength or the priority between formulas is available, hence making possibilistic logic closer to real cognitive situations. Regarding future research, it is possible to use SPL in preference modeling [21], interpreting symbolic weights as priorities. Moreover, it is tempting to extend the syntax of SPL so as to allow for negations and disjunctions of symbolic weighted formulas, so as to provide a generalized possibilistic logic extension of SPL in the style of [20].

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Appendix A. Ordinal semantics of possibilistic bases

In standard possibilistic bases, we recall that $1 - \pi_\Sigma(\omega) = \max_{j:\phi_j \notin \Sigma(\omega)} p_j$ which corresponds to the so-called 'best-out' ordering [2], defined as follows. Let $\trianglerighteq_\Sigma$ be the total pre-order on Ω defined by:

$$\omega \trianglerighteq_\Sigma \omega' \text{ if and only if } \forall \phi_j \in \overline{\Sigma(\omega)}, \exists \psi_i \in \overline{\Sigma(\omega')} \text{ such that } p_i \geq p_j$$

PROPOSITION 11 ([2])
$\pi_\Sigma(\omega) \geq \pi_\Sigma(\omega')$ if and only if $\omega \trianglerighteq_\Sigma \omega'$.

The total pre-order $\trianglerighteq_\Sigma$ allows us to build a totally pre-ordered deductive closure based on a pre-ordering $\succcurlyeq$ as follows:

$$\phi \succcurlyeq \psi \text{ if and only if } \forall \omega \in \overline{[\phi]},\ \exists \omega' \in \overline{[\psi]},\ \omega' \trianglerighteq_\Sigma \omega$$

expressing the relative certainty of propositions in agreement with necessity measures. In particular, we have [12]:

$$\phi \succ \psi \text{ if and only if } N_\Sigma(\phi) > N_\Sigma(\psi).$$

It is important to notice that the pre-order $\succcurlyeq$ does not depend on the precise values of the weights p_i of formulas, if the priority order indicated by the p_i's remains unchanged. In this sense possibilistic logic is not a genuinely numerical uncertainty logic.

As in standard possibilistic logic, we can define an ordinal semantics for symbolic possibilistic bases in SPL. Let Σ be an SPL base, and $\omega, \omega' \in \Omega$ two interpretations. The possibilistic ordering reads

$$\omega >_\Sigma \omega' \text{ iff } C \vDash \iota_\Sigma(\omega) < \iota_\Sigma(\omega').$$

We can again define the best-out ordering [2]:

$$\omega \rhd_\Sigma \omega' \text{ iff } \forall \phi_j \in \overline{\Sigma(\omega)}, \exists \psi_i \in \overline{\Sigma(\omega')} \text{ such that } C \vDash p_i > p_j.$$

The possibilistic ordering $>_\Sigma$ and the best-out ordering coincide in standard possibilistic logic. However, if weights are symbolic, the best-out ordering semantics is at least as demanding, as suggested in [31].

Proposition 12
$\omega \rhd_\Sigma \omega'$ implies $\omega >_\Sigma \omega'$.

Proof. $\omega \rhd_\Sigma \omega'$ reads $\forall \phi_j \in \overline{\Sigma(\omega)}, \exists \phi_i \in \overline{\Sigma(\omega')}, \forall v \models C, v(p_i) > v(p_j)$.

So in this relation, for each ϕ_j there exists some ϕ_i, such that the inequality $v(p_i) > v(p_j)$ should hold for all valuations; so the choice of i depends on j only. The relation $\omega \rhd_\Sigma \omega'$ implies, exchanging the existential and the universal quantifiers:

$$\forall \phi_j \in \overline{\Sigma(\omega)},\ \forall v \vDash C, \exists \phi_i \in \overline{\Sigma(\omega')} \text{ with } v(p_i) > v(p_j).$$

That is equivalent to the following formulation:

$$\forall v \vDash C, \forall \phi_j \in \overline{\Sigma(\omega)}, \exists \phi_{i(v)} \in \overline{\Sigma(\omega')} : v(p_i) > v(p_j). \tag{A.1}$$

Now the choice of ϕ_i depends on v and ϕ_j. But, fixing v, we have $v(p_i) \in]0, 1]$, and we can consider ϕ_{i^*} such that $v(p_{i^*}) = \max\{v(p_i) : \phi_i \notin \Sigma(\omega')\}$ and conclude that (take $i = i^*$, $\forall j$ in (A.1)):

$$\forall v \models C, \exists \phi_i (= \phi_{i^*}) \in \overline{\Sigma(\omega')}, \forall \phi_j \in \overline{\Sigma(\omega)}, v(p_i) > v(p_j)$$

which precisely means $C \models \max_{j:\phi_j \notin \Sigma(\omega)} p_j < \max_{i:\phi_i \notin \Sigma(\omega')} p_i$, i.e. $\omega >_\Sigma \omega'$. ■

Note that Equation (A.1) is an alternative way of extending the best-out ordering to symbolic weights, as it requires that the standard best-out property holds for any valuation, which as we see is equivalent to ordering $>_\Sigma$ on interpretations. It is an open problem whether the converse of Proposition 12 is valid or not.

As a consequence of Proposition 12 we have the following result:

COROLLARY 4
Let Σ be an SPL base with constraints. The property '$\forall\omega \in \overline{[\phi]},\ \exists\omega' \in \overline{[\psi]},\ \omega' \rhd_\Sigma \omega$' implies $C \models N_\Sigma(\phi) > N_\Sigma(\psi)$.

PROOF. Assume $\forall\omega \in \overline{[\phi]}$, $\exists\omega' \in \overline{[\psi]}$, $\omega' \rhd_\Sigma \omega$. Due to Proposition 12, it follows that: $\forall\omega \in [\phi]$, $\exists\omega' \in [\psi]$, $\omega' >_\Sigma \omega$. This is equivalent to: $\forall\omega \in [\phi]$, $\exists\omega' \in [\psi]$, $\forall v \models C$, $v(\iota_\Sigma(\omega')) < v(\iota_\Sigma(\omega))$. This implies, exchanging the existential and the universal quantifiers: $\forall v \models C$, $\forall\omega \in [\phi]$, $\exists\omega' \in [\psi]$, $v(\iota_\Sigma(\omega')) < v(\iota_\Sigma(\omega))$. That means: $\forall v \models C$, $v(\min_{\omega \not\models \phi} \iota_\Sigma(\omega)) > v(\min_{\omega \not\models \psi} \iota_\Sigma(\omega'))$, i.e. $C \models N_\Sigma(\phi) > N_\Sigma(\psi)$. $\blacksquare$