Asymmetric Composition of Possibilistic Operators in Formal Concept Analysis: Application to the Extraction of Attribute Implications from Incomplete Contexts

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Formal concept analysis theory (FCA) classically relies on the use of the Galois powerset operator. Formal similarities between possibility theory and formal concept analysis have led to the use of possibilistic operators in FCA, which were ignored before. In this paper, an approach based on the use of asymmetric composition of the two most usual possibilistic operators is proposed. It enables us to complement the stem base, by deriving attribute implications with disjunctions on both sides of the implications. Besides, the approach is also generalized to incomplete contexts involving explicit positive and negative information. We outline the potential application of these results to the completion of TBoxes in description logic.

1. INTRODUCTION

Formal concept analysis (FCA)[1,2] exploits the classical Galois derivation operator applied to data sets represented by a relation between objects and attributes called a formal context. Formal concepts organized within a hierarchy (i.e., a partial ordering), called the concept lattice, can then be extracted from a formal context. During the past years, FCA has been applied in many different areas like psychology, sociology, anthropology, medicine, biology, linguistics, etc. In such cases, FCA unavoidably deals with relational information structures (formal contexts) derived from human investigation (judgment, observation, measure, etc.).

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In particular, the concept lattice has proved highly useful for extracting attribute implications[3] from data, namely formulas of the form $\{a_1, ..., a_n\} \longmapsto \{b_1, ..., b_m\}$, where both attribute sets $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_m\}$ are implicitly understood as conjunctions. That is, the intended reading of $\{a_1, ..., a_n\} \longmapsto \{b_1, ..., b_m\}$ is "if attributes a_1 and ... and a_n hold then attributes b_1 and ... and b_m also hold."

A stem base (or Guigues–Duquenne base)[2,4] is a set of attribute implications derived from a formal context, that is known to be minimal in the number of implications. There is also the Luxemburger base for partial attribute implications,[5] which admit of exceptions. Missaoui et al.[6,7] generated mixed attribute implications, i.e., attribute implications involving at least one positive attribute and one negative attribute.

About 30 years ago, Gargov et al.[8] observed that the usual modality of necessity in modal logic could be supplemented by another operator they called sufficiency. Twelve years later Düntsch and Orlowska[9,10] caught up this idea and studied the algebraic setting of the modal logic of sufficiency. They provided the sufficiency counterpart of so-called Boolean algebras with operators, originally due to Jónsson and Tarski,[11] and then studied Boolean algebras with both necessity and sufficiency operators. As modal logic semantics relies on accessibility relations, and since the sufficiency operator precisely corresponds to the Galois derivation operator in FCA, it is then clear that four powerset derivation operators can be applied to formal contexts in FCA. This formal algebraic setting was applied by Düntsch and Gediga[12,13] to qualitative data analysis at large (not only FCA but rough sets too; see also Yao and Chen[14] for more details).

About 10 years ago, Dubois et al.[15,16] have given a possibility-theoretic reading of FCA, highlighting the analogies between the four set functions of possibility theory and the four powerset derivation operators originally pointed out by Düntsch and Orlowska.[9] In particular, properties of the hybrid asymmetric composition of possibility and necessity operators were independently rediscovered.[17]

In this paper, we propose to enlarge the underlying semantics of attribute implications by considering "disjunctive attribute implications." Taking advantage of the algebraic structure of the set of all "open-closed" pairs of sets obtained using the asymmetric necessity-possibility operators composition, the first contribution of this paper consists of determining a minimal base of disjunctive attribute implications. The minimality is proved by exploiting the duality existing between the asymmetric necessity-possibility operator and the usual Galois connection.

Another contribution of the paper has to do with the notion of incomplete context. Indeed, it is widely agreed that formal contexts may be incomplete in practical applications.[18,19] Namely, we have to distinguish between the case of not knowing whether an object possesses an attribute or not, and the case when it is well known that the object does not possess this attribute. Like for complete formal contexts, it may be useful to deal with implications extracted from such incomplete formal contexts. For this purpose, we define so-called "certain (or sure) implications" that hold in all possible worlds compatible with the incomplete information, and "possible implications" which hold in at least one situation compatible with the incomplete information. Taking advantage of our previous results, a characterization

Table I. Formal context $\mathcal{K}_{\mathcal{S}}$.

$\mathcal{R}$	a_1	a_2	a_3	a_4	a_5
x_1			×	×	×
x_2	×				×
x_3		×	×		
x_4	×	×	×	×	
x_5			×	×	×
x_6			×		

of possible and certain attribute implications is proposed for both conjunctive and disjunctive semantics.

Finally, we outline an application of our theoretical results to description logics (DLs) when the negation connective is explicitly used in the ABox: we propose to complete a TBox from an Abox with General Concept Inclusion (GCI) by means of certain and possible attribute implications. This paper significantly expands and strengthens preliminary results recently presented in a workshop paper.[20]

The remainder of the paper is organized as follows. Section 2 gives the background on FCA. The possibility-theoretic view of FCA is discussed in Section 3, whereas the next section presents our first contribution, which highlights the interest of using possibility-necessity operators in extracting disjunctive attribute implications, and the corresponding minimal base, from formal contexts. The second contribution, related to certain and possible attribute implications extracted from incomplete formal contexts, is presented in Section 5 and finally, Section 6 outlines the application of our approach to DLs.

2. FORMAL CONCEPT ANALYSIS

FCA[2] essentially relies on a binary relation between a set of objects and a set of attributes. This relation is called a formal context. More formally, a formal context is a triple $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, where $\mathcal{O}$ is a set of objects, $\mathcal{P}$ a set of attributes (or properties[a]), and $\mathcal{R}$ a binary relation s.t. $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{P}$: $x\mathcal{R}a$ means that object x satisfies attribute a, namely $(x, a) \in \mathcal{R}$. Let $\mathcal{R}(x) = \{a \in \mathcal{P} \mid (x, a) \in \mathcal{R}\}$ be the set of properties satisfied by a given object x. Similarly, we can define $\mathcal{R}(a) = \{x \in \mathcal{O} \mid (x, a) \in \mathcal{R}\}$, the set of objects that satisfy (verify) property a.

Example 1. In the following, we consider an example of formal context $\mathcal{K}_{\mathcal{S}}$, where $\mathcal{O} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\mathcal{P} = \{a_1, a_2, a_3, a_4, a_5\}$. The relation $\mathcal{R}$ of $\mathcal{K}_{\mathcal{S}}$ is illustrated in Table I. For instance, object x_1 satisfies properties a_3, a_4, a_5.

2.1. Basic Notions

The paradigm of FCA[1] is classically based on a pair of so-called Galois derivation operators, here denoted by $(.)^{\Delta}_{\mathcal{K}}$, relating the power sets $2^{\mathcal{O}}$ and $2^{\mathcal{P}}$ in the setting

[a] In data analysis, they often call "attribute," which is really a Boolean property.

of a context $\mathcal{K}$. They are defined for two sets $X \in 2^{\mathcal{O}}$ and $A \in 2^{\mathcal{P}}$ as follows:

$$A^{\Delta}_{\mathcal{K}} = \{x \in \mathcal{O} \mid \forall a \in \mathcal{P}(a \in A \Rightarrow x\mathcal{R}a)\} = \{x \in \mathcal{O} \mid A \subseteq \mathcal{R}(x)\} = \bigcap_{a \in A} \mathcal{R}(a)$$

$$X^{\Delta}_{\mathcal{K}} = \{a \in \mathcal{P} \mid \forall x \in \mathcal{O}(x \in X \Rightarrow x\mathcal{R}a)\} = \{a \in \mathcal{P} \mid X \subseteq \mathcal{R}(a)\} = \bigcap_{x \in X} \mathcal{R}(x)$$

In the following, when there is no ambiguity on the underlying formal context $\mathcal{K}$, we drop the subscript in $A^{\Delta}_{\mathcal{K}}$ and $X^{\Delta}_{\mathcal{K}}$, and write A^{Δ}, X^{Δ} for simplicity.

Clearly, A^{Δ} corresponds to the set of objects that satisfy all attributes in A: It is sufficient for an object to possess all properties in A to know it lies in A^{Δ} (hence the name "sufficiency operator" used by Düntsch and Orlowska[9]). Similarly, X^{Δ} corresponds to the set of attributes that are satisfied by all objects in X. In particular, $\{x\}^{\Delta}$ is the set of (known) attributes of object x. A formal concept of $\mathcal{K}$ is a pair of sets (X, A) with $X \subseteq \mathcal{O}$, $A \subseteq \mathcal{P}$ s.t. $X^{\Delta} = A$ and $A^{\Delta} = X$. For instance, $(\{x_3, x_4\}, \{a_2, a_3\})$ is a formal concept of $K_{\mathcal{S}}$. The set of objects X is called the extent, and the set of attributes A, the intent of the formal concept (X, A). These sets are closed in the sense that $A^{\Delta\Delta} = A$ and $X^{\Delta\Delta} = X$. The set of all formal concepts (denoted by $\mathcal{C}^{\mathcal{K}}$), equipped with a partial order $\preceq$ defined as $(X_1, A_1) \preceq (X_2, A_2)$ if $X_1 \subseteq X_2$ (or equivalently, $A_2 \subseteq A_1$), forms a complete lattice (denoted by $\mathfrak{L}^{\mathcal{K}}$). Let us denote by $INT^{\mathcal{K}}$ the set of all intents of $\mathcal{C}^{\mathcal{K}}$.

FCA has proved to be instrumental for extracting (conjunctive) attribute implications from formal contexts, as mentioned in the Introduction. The following subsection gives a brief survey on this issue.

2.2. Conjunctive Attribute Implications

In the FCA paradigm[2,4], an attribute implication, denoted by $A \longmapsto B$ asserts a sure relationship between two subsets A, B of attributes ($A, B \in 2^{\mathcal{P}}$). The definition given below[2] expresses the satisfaction of such an implication intensionally.

DEFINITION 1. *A subset $M \subseteq \mathcal{P}$ satisfies an attribute implication $A \longmapsto B$ (denoted by $M \models A \longmapsto B$) iff $A \not\subseteq M$ or $B \subseteq M$.*

In the above definition, M can be replaced by $\{x\}^{\Delta}$ using an object x satisfying all attributes in M and falsifying other attributes. The satisfaction of an attribute implication may also be expressed in an extensional form as[2]

DEFINITION 2. *A formal context $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ satisfies an attribute implication $A \longmapsto B$, where $A, B \subseteq \mathcal{P}$ (denoted by $\mathcal{K} \models A \longmapsto B$) iff for each $x \in \mathcal{O}$, $\{x\}^{\Delta} \models A \longmapsto B$.*

From this definition, it may be remarked that an attribute implication of the form $A \longmapsto B$ holds in $\mathcal{K}$ iff $A^{\Delta} \subseteq B^{\Delta}$ or equivalently $B \subseteq A^{\Delta\Delta}$.

The semantics of such an attribute implication is that, for every object $x \in \mathcal{O}$, if every attribute from A applies to the object x, then every attribute from B also applies to x. For instance, in the formal context $K_{\mathcal{S}}$ given in Table I, all objects that have attributes "a_3" and "a_5," namely "x_1 and x_5," also have attribute "a_4." Such a piece of knowledge is represented through an attribute implication as

$$\{a_3, a_5\} \longmapsto \{a_4\}$$

Let $\mathcal{B}$ be a set of attribute implications and denote by $\delta(\mathcal{B})$ the set of subsets of $\mathcal{P}$ that satisfy every attribute implication in $\mathcal{B}$, i.e., $\delta(\mathcal{B}) = \{M \subseteq \mathcal{P} \mid M \models \mathcal{B}\}$, where $M \models \mathcal{B}$ stands for the satisfaction by M of every attribute implication in $\mathcal{B}$.

DEFINITION 3. [2] *A set of attribute implications $\mathcal{B}$ is*

- *sound for a context $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ iff this context satisfies every implication from $\mathcal{B}$.*
- *complete for a context $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ iff every subset of $\mathcal{P}$ that satisfies $\mathcal{B}$ is an intent of this context: $\delta(\mathcal{B}) \subseteq INT^{\mathcal{K}}$.*
- *irredundant if none of the attribute implications $A \longmapsto B \in \mathcal{B}$ follows from $\mathcal{B} \setminus \{A \longmapsto B\}$, i.e., there is a subset of $\mathcal{P}$ that satisfies $\mathcal{B} \setminus \{A \longmapsto B\}$ but does not satisfy $A \longmapsto B$.*

Then we can define a base for a context:

DEFINITION 4. *Let $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ be a formal context and $\mathcal{B} = \{A \longmapsto B \mid A, B \subseteq \mathcal{P}\}$ be a set of attribute implications. $\mathcal{B}$ is called a base for $\mathcal{K}$ iff it is sound, complete, and irredundant for this context. It is denoted by $\mathcal{B}^{\mathcal{K}}$.*

According to this definition, a base for a context $\mathcal{K}$ is minimal for inclusion, but we may have several such bases of attribute implications. The stem base[2,4] is one such minimal set of implications. We give hereafter its definition. To this end, we need the notion of pseudointent:

DEFINITION 5. *Given a formal context $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, a set $A \subseteq \mathcal{P}$ is called a pseudo-intent iff it is not an intent, i.e. $A^{\Delta\Delta} \neq A$, and it contains the $\Delta\Delta$-closures of all its subsets that are pseudointents.*

This rather unintuitive recursive definition must be applied staring from the empty set up. In particular, we must first look for singleton attributes that are not closed $((\{a\}^{\Delta})^{\Delta} \neq \{a\})$ as all of them comply with the definition of a pseudointent. For instance, in Example 1 it is clear that $\{a_2\}$ and $\{a_4\}$ are not closed and are pseudointents. Other singletons are closed. As a consequence, the set $\{a_1, a_3\}$, which is not an intent and does not contain any pseudointent, is itself a pseudointent. Additional pseudointents in this example are $\{a_3, a_5\}$ and $\{a_2, a_3, a_4\}$

Then a minimal base for a context is obtained as follows[4]:

PROPOSITION 1. *The set of implications $\{A \longmapsto A^{\Delta\Delta} \mid A$ is a pseudointent of $\mathcal{K}\}$ is minimal. It is called a stem base.*

We can replace $A \longmapsto A^{\Delta\Delta}$ by $A \longmapsto A^{\Delta\Delta} \setminus A$. In Example 1, the minimal base is

$$\Big\{a_2 \longmapsto a_3,\, a_4 \longmapsto a_3,\, \{a_1, a_3\} \longmapsto \{a_2, a_4\},\, \{a_3, a_5\} \longmapsto \{a_4\},\, \{a_2, a_3, a_4\} \longmapsto \{a_1\}\Big\}.$$

It is important to recall that the underlying semantics of rules is a conjunctive one. For $\{a_1, a_3\} \longmapsto \{a_2, a_4\}$, the reading is a_1 *and* $a_3 \longmapsto a_2$ *and* a_4. This kind of rule is enforced by the exclusive use of the classical adjoint pair $((.)^{\Delta}, (.)^{\Delta})$. In the next section, we consider alternative operators in FCA that lead us to rules of the form $a_5 \longmapsto a_1$ or a_4.

3. ASYMMETRIC COMPOSITIONS OF POSSIBILISTIC OPERATORS

As recalled in the preceding section, the Galois derivation operator in FCA is the sufficiency operator $(.)^{\Delta}$. At the end of the last century, this operator was identified as an alternative to modal possibility and necessity operators in modal logics by Düntsch and Orlowska.[9] These usual modalities are thus naturally applicable to FCA. The same authors investigated an algebraic setting for these operators.[10] Some years later, Dubois et al.[15,16] have pointed out the close links existing between the sufficiency operator in FCA and one of the four set functions of possibility theory,[21] namely the guaranteed possibility function.[22] These findings led to considering three other powerset derivation operators in FCA, namely the possibility operator (denoted $(.)^{\Pi}$), the necessity operator (denoted $(.)^{\mathrm{N}}$), and the dual sufficiency operator (denoted $(.)^{\nabla}$). The two former operators A^{Π} and A^{N} are recalled in the following.

A^{Π} corresponds to the set of objects that possess at least one attribute in A. Formally, we have

$$\begin{aligned} A^{\Pi} &= \{x \in \mathcal{O} \mid \exists a \in A, x\mathcal{R}a\} \\ &= \{x \in \mathcal{O} \mid A \cap \mathcal{R}(x) \neq \emptyset\} \\ &= \textstyle\bigcup_{a\in A} \mathcal{R}(a) \end{aligned}$$

A^{N} corresponds to the set of objects whose set of attributes is a subset of A.

$$\begin{aligned} A^{\mathrm{N}} &= \{x \in \mathcal{O} \mid \forall a \in \mathcal{P}(x\mathcal{R}a \Rightarrow a \in A)\} \\ &= \{x \in \mathcal{O} \mid \mathcal{R}(x) \subseteq A\} \end{aligned}$$

Sets of attributes X^{Π} and X^{N} are similarly obtained from a set of objects X, exchanging $\mathcal{O}$ and $\mathcal{P}$, and changing A into X in the above expressions. Notice that for singletons, $\{a\}^{\Pi} = \{a\}^{\Delta} = \mathcal{R}(a)$ and $\{x\}^{\Pi} = \{x\}^{\Delta} = \mathcal{R}(x)$.

Exemple 1 (continued). For the formal context $\mathcal{K}_s$ illustrated in Table I, we have

$\{a_3, a_4, a_5\}^{\Pi} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$,

since $\bigcup_{a\in\{a_3,a_4,a_5\}} \mathcal{R}(a) = \mathcal{R}(a_3) \cup \mathcal{R}(a_4) \cup \mathcal{R}(a_5) = \{x_1, x_3, x_4, x_5, x_6\} \cup \{x_1, x_4, x_5\} \cup \{x_1, x_2, x_5\}$.

$\{a_3, a_4, a_5\}^{\mathrm{N}} = \{x_1, x_5, x_6\}$,

since $\mathcal{R}(x_1), \mathcal{R}(x_5), \mathcal{R}(x_6) \subseteq \{a_3, a_4, a_5\}$ and $\mathcal{R}(x_2), \mathcal{R}(x_3), \mathcal{R}(x_4) \not\subseteq \{a_3, a_4, a_5\}$.
Let $x\overline{\mathcal{R}}a$ denote the fact that object x does not satisfy attribute a. Based on the closed world assumption (CWA), we define the complementary formal context $\overline{\mathcal{K}}$ of $\mathcal{K}$ as $\overline{\mathcal{K}} = (\mathcal{O}, \mathcal{P}, \overline{\mathcal{R}})$, where $\overline{\mathcal{R}} = \{(x, a) \in \mathcal{O} \times \mathcal{P} \mid (x, a) \notin \mathcal{R}\}$. In the rest of the paper, for reader convenience, when the derivation operators $(.)^{\Pi}$, $(.)^{\mathrm{N}}$, $(.)^{\Delta}$ are applied to the complementary context $\overline{\mathcal{K}}$ we use the explicit notation $(.)^{\Pi}_{\overline{\mathcal{K}}}$, $(.)^{\mathrm{N}}_{\overline{\mathcal{K}}}$, $(.)^{\Delta}_{\overline{\mathcal{K}}}$.

Given $X \subseteq \mathcal{O}$ and $\overline{X}$ its complementary set (i.e., $\mathcal{O} \setminus X$), the following recalls some useful easy to prove properties[12,17] needed in the rest of the paper.

PROPOSITION 2.

$$
\begin{array}{ll}
P_1 : X^{\Delta}_{\overline{\mathcal{K}}} = \overline{(X^{\Pi})} & P_5 : X \subseteq (X^{\Pi})^{\mathrm{N}} \\
P_2 : X^{\Delta}_{\overline{\mathcal{K}}} = (\overline{X})^{\mathrm{N}} & P_6 : (X^{\mathrm{N}})^{\Pi} \subseteq X \\
P_3 : \ If\ X_1 \subseteq X_2\ then\ (X_1)^{\Pi} \subseteq (X_2)^{\Pi} & P_7 : X^{\mathrm{N}} = \overline{(\overline{X})^{\Pi}} \\
P_4 : \ If\ X_1 \subseteq X_2\ then\ (X_1)^{\mathrm{N}} \subseteq (X_2)^{\mathrm{N}} & P_8 : X^{\Pi} = ((X^{\Pi})^{\mathrm{N}})^{\Pi}
\end{array}
$$

P_1 and P_2 express possibility and necessity operators in terms of the sufficiency one for the complementary context. P_7 expresses the duality between X^{Π} and X^{N}. P_3 and P_4 state that the possibility and necessity operators are isotone. P_5 claims that the combined operation $\mathrm{N} \circ \Pi$ is extensive, whereas P_6 claims that the combined operation $\Pi \circ \mathrm{N}$ is contractive. P_8 expresses that $\Pi \circ \mathrm{N}$ is an idempotent operator.

These properties are dually satisfied for subsets of attributes $A \subseteq \mathcal{P}$.

Let us denote by NΠ-pair, a pair of sets (X, A) s.t. $X = A^{\Pi}$ and $A = X^{\mathrm{N}}$. The set of objects X (resp. of attributes A) will be called NΠ-extent (resp. NΠ-intent). The set of all NΠ-pairs is denoted by $\mathcal{C}^{\mathcal{K}}_{\mathrm{N}\Pi}$, whereas the set $INT^{\mathcal{K}}_{\mathrm{N}\Pi}$ corresponds to the set of all NΠ-intents. For instance, $(\{x_1, x_2, x_4, x_5\}, \{a_1, a_4, a_5\}) \in \mathcal{C}^{\mathcal{K}}_{\mathrm{N}\Pi}$ of $\mathcal{K}_s$, whereas $\{a_1, a_4, a_5\} \in INT^{\mathcal{K}}_{\mathrm{N}\Pi}$.

Proposition 3 establishes a characterization of NΠ-pairs, in terms of usual formal concepts and corresponds to Corollary 1 by Düntsch and Gediga[12] However, we give the full proof for the sake of clarity.

PROPOSITION 3. *Let $X \in 2^{\mathcal{O}}$ and $A \in 2^{\mathcal{P}}$, (X, A) is an* NΠ*-pair in $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ iff $(\overline{X}, A)$ is a formal concept in $\overline{\mathcal{K}} = (\mathcal{O}, \mathcal{P}, \overline{\mathcal{R}})$.*

Proof. It is proved using properties P_1 and P_7.

$$
\begin{array}{lll}
& (\overline{X}, A)\ \text{a formal concept in}\ \overline{\mathcal{K}} & \\
\Longleftrightarrow & \overline{X} = (A)^{\Delta}_{\overline{\mathcal{K}}} \ \text{ and } \ A = (\overline{X})^{\Delta}_{\overline{\mathcal{K}}} & \text{(by definition)} \\
\Longleftrightarrow & \overline{X} = \overline{A^{\Pi}} \ \text{ and } \ A = \overline{((\overline{X})^{\Pi})}\ \text{in}\ \mathcal{K} & \text{(using } P_1) \\
\Longleftrightarrow & \overline{\overline{X}} = \overline{\overline{A^{\Pi}}} \ \text{ and } \ A = \overline{\overline{X}}^{\mathrm{N}} & \text{(using} P_7) \\
\Longleftrightarrow & X = A^{\Pi} \quad \text{ and } \ A = X^{\mathrm{N}} & \\
\Longleftrightarrow & (X, A)\ \text{is a}\ \mathrm{N}\Pi\text{-pair in}\ \mathcal{K}. &
\end{array}
$$

□

Since the formal concept $(\overline{X}, A)$ in $\overline{\mathcal{K}}$ is a pair of closed sets, the NΠ-pair (X, A) can be viewed as an "open-closed" pair. It is then clear that the set $\mathcal{C}^{\mathcal{K}}_{\mathrm{N}\Pi}$ equipped with a partial order (denoted by $\preceq$) defined as $(X_1, A_1) \preceq (X_2, A_2)$ if $X_1 \subseteq X_2$ (or, equivalently, $A_1 \subseteq A_2$) forms a complete lattice, called the NΠ-lattice and denoted by $\mathfrak{L}_{\mathrm{N}\Pi}$. The following proposition gives the supremum (least upper bound) and the infimum (greatest lower bound) for a given subset of $\mathfrak{L}_{\mathrm{N}\Pi}$. The next result gives the algebraic structure of the set $\mathcal{C}^{\mathcal{K}}_{\mathrm{N}\Pi}$. It mirrors the structure of the usual concept lattice.

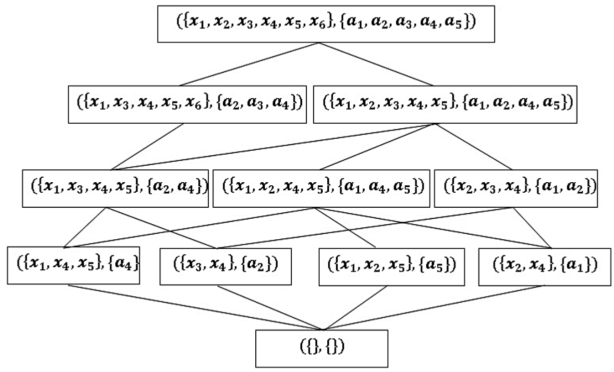

Figure 1. Lattice $\mathfrak{L}_{\mathrm{N}\Pi}$ for $\mathcal{K}_S$.

PROPOSITION 4. *The supremum (denoted) and infimum (denoted) of a subset* $\{(X_j, A_j) \mid j \in J\}$ *of* $\mathfrak{L}_{\mathrm{N}\Pi}$ *are given by*

$$\underset{j\in J}{}(X_j, A_j) = \Big(\underset{j\in J}{} X_j, \Big(\Big(\underset{j\in J}{} A_j \quad ^{\Pi} \quad ^{\mathrm{N}}$$

$$\prod_{j\in J}(X_j, A_j) = \Big(\Big(\Big(\bigcap_{j\in J} X_j \quad ^{\mathrm{N}} \quad ^{\Pi}, \bigcap_{j\in J} A_j \quad .$$

The proof is given in the Appendix for the sake of completeness.

Exemple 1 (continued). Figure 1 illustrates the $\mathfrak{L}_{\mathrm{N}\Pi}$ lattice corresponding to the formal context given in Table I. It can be derived from the usual concept lattice of the complementary context by duality. One can read the infimum and supremum from the $\mathfrak{L}_{\mathrm{N}\Pi}$ lattice. Choose any two NΠ-pairs from Figure 1 and follow the descending paths from the corresponding nodes in the $\mathfrak{L}_{\mathrm{N}\Pi}$ lattice. There is always a highest point where these paths meet, that is, a highest NΠ-pair that is below both, namely, the infimum. Similarly, for any two NΠ-pairs, there is always a lowest node (the supremum of the two) that can be reached from both NΠ-pairs via ascending paths.

It may also be computed using Proposition 4. For instance, the supremum and infimum of the two NΠ-pairs $(\{x_1, x_4, x_5\}, \{a_4\})$, $(\{x_1, x_2, x_5\}, \{a_5\})$ are obtained as follows:

$$\Big((\{x_1, x_4, x_5\}, \{a_4\}), (\{x_1, x_2, x_5\}, \{a_5\})$$
$$= (\{\{x_1, x_4, x_5\} \cup \{x_1, x_2, x_5\}\}, \{\{\{a_4\} \cup \{a_5\}\}^{\Pi}\}^{\mathrm{N}})$$
$$= (\{x_1, x_2, x_4, x_5\}, \{\{a_4, a_5\}^{\Pi}\}^{\mathrm{N}})$$
$$= (\{x_1, x_2, x_4, x_5\}, \{a_1, a_4, a_5\})$$

$$\prod\Big((\{x_1, x_4, x_5\}, \{a_4\}), (\{x_1, x_2, x_5\}, \{a_5\})$$
$$= (\{\{\{x_1, x_4, x_5\} \cap \{x_1, x_2, x_5\}\}^{\mathrm{N}}\}^{\Pi}, \{\{a_4\} \cap \{a_5\}\})$$
$$= (\{\{x_1, x_5\}^{\mathrm{N}}\}^{\Pi}, \{\})$$
$$= (\{\}, \{\})$$

□

4. MINIMAL BASE OF DISJUNCTIVE ATTRIBUTE IMPLICATIONS

Taking advantage of the use of pairs of modal or possibilistic operators in FCA recalled in Section 3, the first contribution of this paper is to give a well-founded approach to extracting so-called disjunctive attribute implications and obtaining a minimal base thereof. This approach follows directly from the duality existing between the usual Galois connection in FCA and the hybrid asymmetric composition of possibility and necessity operators just described, applying the standard theory to the complementary context. For the sake of self-containedness, we outline the counterpart to Guigues–Duquenne theory for disjunctive attribute implications in the Appendix, relying on the notion of NΠ-pair, that is the counterpart of formal concept.

4.1. Disjunctive Attribute Implications

The definition expressing the satisfaction of an attribute implication in an intensional way has been already given (see Definition 1). In the following, we extend this definition to disjunctive attribute implications of the form $a_1 \vee ... \vee a_n \longmapsto b_1 \vee ... \vee b_m$ (equivalently denoted by $\bigvee A \longmapsto \bigvee B$ with $A = \{a_1, ..., a_n\}$, and $B = \{b_1, ..., b_m\}$).

DEFINITION 6. *(Intension) A subset $M \subseteq \mathcal{P}$ satisfies $\bigvee A \longmapsto \bigvee B$ (denoted by $M \models \bigvee A \longmapsto \bigvee B$), if $B \not\subseteq M$ or $A \subseteq M$. We say that M satisfies a set $\mathcal{D}$ of disjunctive attribute implications (denoted by $M \models \mathcal{D}$), if M satisfies every disjunctive attribute implication in $\mathcal{D}$.*

In the scope of a formal context, the satisfaction of a disjunctive attribute implication is adapted as follows: replacing sets of attributes by their extents.

DEFINITION 7. *(Extension) Given a formal context $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, $\mathcal{K} \models \bigvee A \longmapsto \bigvee B$ iff $\forall x \in \mathcal{O}$, if $b_1 \notin \{x\}^{\Pi} \wedge ... \wedge b_m \notin \{x\}^{\Pi}$ then $a_1 \notin \{x\}^{\Pi} \wedge ... \wedge a_n \notin \{x\}^{\Pi}$.*

For example, the formal context $K_{\mathcal{S}}$ given in Table I satisfies the disjunctive attribute implication $a_4 \longmapsto a_1 \vee a_5$ ($K_{\mathcal{S}} \models a_4 \longmapsto a_1 \vee a_5$), since every object that satisfies neither a_1 nor a_5 does not satisfy a_4 either.

A more compact way is to assert the satisfaction of a disjunctive attribute implication based on the possibility operator $(.)^{\Pi}$, given a formal context $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$. Namely, $\mathcal{K} \models \bigvee A \longmapsto \bigvee B$ iff for each $x \in \mathcal{O}$, $B \not\subseteq \overline{\{x\}^{\Pi}}$ or $A \subseteq \overline{\{x\}^{\Pi}}$.

Table II. Complementary formal context.

$\mathcal{R}$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$	$\neg a_5$
x_1	×	×			
x_2		×	×	×	
x_3	×			×	×
x_4					×
x_5	×	×			
x_6	×	×		×	×

This is because the expression

$$\forall x \in \mathcal{O} \text{ if } b_1 \notin \{x\}^{\Pi} \wedge ... \wedge b_m \notin \{x\}^{\Pi} \text{ then } a_1 \notin \{x\}^{\Pi} \wedge ... \wedge a_n \notin \{x\}^{\Pi}$$

can be equivalently written as $\forall x \in \mathcal{O}$ if $\{b_1, ..., b_m\} \subseteq \overline{\{x\}^{\Pi}}$ then $\{a_1, ..., a_n\} \subseteq \overline{\{x\}^{\Pi}}$. It is then clear that a formal context $\mathcal{K}$ satisfies a disjunctive attribute implication $\bigvee A \longmapsto \bigvee B$ iff every object that satisfies no attribute from B also does not satisfy any attribute from A. Equivalently, any object that satisfies at least one attribute in A also satisfies at least one attribute in B.

In the following, we shall use the complementary formal context $\overline{\mathcal{K}}$. The conjunctive attribute implications obtained from $\overline{\mathcal{K}}$ involve negations of attributes. Thus, for the sake of clarity, if $A \subseteq \mathcal{P}$, we denote by $A^{\neg}$ the set $\{\neg a \mid a \in A\}$. For instance, considering Example 1, the complementary formal context of $\mathcal{K}_{\mathcal{S}}$ is in Table II. If $A = \{a_1, a_4, a_5\}$, we write $A^{\neg} = \{\neg a_1, \neg a_4, \neg a_5\}$. We can check that $\{\neg a_1, \neg a_5\} \longmapsto \{\neg a_4\}$ since $\{\neg a_1, \neg a_5\}^{\Delta}_{\overline{\mathcal{K}_{\mathcal{S}}}} = \{\neg a_1\}^{\Delta}_{\overline{\mathcal{K}_{\mathcal{S}}}} \cap \{\neg a_5\}^{\Delta}_{\overline{\mathcal{K}_{\mathcal{S}}}} \subseteq \{\neg a_4\}^{\Delta}_{\overline{\mathcal{K}_{\mathcal{S}}}}$.

The following proposition gives a useful result.

PROPOSITION 5. *The four following statements are equivalent:*

- *The disjunctive attribute implication* $\bigvee A \longmapsto \bigvee B$ *is valid in formal context* $\mathcal{K}$.
- *The (conjunctive) attribute implication* $B^{\neg} \longmapsto A^{\neg}$ *is valid in formal context* $\overline{\mathcal{K}}$.
- $A \subseteq (B^{\Pi})^{\mathrm{N}}$
- $A^{\Pi} \subseteq B^{\Pi}$

Proof. Suppose $B^{\neg} \longmapsto A^{\neg}$ is valid in $\overline{\mathcal{K}}$. In logical terms, it means $\wedge_{b \in B} \neg b \rightarrow \wedge_{a \in A} \neg a$, which is logically equivalent to $\vee_{a \in A} a \rightarrow \vee_{b \in B} b$. Now, $B^{\neg} \longmapsto A^{\neg}$ is valid in $\overline{\mathcal{K}}$ means $A \subseteq (B^{\Delta}_{\overline{\mathcal{K}}})^{\Delta}_{\overline{\mathcal{K}}}$, that is, $A \subseteq (\overline{B^{\Pi}})^{\Delta}_{\overline{\mathcal{K}}}$ iff $A \subseteq (B^{\Pi})^{\mathrm{N}}$. For the fourth item: $A \subseteq ((B)^{\Pi})^{\mathrm{N}}$ iff $(A)^{\Pi} \subseteq (((B)^{\Pi})^{\mathrm{N}})^{\Pi}$ (using P_3) iff $(A)^{\Pi} \subseteq (B)^{\Pi}$ (using P_8). □

For instance, $K_{\mathcal{S}} \models a_4 \longmapsto a_1 \vee a_5$ also reads $\{a_4\}^{\Pi} \subseteq \{a_1\}^{\Pi} \cup \{a_5\}^{\Pi}$. It is interesting to highlight that $(.)^{\Delta}$ and $(.)^{\Pi}$ play similar roles for extracting conjunctive and disjunctive rules, respectively. The condition for $A \longmapsto B$ to be a conjunctive implication rule for a context is $A^{\Delta} \subseteq B^{\Delta}$, and it is $A^{\Pi} \subseteq B^{\Pi}$ for disjunctive rules. It can also be shown directly as follows using property P_1:

$$P_9 : B^{\Delta}_{\mathcal{K}} \subseteq A^{\Delta}_{\mathcal{K}} \Longleftrightarrow \overline{B^{\Pi}_{\overline{\overline{\mathcal{K}}}}} \subseteq \overline{A^{\Pi}_{\overline{\overline{\mathcal{K}}}}} \Longleftrightarrow \overline{B^{\Pi}_{\mathcal{K}}} \subseteq \overline{A^{\Pi}_{\mathcal{K}}} \Longleftrightarrow A^{\Pi}_{\mathcal{K}} \subseteq B^{\Pi}_{\mathcal{K}}$$

Note that this proposal relies on a specific understanding of a context whereby a blank space at place (x, a) stands for the knowledge that x *does not* possess the attribute a. Should it be understood as ignorance, no such disjunctive rule would be meaningful for the context.

The following corollary, which follows directly from Proposition 5, will be used in the sequel. It establishes a link between the satisfaction of conjunctive and disjunctive attribute implications.

COROLLARY 1. *Let* $A, B, M \subseteq \mathcal{P}$. $M^{\neg} \models B^{\neg} \longmapsto A^{\neg}$ *iff* $M \models \bigvee A \longmapsto \bigvee B$

Proof. $M^{\neg} \models B^{\neg} \longmapsto A^{\neg}$ is equivalent to if $B^{\neg} \subseteq M^{\neg}$ then $A^{\neg} \subseteq M^{\neg}$, that is equivalent to if $B \subseteq M$ then $A \subseteq M$, which is precisely $M \models \bigvee A \longmapsto \bigvee B$. □

4.2. Minimal Base of Disjunctive Attribute Implications

Here we provide a method for constructing a minimal base of disjunctive attribute implications. Let $\mathcal{D} = \{\bigvee A \longmapsto \bigvee B \mid A, B \subseteq \mathcal{P}\}$ be a set of disjunctive attribute implications for $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, and let $\mathcal{D}^{\neg} = \{B^{\neg} \longmapsto A^{\neg} \mid B^{\neg}, A^{\neg} \subseteq \mathcal{P}^{\neg}, \bigvee A \longmapsto \bigvee B \in \mathcal{D}\}$ be the corresponding set of conjunctive attribute implications for $\overline{\mathcal{K}}$.

The following theorem provides a means to derive a sound, complete and irredundant set of disjunctive attribute implications from a formal context, from a stem base of its complementary context.

THEOREM 1. *Let* $\mathcal{D}^{\neg}$ *a base for* $\overline{\mathcal{K}}$*,*

(i) $\mathcal{D}^{\neg}$ *is sound for* $\overline{\mathcal{K}}$ *iff* $\mathcal{D}$ *is sound for* $\mathcal{K}$*.*
(ii) $\mathcal{D}^{\neg}$ *is complete for* $\overline{\mathcal{K}}$ *iff* $\mathcal{D}$ *is complete for* $\mathcal{K}$*.*
(iii) $\mathcal{D}^{\neg}$ *is irredundant for* $\overline{\mathcal{K}}$ *iff* $\mathcal{D}$ *is irredundant for* $\mathcal{K}$*.*

Proof. (i) $\forall B^{\neg} \longmapsto A^{\neg} \in \mathcal{D}^{\neg}$: $\overline{\mathcal{K}} \models B^{\neg} \longmapsto A^{\neg}$ iff $\mathcal{K} \models \bigvee A \longmapsto \bigvee B$ (by Proposition 5)
(ii) $\mathcal{D}^{\neg}$ is complete for $\overline{\mathcal{K}} \Longleftrightarrow \delta(\mathcal{D}^{\neg}) \subseteq INT^{\overline{\mathcal{K}}} \Longleftrightarrow \forall M^{\neg} \in \delta(\mathcal{D}^{\neg}) : M^{\neg} = ((M^{\neg})^{\triangle}_{\overline{\mathcal{K}}})^{\triangle}_{\overline{\mathcal{K}}}$

$\Longleftrightarrow \forall M^{\neg} \subseteq \mathcal{P}^{\neg} : (M^{\neg} \models \mathcal{D}^{\neg}) \Longrightarrow M^{\neg} = ((M^{\neg})^{\triangle}_{\overline{\mathcal{K}}})^{\triangle}_{\overline{\mathcal{K}}}$
$\Longleftrightarrow \forall M^{\neg} \subseteq \mathcal{P}^{\neg} : (M^{\neg} \models \mathcal{D}^{\neg}) \Longrightarrow M = (M^{\Pi})^{\mathrm{N}}$ (by P_1 and P_2)
$\Longleftrightarrow \forall M \subseteq \mathcal{P} : (M \models \mathcal{D}) \Longrightarrow M = (M^{\Pi})^{\mathrm{N}}$ (by Corollary 1)
$\Longleftrightarrow \delta(\mathcal{D}) \subseteq INT^{\mathcal{K}}_{\mathrm{N}\Pi}$, which expresses the fact that the base of disjunctive attribute implications $\mathcal{D}$ is complete for the context $\mathcal{K}$.

(iii) ($\mathcal{D}^{\neg}$ is irredundant for $\overline{\mathcal{K}}$) $\Longleftrightarrow (M^{\neg} \models \mathcal{D}^{\neg} \backslash \{B^{\neg} \longmapsto A^{\neg}\} \Longrightarrow M^{\neg} \not\models B^{\neg} \longmapsto A^{\neg}) \Longleftrightarrow (M \models \mathcal{D} \backslash \{A \longmapsto B\} \Longrightarrow M \not\models A \longmapsto B) \Longleftrightarrow$ ($\mathcal{D}$ is irredundant for $\mathcal{K}$) (by Proposition 5). □

In the Appendix can be found the study of minimal bases of disjunctive attribute implications using a counterpart of pseudointents. The above result shows that the minimal base of disjunctive attribute implications can be obtained by contraposition from the stem base of the complementary context.

Exemple 1 (continued). Let us find the disjunctive attribute implications for the formal context $\mathcal{K}_\mathcal{S}$ already illustrated. It comes down to finding attribute implications for the complementary formal context $\overline{\mathcal{K}_\mathcal{S}}$, as on Table II. The reader can check that the only singleton pseudointent is $\{\neg a_3\}$; there are four pseudointents with two attributes, namely, $\{\neg a_1, \neg a_4\}$, $\{\neg a_1, \neg a_5\}$, $\{\neg a_4, \neg a_5\}$, and $\{\neg a_2, \neg a_5\}$. In contrast, for instance, $\{\neg a_2, \neg a_4\}$ is closed; $\{\neg a_1, \neg a_3\}$ does not contain the closure of a_3, etc.) There are no pseudointents with three nor four attributes. As the closures of the pseudointents are, respectively,

$$\{\neg a_3\}^{\Delta\Delta}_{\overline{\mathcal{K}_\mathcal{S}}} = \{\neg a_2, \neg a_3, \neg a_4\},$$

$$\{\neg a_1, \neg a_4\}^{\Delta\Delta}_{\overline{\mathcal{K}_\mathcal{S}}} = \{\neg a_1, \neg a_5\}^{\Delta\Delta}_{\overline{\mathcal{K}_\mathcal{S}}} = \{\neg a_4, \neg a_5\}^{\Delta\Delta}_{\overline{\mathcal{K}_\mathcal{S}}} = \{\neg a_1, \neg a_4, \neg a_5\},$$

$$\{\neg a_2, \neg a_5\}^{\Delta\Delta}_{\overline{\mathcal{K}_\mathcal{S}}} = \{\neg a_1, \neg a_2, \neg a_4, \neg a_5\}$$

The minimal base of context $\overline{\mathcal{K}_\mathcal{S}}$ is made of the following conjunctive rules:

$$\neg a_3 \longmapsto \{\neg a_2, \neg a_4\}, \{\neg a_1, \neg a_4\} \longmapsto \neg a_5, \{\neg a_1, \neg a_5\} \longmapsto \neg a_4, \{\neg a_4, \neg a_5\} \longmapsto \neg a_1, \{\neg a_2, \neg a_5\} \longmapsto \{\neg a_1, \neg a_4\}$$

The minimal disjunctive rule base of context $\mathcal{K}_\mathcal{S}$ is thus obtained by contraposition of the above rules:

$$\mathcal{D}^{\mathcal{K}_\mathcal{S}}_{\mathrm{N\Pi}} = \{a_2 \vee a_4 \longmapsto a_3, a_5 \longmapsto a_1 \vee a_4,\ a_4 \longmapsto a_1 \vee a_5,\ a_1 \longmapsto a_4 \vee a_5,\ a_1 \vee a_4 \longmapsto a_2 \vee a_5\}$$

Note that rule $a_2 \vee a_4 \longmapsto a_3$ is logically equivalent to two elementary rules $a_2 \longmapsto a_3$ and $a_4 \longmapsto a_3$, which are both conjunctive and disjunctive and were already found in the stem base of $\mathcal{K}_\mathcal{S}$.□

To conclude, this section provides useful theoretical results that allow to obtain a minimal base of disjunctive attribute implications from binary formal contexts. These pieces of knowledge nontrivially complement the set of conjunctive attribute implications usually derived.

5. POSSIBLE AND CERTAIN IMPLICATIONS IN INCOMPLETE CONTEXTS

It is widely agreed that in many areas, knowledge may be incomplete. Thus, it seems to be important to distinguish between the case where it is known that an object does not possess an attribute and cases when it is not known whether it possesses the attribute or not, a distinction that usual contexts cannot make. Hence, it seems useful to see what becomes of implications resulting from incomplete formal contexts. The case of partial ignorance in contexts has been considered by Obiedkov[19] using a modal logic and by Burmeister and Holzer[18] using Kleene three–valued logic. They have proposed to extend formal contexts using a third value, denoted by "?," which leads to the notion of *incomplete context*, sometimes also called three-valued context. In this section, we use possibility theory to model incomplete information.

More formally, an incomplete context is a 4-tuple $\mathbf{K}^? = (\mathcal{O}, \mathcal{P}, \{+, -, ?\}, \mathbf{R})$ where $\mathcal{O}$ is the set of objects; $\mathcal{P}$ the set of attributes; "+," "-,", and "?" are the three possible entries of the incomplete context; and $\mathbf{R}$ is a ternary relation $\mathbf{R} \subseteq \mathcal{O} \times \mathcal{P} \times \{+, -, ?\}$. The interpretation of the relation $\mathbf{R}$ is as follows. Let $x \in \mathcal{O}$ and $a \in \mathcal{P}$:

- $(x, a, +) \in \mathbf{R}$: it is known that the object x has the attribute a;
- $(x, a, -) \in \mathbf{R}$: it is known that the object x does not have the attribute a;
- $(x, a, ?) \in \mathbf{R}$: it is unknown, whether the object x has the attribute a or not.

An incomplete formal context may be viewed as the family of all complete Boolean formal contexts obtained by changing unknown entries $(x, a, ?)$ into known ones ($(x, a, +)$ or $(x, a, -)$). The two extreme cases where all such unknown entries $(x, a, ?)$ are changed into $(x, a, -)$ and the case where all such unknown entries $(x, a, ?)$ are changed into $(x, a, +)$ give birth to lower and upper completions, respectively.[23,24]

In this way, two classical (Boolean) formal contexts, denoted by $\mathcal{K}_*$ and $\mathcal{K}^*$ are obtained as extreme results of the two replacements. More formally:

- $\mathcal{K}_* = (\mathcal{O}, \mathcal{P}, \mathcal{R}_*)$ is a Boolean formal context such that $\mathcal{R}_* = \{(x, a) \mid (x, a, +) \in \mathbf{R}\}$ where the entries "?" are replaced by $-$, interpreting lack of knowledge about (x, a) as the fact that x does not possess attribute a.
- $\mathcal{K}^* = (\mathcal{O}, \mathcal{P}, \mathcal{R}^*)$ is a Boolean formal context such that $\mathcal{R}^* = \{(x, a) \mid (x, a, +) \in \mathbf{R}$ or $(x, a, ?) \in \mathbf{R}\}$ where the entries "?" are replaced by $+$, interpreting lack of knowledge about (x, a) as the fact that x possesses attribute a.

There exist exactly 2^n formal contexts obtained by arbitrarily replacing each "?" by "+" or "-" (where n is the number of "?" in the incomplete formal context). They are called *possible contexts*. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are possible contexts from $\mathbf{K}^?$, one defines an ordering between them as follows: $\mathcal{K}_1 \prec \mathcal{K}_2$ if $\mathcal{R}_1 \subset \mathcal{R}_2$, i.e., there are more 1s in $\mathcal{R}_2$ than in $\mathcal{R}_1$. Then it is easy to check that:

LEMMA 1. $\mathcal{K}_1 \prec \mathcal{K}_2$ *implies that for any subset A of attributes,* $A^{\Delta}_{\mathcal{K}_1} \subseteq A^{\Delta}_{\mathcal{K}_2}$.

Proof. It is obvious that since all 1s in $\mathcal{R}_1$ are 1s in $\mathcal{R}_2$ there cannot be less objects satisfying all attributes in A for $\mathcal{K}_2$ than for $\mathcal{K}_1$. □

Clearly, the minimal context for $\prec$ is $\mathcal{K}_*$ and the maximal context is $\mathcal{K}^*$.

Example 2. The formal contexts in Table IV illustrate all possible formal contexts obtained from the incomplete formal context $\mathbf{K}^?$ given in Table III. In the case of an incomplete formal context $\mathbf{K}^?$, the attribute implication $a_3 \longmapsto a_2$ holds in formal contexts $\mathcal{K}_*$, $\mathcal{K}^*$, $\mathcal{K}_1$, and $\mathcal{K}_2$; this attribute implication holds independently of what the question marks stand for. Considering the attribute implication $a_1 \longmapsto a_2$, it is

Table III. Incomplete formal context $\mathbf{K}^?$.

$\mathbf{K}^?$	a_1	a_2	a_3
x_1	+	+	+
x_2	−	+	?
x_3	+	?	−

Table IV. All possible formal contexts of $\mathbf{K}^?$.

$\mathcal{K}_*$	a_1	a_2	a_3	$\mathcal{K}^*$	a_1	a_2	a_3	$\mathcal{K}_1$	a_1	a_2	a_3
x_1	+	+	+	x_1	+	+	+	x_1	+	+	+
x_2	−	+	−	x_2	−	+	+	x_2	−	+	−
x_3	+	−	−	x_3	+	+	−	x_3	+	+	−
				$\mathcal{K}_2$	a_1	a_2	a_3				
				x_1	+	+	+				
				x_2	−	+	+				
				x_3	+	−	−				

only valid in $\mathcal{K}^*$ and $\mathcal{K}_1$: It depends on the value with which the question marks will be replaced.

Note that, in the terminology of possibility theory, $A^{\Delta}_{\mathcal{K}_*} = \{x \mid A \subseteq \mathcal{R}_*(x)\}$ is the set of objects certainly having all attributes in A while $A^{\Delta}_{\mathcal{K}^*} = \{x \mid A \subseteq \mathcal{R}^*(x)\}$ is the set of objects possibly having all attributes in A.

Attribute implications obtained from incomplete formal contexts are either certain attribute implications or possible attribute implications.

DEFINITION 8. *An attribute implication is said to be certain in an incomplete context* $\mathbf{K}^?$ *if and only if it is valid in each formal context* $\mathcal{K}$ *such that* $\mathcal{K}_* \preceq \mathcal{K} \preceq \mathcal{K}*$.

This definition may seem hard to verify using explicit enumeration of possible contexts. The following theorem provides a solution to this problem.

THEOREM 2. $A \longmapsto B$ *is a certain attribute implication in* $\mathbf{K}^?$ *iff* $A^{\Delta}_{\mathcal{K}^*} \subseteq B^{\Delta}_{\mathcal{K}_*}$

Proof.(a) *Necessary condition*

Let $A \longmapsto B$ be a certain attribute implication in $\mathbf{K}^?$ and suppose $A^{\Delta}_{\mathbf{K}^*} \nsubseteq B^{\Delta}_{\mathbf{K}_*}$. $A^{\Delta}_{\mathbf{K}^*} \nsubseteq B^{\Delta}_{\mathbf{K}_*} \Longrightarrow \exists x \in \mathcal{O} \mid x \in A^{\Delta}_{\mathbf{K}^*}$ and $x \notin B^{\Delta}_{\mathbf{K}_*} \Longrightarrow \exists$ formal context $\mathcal{K}_j(\mathcal{O}, \mathcal{P}, \mathcal{R}_j)$ s.t. $\mathcal{R}_j = \mathcal{R}_* \cup \mathcal{Q}$ where $\mathcal{Q} = \{(x, a) \in \mathcal{R}^* \mid a \in A \wedge (x, a, ?) \in \mathbf{R}\}$

Thus, $x \in A^{\Delta}_{\mathcal{K}_j}$ and $x \notin B^{\Delta}_{\mathcal{K}_j} \Longrightarrow A^{\Delta}_{\mathcal{K}_j} \nsubseteq B^{\Delta}_{\mathcal{K}_j} \Longrightarrow A \longmapsto B$ does not hold in $\mathcal{K}_j \Longrightarrow A \longmapsto B$ is not certain attribute implication.

(b) *Sufficient condition*

If $A^{\Delta}_{\mathcal{K}^*} \subseteq B^{\Delta}_{\mathcal{K}_*}$ then $A \longmapsto B$ is a certain attribute implication in $\mathbf{K}^?$. Indeed, for any possible context $\mathcal{K}_j$ it holds that $A^{\Delta}_{\mathcal{K}_j} \subseteq A^{\Delta}_{\mathcal{K}^*} \subseteq B^{\Delta}_{\mathcal{K}_*} \subseteq B^{\Delta}_{\mathcal{K}_j}$ due to Lemma 1. Hence $A \longmapsto B$ is a certain attribute implication in $\mathcal{K}_j$ for all possible contexts of $\mathbf{K}^?$. □

Remark 1. A direct proof of the necessary condition can also be given. Note that $A \longmapsto B$ is a certain attribute implication if and only if $A \longmapsto b$ is a certain attribute implication for all $b \in B \setminus A$. In the following, we can restrict to certain attribute implications $A \longmapsto b$. It means that for any context $\mathcal{K}$ compatible with $\mathbf{K}^?$, $A^{\Delta}_{\mathcal{K}} \subseteq \{b\}^{\Delta}_{\mathcal{K}}$ holds. Consider the set of objects such that $\mathcal{O}_A = \{x \in \mathcal{O} : \forall a \in A, (x, a, -) \notin \mathbf{R}\}$. So, lines restricted to A of the incomplete context corresponding to $\mathcal{O}_A$ contain only 1 or "?." The inclusion $A^{\Delta}_{\mathcal{K}} \subseteq \{b\}^{\Delta}_{\mathcal{K}}, \forall \mathcal{K}$ compatible with $\mathbf{K}^?$ means that if $x \in \mathcal{O}_A$ then $(x, b, +) \in \mathbf{R}$. So we can see that $A^{\Delta}_{\mathcal{K}^*} = \mathcal{O}_A$, while $\{b\}^{\Delta}_{\mathcal{K}_*} \supseteq \mathcal{O}_A$.

Another problem is to determine all possible attribute implications.

DEFINITION 9. *An attribute implication is said to be possible in an incomplete context* $\mathbf{K}^?$ *if and only if it is valid in at least one formal context* $\mathcal{K}$ *such that* $\mathcal{K}_* \preceq \mathcal{K} \preceq \mathcal{K}*$.

Clearly a certain attribute implication is also a possible one in the sense of this definition. Obviously, the converse does not hold. Again, checking if an attribute implication is possible seems to require an enumeration of possible contexts. The following theorem facilitates this determination.

THEOREM 3. $A \longmapsto B$ *is a possible attribute implication in* $\mathbf{K}^?$ *iff* $A^{\Delta}_{\mathcal{K}_*} \subseteq B^{\Delta}_{\mathcal{K}^*}$

Proof. (a) Necessary condition: $A \longmapsto B$ is a possible attribute implications in $\mathbf{K}^?$ means that there is a formal context $\mathcal{K}_j$ such that $\mathcal{K}_* \preceq \mathcal{K}_j \preceq \mathcal{K}^*$ and $A^{\Delta}_{\mathbf{K}_j} \subseteq B^{\Delta}_{\mathbf{K}^j}$.
(a)

It is clear that $A^{\Delta}_{\mathcal{K}_*} \subseteq A^{\Delta}_{\mathbf{K}_j}$ and $B^{\Delta}_{\mathbf{K}_j} \subseteq B^{\Delta}_{\mathcal{K}^*}$ by Lemma 1

Putting these inclusions together: $A^{\Delta}_{\mathcal{K}_*} \subseteq A^{\Delta}_{\mathbf{K}_j} \subseteq B^{\Delta}_{\mathbf{K}_j} \subseteq B^{\Delta}_{\mathcal{K}^*}$, finally $A^{\Delta}_{\mathcal{K}_*} \subseteq B^{\Delta}_{\mathcal{K}^*}$.

(b) Sufficient condition: suppose $A^{\Delta}_{\mathcal{K}_*} \subseteq B^{\Delta}_{\mathcal{K}^*}$. Two cases are easily done with:

$B \subseteq A$: then $A \longmapsto B$ is clearly a valid rule for any context; $B \cap A \neq \emptyset$ and $B \not\subseteq A$: then $A \longmapsto B$ is a valid rule iff $A \longmapsto B \setminus A$ is a valid rule for the same context.

So we only have to study the case when A and B are disjoint. Define the context $\mathcal{K}_{AB}$ as follows: replace all (?) in columns of A by 0, and all (?) in columns of B by 1. Then, it is clear that $A^{\Delta}_{\mathbf{K}_*} = A^{\Delta}_{\mathbf{K}_{AB}}$ and $B^{\Delta}_{\mathbf{K}^*} = B^{\Delta}_{\mathbf{K}_{AB}}$. Hence $A^{\Delta}_{\mathbf{K}_{AB}} \subseteq B^{\Delta}_{\mathbf{K}_{AB}}$, so $A \longmapsto B$ a valid implication for $\mathcal{K}_{AB}$, and since $\mathcal{K}_* \preceq \mathcal{K}_{AB} \preceq \mathcal{K}^*$, $A \longmapsto B$ is a possible implication for $\mathbf{K}^?$. □

This section also considers disjunctive attribute implications, presented in Section 4, for incomplete formal contexts. As in the case of conjunctive attribute implications, we distinguish certain disjunctive attribute implications and possible disjunctive attribute implications. Note that $(A)^{\Pi}_{\mathcal{K}_*}$ is the set of objects certainly having at least one attribute in A and $(A)^{\Pi}_{\mathcal{K}^*}$ is the set of objects possibly having at least one attribute in A. Then, $\overline{(A)^{\Pi}_{\mathcal{K}_*}}$ is the set of objects that certainly never have any attribute in A and $\overline{(A)^{\Pi}_{\mathcal{K}^*}}$ is the set of objects that have all their attributes for sure outside A.

We denote by $\overline{\mathbf{K}^?} = (\mathcal{O}, \mathcal{P}^{\neg}, \{+, -, ?\}, \overline{\mathbf{R}})$ the complement of the incomplete context $\mathbf{K}^?$. It is defined using Kleene negation, namely $(x, a, +) \in \overline{\mathbf{R}}$ if and only if $(x, a, -) \in \mathbf{R}$, $(x, a, -) \in \overline{\mathbf{R}}$ if and only if $(x, a, +) \in \mathbf{R}$, and $(x, a, ?) \in \overline{\mathbf{R}}$ if and only if $(x, a, ?) \in \mathbf{R}$. It is easy to see that the lower completion of $\overline{\mathbf{K}^?}$ is $\overline{\mathcal{K}^*}$, the complement of the upper completion of $\mathbf{K}^?$, and the upper completion of $\overline{\mathbf{K}^?}$ is $\overline{\mathcal{K}_*}$, the complement of the lower completion of $\mathbf{K}^?$. We get two more results.

THEOREM 4. *$\bigvee A \longmapsto \bigvee B$ is a certain disjunctive attribute implication iff $A^{\Pi}_{\mathcal{K}^*} \subseteq B^{\Pi}_{\mathcal{K}_*}$*

Proof. On the one hand, by Theorem 2, $B^{\neg} \longmapsto A^{\neg}$ is a certain attribute implication in $\overline{\mathbf{K}^?}$ iff $B^{\Delta}_{\overline{\mathcal{K}_*}} \subseteq A^{\Delta}_{\overline{\mathcal{K}^*}}$, since $\overline{\mathcal{K}_*}$ and $\overline{\mathcal{K}^*}$ are, respectively, the upper and the lower completions in $\overline{\mathbf{K}^?}$; moreover, it also means that $\bigvee A \longmapsto \bigvee B$ is a certain disjunctive attribute implication in $\mathbf{K}^?$.

On the other hand, we have: $B^{\Delta}_{\overline{\mathcal{K}_*}} \subseteq A^{\Delta}_{\overline{\mathcal{K}^*}}$ iff $A^{\Pi}_{\mathcal{K}^*} \subseteq B^{\Pi}_{\mathcal{K}_*}$ (after Proposition 5 and P_9). It follows that $\bigvee A \longmapsto \bigvee B$ is a certain disjunctive attribute implication iff $A^{\Pi}_{\mathcal{K}^*} \subseteq B^{\Pi}_{\mathcal{K}_*}$. □

THEOREM 5. *$\bigvee A \longmapsto \bigvee B$ is a possible disjunctive attribute implication iff $A^{\Pi}_{\mathcal{K}_*} \subseteq B^{\Pi}_{\mathcal{K}^*}$*

Proof. On the one hand, by Theorem 3, $B^{\neg} \longmapsto A^{\neg}$ is a possible attribute implication iff $B^{\Delta}_{\overline{\mathcal{K}^*}} \subseteq A^{\Delta}_{\overline{\mathcal{K}_*}}$. It also means that $\bigvee A \longmapsto \bigvee B$ is a possible disjunctive attribute implication in $\mathbf{K}^?$.

On the other hand, $B^{\Delta}_{\overline{\mathcal{K}^*}} \subseteq A^{\Delta}_{\overline{\mathcal{K}_*}}$ is equivalent to $A^{\Pi}_{\mathcal{K}_*} \subseteq B^{\Pi}_{\mathcal{K}^*}$(after Proposition 5 and P_9). Consequently, we get that $\bigvee A \longmapsto \bigvee B$ is a possible disjunctive attribute implication iff $A^{\Pi}_{\mathcal{K}_*} \subseteq B^{\Pi}_{\mathcal{K}^*}$. □

The above results could be couched in the setting of gradual possibility theory. Namely, adopting usual conventions in possibility theory, let "−" be encoded by 0, "+" be encoded by 1, and equip the set $\{0, ?, 1\}$ with the total order $0 < ? < 1$, and, for a subset A of attributes, define the extent $A^{\Delta}_{\mathbf{K}^?}$ in an incomplete context $\mathbf{K}^?$ as a fuzzy set, with membership function $\mu_{A^{\Delta}_{\mathbf{K}^?}} : \mathcal{O} \to \{0, ?, 1\}$, using the usual definition of guaranteed possibility:[22]

$$\forall x \in \mathcal{O}, \mu_{A^{\Delta}_{\mathbf{K}^?}}(x) = \min_{a \in A} R(x, a)$$

where $R(x, a) = \alpha$ iff $(x, a, \alpha) \in \mathbf{R}$. The above results can be expressed in terms of fuzzy extents $A^{\Delta}_{\mathbf{K}^?}$, more precisely in terms of their cuts:

$$\text{the core } C(A^{\Delta}_{\mathbf{K}^?}) = \{x \in \mathcal{O} : \mu_{A^{\Delta}_{\mathbf{K}^?}}(x) = 1\},$$

$$\text{the support } S(A^{\Delta}_{\mathbf{K}^?}) = \{x \in \mathcal{O} : \mu_{A^{\Delta}_{\mathbf{K}^?}}(x) > 0\}.$$

Indeed notice that for any incomplete context, $C(A^{\Delta}_{\mathbf{K}^?}) = A^{\Delta}_{\mathcal{K}_*}$ and $S(A^{\Delta}_{\mathbf{K}^?}) = A^{\Delta}_{\mathcal{K}^*}$. For instance, $A \mapsto B$ is a certain attribute implication for $\mathbf{K}^?$ iff $S(A^{\Delta}_{\mathbf{K}^?}) \subseteq C(B^{\Delta}_{\mathbf{K}^?})$ (a strong form of inclusion of $A^{\Delta}_{\mathbf{K}^?}$ in $B^{\Delta}_{\mathbf{K}^?}$), and $A \mapsto B$ is a possible attribute implication for $\mathbf{K}^?$ iff $C(A^{\Delta}_{\mathbf{K}^?}) \subseteq S(B^{\Delta}_{\mathbf{K}^?})$ (a weak form of inclusion of $A^{\Delta}_{\mathbf{K}^?}$ in $B^{\Delta}_{\mathbf{K}^?}$).

For instance, in the incomplete context of Example 2, consider the certain attribute implication $a_3 \mapsto a_2$. The fuzzy set $\{a_i\}^{\Delta}_{\mathbf{K}^?}$ is the ith column of the matrix in Table III, and it can be checked that $S(\{a_3\}^{\Delta}_{\mathbf{K}^?}) \subseteq C(\{a_2\}^{\Delta}_{\mathbf{K}^?})$.

6. APPLICATION TO DESCRIPTION LOGICS

This section tries to suggest that the results obtained above, concerning disjunctive attribute implications and incomplete contexts, may be relevant for applying FCA to DLs. In the following, we briefly recall some notions regarding DLs. Then, we show how to enrich the terminological component of a description logic in terms of GCIs by exploiting its assertional component, using asymmetric FCA operators studied in the previous sections, especially the symmetric composition of sufficiency operators and the asymmetric composition of necessity and possibility operators in FCA.

6.1. Description Logics and Incomplete Formal Contexts

DLs[25] are a class of knowledge representation formalisms. A knowledge base in DL consists of an ABox and a TBox. An ABox is a finite set of assertions. A TBox represents intensional knowledge usually represented by means of GCI axioms of the form $C \sqsubseteq D$, where both C and D are so-called concept descriptions. The semantics of concept descriptions is defined in terms of an interpretation $\mathcal{I} =< \triangle^{\mathcal{I}}, (.)^{\mathcal{I}} >$. The domain $\triangle^{\mathcal{I}}$ of $\mathcal{I}$ is a nonempty set of individuals (objects), and the interpretation function $(.)^{\mathcal{I}}$ maps each concept name A to a subset $A^{\mathcal{I}}$ of $\triangle^{\mathcal{I}}$ and each role r to a binary relation $r^{\mathcal{I}} \subseteq \triangle^{\mathcal{I}} \times \triangle^{\mathcal{I}}$. Given a fixed ABox $\mathcal{A}$, let us denote by N_C the set of concept names, by N_R the set of role names and by $N_O = \triangle^{\mathcal{I}}$ the set of object names occurring in $\mathcal{A}$.

An example of ABox is illustrated below.

Example 3. We consider the set $\mathcal{A}$ of assertions:

$\mathcal{A}$:={Man(John),\; Man(Peter), ¬Man(Maria), ¬Man(Clara) Woman(Maria), Woman(Clara), ¬Woman(John), ¬Woman(Peter), ¬Father(John),
¬Father(Maria),¬Father(Clara), Mother(Clara), ¬Mother(John), ¬Mother(Maria),
¬Mother(Peter), Parent(Clara), Parent(Peter), ¬Parent(John)}.

We can check that N_O = {John, Peter, Maria, Clara}, N_C = {Man, Woman, Father}.

We consider also the interpretation $\mathcal{I} =< \{John, Maria, Peter, Clara\}, (.)^{\mathcal{I}} >$ with

$\text{Man}^{\mathcal{I}}$ = {John, Peter}
$\text{Woman}^{\mathcal{I}}$ = {Maria, Clara}
$(\neg\text{Father})^{\mathcal{I}}$ = {John, Maria, Clara}
$\text{Mother}^{\mathcal{I}}$ = {Clara}

Table V. Incomplete formal context $\mathbf{K}^?$ for the ABox of Example 3.

R	Man	Woman	Father	Mother	Parent
John	+	−	−	−	−
Maria	−	+	−	−	?
Peter	+	−	?	−	+
Clara	−	+	−	+	+

$\text{Parent}^{\mathcal{I}} = \{\text{Peter, Clara}\}$
$(\neg\text{Man})^{\mathcal{I}} = \{\text{Maria, Clara}\}$
$(\neg\text{Woman})^{\mathcal{I}} = \{\text{John, Peter}\}$
$(\neg\text{Mother})^{\mathcal{I}} = \{\text{John, Maria, Peter}\}$
$(\neg\text{Parent})^{\mathcal{I}} = \{\text{John}\}$

It has been shown[26–28] that learning GCIs from an ABox needs, in a first step, to map the considered ABox into a formal context to extract all attribute implications. In a second step, a formal correspondence is established between a given attribute implication and a related GCI.

However, it is important to remember that the CWA (a fact whose truth we know nothing about is assumed to be false) is often associated with formal contexts, where blank places are then supposed to express falsity. Whereas, the open world assumption (a fact whose truth we know nothing about can be either true or false) is the natural understanding of an ABox. This discrepancy prevents us from a straightforward mapping between DLs and FCA.

The learning process is based on an intermediate representation, namely a formal context that must consider the open world assumption, and a description logic where explicit negation is allowed. To take into account this assumption for learning correct GCIs, we propose to consider formal contexts dealing with partial ignorance as it will be seen in the next subsection.

Let us consider the incomplete formal context $\mathbf{K}^? = (N_O, N_C, \{+, -, ?\}, \mathbf{R})$ whose set of objects is N_O, whose attributes are the concept names N_C, and $\mathbf{R}$ is defined as follows:

- $(x, a, +) \in \mathbf{R}$ if $a(x)$ is in the ABox;
- $(x, a, -) \in \mathbf{R}$ if $\neg a(x)$ is in the ABox;
- $(x, a, ?) \in \mathbf{R}$ otherwise.

The incomplete formal context $\mathbf{K}^?$ induced from the set $\mathcal{A}$ of assertions and the interpretation $\mathcal{I}$ given in Example 3 is shown in Table V.

As seen in Section 5, dealing with incomplete formal contexts leads to possible and certain attribute implications (in both disjunctive and conjunctive semantics). Since these kinds of implications are at the basis of the learning process, different cases may arise depending on whether the corresponding attribute implication is possible or certain. It is well agreed in the literature that a subsumption of concepts (GCI) is universally verified (i.e., in all interpretations). Consequently, we will consider that a certain attribute implication, which is verified in all formal contexts, corresponds to a GCI in its strict definition.

6.2. Extracting Conjunctive GCIs Using FCA in Incomplete Environments

The learning process takes advantage from the previous results to determine all subsumption relationships between conjunctions of concept names (also called conjunctive GCI and denoted by $GCI_{\sqcap}$), which are of the form $a_1 \sqcap ... \sqcap a_n \sqsubseteq b_1 \sqcap ... \sqcap b_m$ s.t. $a_i, b_j \in N_C$ (equivalently denoted by $\bigsqcap A \sqsubseteq \bigsqcap B$) s.t. $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$.

Let $\mathcal{K}_j = (N_O, N_C, \mathcal{R}_j)$ be a possible Boolean formal context of the incomplete formal context $\mathbf{K}^? = (N_O, N_C, \{+, -, ?\}, \mathbf{R})$ (obtained from the interpretation $\mathcal{I}$). It is clear that there exists a possible interpretation $\mathcal{I}_j$ that corresponds to $\mathcal{K}_j$ (the ABox associated with $\mathcal{K}_j$ strictly contains the one associated with $\mathbf{K}^?$).

The following lemma highlights a correspondence between the sufficiency derivation operators $(.)^{\Delta}$ in the formal context $\mathcal{K}_j$ and the interpretation $\mathcal{I}_j$.

LEMMA 2. *Let $A \subseteq N_C$, then $A^{\Delta}_{\mathcal{K}_j} = (\bigsqcap A)^{\mathcal{I}_j}$.*

Proof. By definition of the sufficiency derivation operator $(.)^{\Delta}$

$$A^{\Delta}_{\mathcal{K}_j} = \{x \in N_O \;\mid\; \forall a \in A, x\mathcal{R}_j a\} = \bigcap_{a \in A} \{a\}^{\Delta}_{\mathcal{K}_j}$$

and by definition of the interpretation of concepts conjunction

$$(\bigsqcap A)^{\mathcal{I}_j} = \{x \in \Delta^{\mathcal{I}} \mid \forall a \in A, x \in (a)^{\mathcal{I}_j}\} = \bigcap_{a \in A} (a)^{\mathcal{I}_j}$$

It is easy to see that $(a)^{\mathcal{I}_j} = \{a\}^{\Delta}_{\mathcal{K}_j}$, then $A^{\Delta}_{\mathcal{K}_j} = (\bigsqcap A)^{\mathcal{I}_j}$. □

The following theorem gives a method for inducing conjunctive GCIs (i.e., GCIs valid in each interpretation $\mathcal{I}_j$) from a certain conjunctive attribute implication for which a simple characterization is given in Theorem 2.

THEOREM 6. *$\bigsqcap A \sqsubseteq \bigsqcap B$ is a conjunctive GCI iff $A \longmapsto B$ is a certain attribute implication in $\mathbf{K}^?$.*

Proof. $A \longmapsto B$ is a certain attribute implication in $\mathbf{K}^? \Longleftrightarrow A \longmapsto B$ holds in all $\mathcal{K}_j \Longleftrightarrow A^{\Delta}_{\mathcal{K}_j} \subseteq B^{\Delta}_{\mathcal{K}_j}$ in all $\mathcal{K}_j$ (by definition) $\Longleftrightarrow (\bigsqcap A)^{\mathcal{I}_j} \subseteq (\bigsqcap B)^{\mathcal{I}_j}$ in all possible interpretations $\mathcal{I}_j$ (by Lemma 2) $\Longleftrightarrow \bigsqcap A \sqsubseteq \bigsqcap B$ holds in all $\mathcal{I}_j \Longleftrightarrow \bigsqcap A \sqsubseteq \bigsqcap B$ is a conjunctive GCI. □

Thus, it is easy to obtain a set $\mathcal{B}_{GCI_{\sqcap}}$ of conjunctive GCIs from interpretation $\mathcal{I}$, given in Example 3, using Theorems 2 and 6:

$$\mathcal{B}_{GCI_{\sqcap}} = \{Mother \sqsubseteq Woman \sqcap Parent, Woman \sqcap Parent \sqsubseteq Mother\}$$

In contrast, when the attribute implication is a possible one, namely verified in some formal context, we may consider that the corresponding GCI is plausible as defined below.[29]

DEFINITION 10. *A GCI is said to be plausible if it is valid at least one possible interpretation $\mathcal{I}_j$, where a possible interpretation corresponds to a possible formal context $\mathcal{K}_j$.*

The following theorem gives a method for inducing plausible conjunctive GCIs from a possible attribute implication for which a simple characterization is given in Theorem 3.

THEOREM 7. *$\bigsqcap A \sqsubseteq \bigsqcap B$ is a plausible conjunctive GCI iff $A \longmapsto B$ is a possible attribute implication in $\mathbf{K}^?$.*

Proof. $A \longmapsto B$ is a possible attribute implication in $\mathbf{K}^? \Longleftrightarrow A \longmapsto B$ holds in at least one possible Boolean formal context $\mathcal{K}_j \Longleftrightarrow A^{\Delta}_{\mathcal{K}_j} \subseteq B^{\Delta}_{\mathcal{K}_j}$ (by definition) $\Longleftrightarrow (\bigsqcap A)^{\mathcal{I}_j} \subseteq (\bigsqcap B)^{\mathcal{I}_j}$ (by Lemma 2) $\Longleftrightarrow \bigsqcap A \sqsubseteq \bigsqcap B$ holds in $\mathcal{I}_j \Longleftrightarrow \bigsqcap A \sqsubseteq \bigsqcap B$ is a plausible conjunctive GCI. □

Note that taking a plausible GCI for granted is risky as it may be conflicting with other ones, which may damage the consistency of the Tbox. So this kind of knowledge item should be handled with care and rejected if in contradiction with other already accepted ones for instance.

6.3. Toward Extracting Disjunctive GCIs

A number of publications are concerned by the induction of a TBox from an ABox using knowledge in the form of attribute implications, by means of FCA techniques. For instance, Baader et al.[30] use FCA for completing description logic knowledge bases. Baader[31] uses attribute exploration for computing the subsumption hierarchy of all conjunctions of a set of DL concepts. Baader and Sertkaya[32] are interested in computing the subsumption hierarchy of all least common subsumers of subsets of DL concepts in an efficient way using methods from FCA. Rudolph[26] considers an infinite family of contexts obtained by restricting the so-called role depth of the concepts. An attribute exploration[33] is then applied in each step by increasing the role depths until a certain termination condition applies. The main problem with this approach is that the number of attributes grows very fast when the role depth grows, and the implication bases do not appear to yield a basis for all the GCIs holding in the given finite model. Another approach has been discussed by Baader and Distel,[34] who address the problem of how to compute the basis of a set of GCIs holding in a finite model, efficiently, by adapting methods from FCA.

Bazin and Ganascia[27] adapt classical FCA algorithms to build sets of concept definitions from object descriptions. In this spirit, we have already proposed an approach to generate GCIs from objects descriptions in DLs $\mathcal{EL}$.[28] This approach considers all concepts up to the maximum role depth instead of using a different learning phase for each depth, and it allows to obtain a minimal base of GCIs.

It may be remarked that all above-mentioned approaches consider attribute implication limited to their conjunctive form. As an obvious consequence, these proposed approaches do not allow to extract disjunctive GCIs. This limitation is a consequence of the use of the classical Galois derivation operator, namely the sufficiency operator which generates formal concepts in a conjunctive semantics.

We can make a similar study to determine all subsumption relationships between disjunctions of concept names (also called disjunctive GCI and denoted by $GCI_{\sqcup}$). Disjunctive GCIs ($GCI_{\sqcup}$) are of the form $a_1 \sqcup ... \sqcup a_n \sqsubseteq b_1 \sqcup ... \sqcup b_m$ s.t. $a_i, b_j \in N_C$. $\big($equivalently denoted by $\bigsqcup A \sqsubseteq \bigsqcup B\big)$ s.t. $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$.

The following lemma highlights a correspondence between the possibility derivation operators $(.)^{\Pi}$ in the possible formal context $\mathcal{K}_j$ compatible with the incomplete context $\mathbf{K}^?$ and the interpretation $\mathcal{I}_j$.

LEMMA 3. *Let $A \subseteq N_C$, then $A^{\Pi}_{\mathcal{K}_j} = (\bigsqcup A)^{\mathcal{I}_j}$.*

Proof. By definition of the possibility derivation operator $(.)^{\Pi}$

$$A^{\Pi}_{\mathcal{K}_j} = \{x \in N_O \mid \exists a \in A, x\mathcal{R}_j a\} = \bigcup_{a \in A} \{a\}^{\Pi}_{\mathcal{K}_j}$$

and by definition of the interpretation of concepts disjunction

$$(\bigsqcup A)^{\mathcal{I}_j} = \{x \in \Delta^{\mathcal{I}} \mid \exists a \in A, x \in (a)^{\mathcal{I}_j}\} = \bigcup_{a \in A} (a)^{\mathcal{I}_j}$$

It is easy to see that $(a)^{\mathcal{I}_j} = \{a\}^{\Pi}_{\mathcal{K}_j}$, then $A^{\Pi}_{\mathcal{K}_j} = (\bigsqcup A)^{\mathcal{I}_j}$. □

Since a disjunctive general concept inclusion ($GCI_{\sqcup}$) is valid in each interpretation $\mathcal{I}_j$, it will now be established that to each certain disjunctive attribute implication corresponds a disjunctive GCI (intensional knowledge).

THEOREM 8. *$\bigsqcup A \sqsubseteq \bigsqcup B$ is a disjunctive GCI iff $\bigvee A \longmapsto \bigvee B$ is a certain disjunctive attribute implication in $\mathbf{K}^?$.*

Proof. $\bigvee A \longmapsto \bigvee B$ is a certain disjunctive attribute implication in $\mathbf{K}^?$ $\Longleftrightarrow \bigvee A \longmapsto \bigvee B$ holds in all $\mathcal{K}_j$ $\Longleftrightarrow A^{\Pi}_{\mathcal{K}_j} \subseteq B^{\Pi}_{\mathcal{K}_j}$ in all $\mathcal{K}_j$ (by Proposition 5) $\Longleftrightarrow (\bigsqcup A)^{\mathcal{I}_j} \subseteq (\bigsqcup B)^{\mathcal{I}_j}$ in all possible interpretations $\mathcal{I}_j$ (by Lemma 3) $\Longleftrightarrow \bigsqcup A \sqsubseteq \bigsqcup B$ holds in all $\mathcal{I}_j \Longleftrightarrow \bigsqcup A \sqsubseteq \bigsqcup B$ is a disjunctive GCI. □

Thus, it is easy to induce a set $\mathcal{D}_{GCI_{\sqcup}}$ of disjunctive GCIs from the interpretation $\mathcal{I}$, given in Example 3, using Theorems 4 and 8,

$$\mathcal{D}_{GCI_{\sqcup}} = \{Father \sqsubseteq Man, Mother \sqsubseteq Woman, Father \sqcup Mother \sqsubseteq Parent, \\ Parent \sqsubseteq Father \sqcup Mother\}$$

A disjunctive general concept inclusion ($GCI_{\sqcup}$) is said to be plausible iff it is valid in at least one possible interpretation $\mathcal{I}_j$. The following theorem gives a method for inducing plausible disjunctive GCIs from a possible disjunctive attribute implications for which a simple characterization is given in Theorem 5.

THEOREM 9. *$\bigsqcup A \sqsubseteq \bigsqcup B$ is a plausible disjunctive GCI iff $\bigvee A \longmapsto \bigvee B$ is a possible disjunctive attribute implication in $\mathbf{K}^?$.*

Proof. $\bigvee A \longmapsto \bigvee B$ is a possible disjunctive attribute implication in $\mathbf{K}^? \Longleftrightarrow \bigvee A \longmapsto \bigvee B$ holds in at least one possible Boolean formal context $\mathcal{K}_j \Longleftrightarrow A^{\Pi}_{\mathcal{K}_j} \subseteq B^{\Pi}_{\mathcal{K}_j}$ (by Proposition 5) $\Longleftrightarrow (\bigsqcup A)^{\mathcal{I}_j} \subseteq (\bigsqcup B)^{\mathcal{I}_j}$ (by Lemma 3) $\Longleftrightarrow \bigsqcup A \sqsubseteq \bigsqcup B$ holds in $\mathcal{I}_j \Longleftrightarrow \bigsqcup A \sqsubseteq \bigsqcup B$ is a plausible disjunctive GCI. □

Putting together results in this section indicate clearly that it is possible to extend FCA-based induction techniques in DLs to the extraction of disjunctive rules and a more explicit handling of incompleteness of information. This is a topic for further research.

7. CONCLUSION

All existing works and approaches pertaining to FCA rely on the use of the classical Galois derivation operator (i.e., sufficiency operator) and exploit the complete lattice of all formal concepts obtained using the composition of sufficiency operators. Consequently, induced implications are in the conjunctive form. In this paper, we propose an approach that enlarges the FCA framework to disjunctive attribute implications, using a closed world interpretation of contexts. The proposed approach considers "open-closed" pairs obtained by means of the asymmetric composition ($\mathrm{N} \circ \Pi$) of necessity and possibility operators. Besides, we have shown that we can extend the notions of conjunctive and disjunctive attribute implications to incomplete contexts containing positive and negative information. It is worth noticing that basic concepts from possibility theory are at work on the one hand in basic FCA operators (as well as in modal logic operators) and on the other hand for the handling of incomplete contexts.

These results open the way to further research:

1. We have only focused on the hybrid composition of operators $(.)^{\mathrm{N}\Pi}$. Further research should focus on the study of the composition of other operators, such as $(.)^{\Pi\Delta}$, $(.)^{\nabla\Delta}$, etc.
2. Incomplete contexts can be refined by introducing grades of certainty that an object satisfies an attribute or does not satisfy it, using the full-fledged graded version of possibility theory. This representation was previously introduced by the three last authors.[24] Preliminary steps toward assigning grades of possibility and certainty to attribute implications extracted from such generalized incomplete contexts can be found in a recent workshop paper.[20]
3. Regarding algorithmic aspects, it is clear that the same FCA methods can be used to derive bases of conjunctive and disjunctive rules since the disjunctive rules from a context are in one-to-one correspondence with the conjunctive rules induced from the complementary context. As to incomplete contexts, it has been shown that part of the complexity can be tamed by using only two formal contexts (the upper and the lower) instead of all contexts compatible with the available information. Nevertheless, designing efficient algorithms for the induction of possible and certain rules need further effort beyond existing methods.
4. The application to DLs outlined in Section 6 is certainly worth developing at the practical level.

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APPENDIX

We prove some technical results regarding the NΠ-pairs. First, some results pertaining the lattice of NΠ-pairs.

PROOF OF PROPOSITION 4. Using Proposition 3, $(\overline{X}, A)$ is a formal concept of $\overline{\mathcal{K}} = (\mathcal{O}, \mathcal{P}, \overline{\mathcal{R}})$, then

$$\prod_{j\in J}\big(\overline{X}_j, A_j\big) = \big(\bigcap_{j\in J}\overline{X}_j, (\underset{j\in J}{\quad} A_j)^{\Delta\Delta}_{\overline{\mathcal{K}}}\big)$$

$$\Longleftrightarrow \prod_{j\in J}\big(\overline{X}_j, A_j\big) = \big(\overline{X}_1 \cap \ldots \cap \overline{X}_j, \big(\overline{(\underset{j\in J}{\quad} A_j)^{\Pi}}\big)^{\Delta}_{\overline{\mathcal{K}}}\big)$$

$$\Longleftrightarrow \underset{j\in J}{\quad} \big(\overline{\overline{X}}_j, A_j\big) = \big(\overline{\overline{X}_1 \cap ... \cap \overline{X}_j}, \overline{\overline{\big((\underset{j\in J}{\quad} A_j)^{\Pi}}})^{\mathrm{N}}\big)$$

$$\Longleftrightarrow \underset{j\in J}{\quad} \big(X_j, A_j\big) = \big(\overline{\overline{X_1}} \cup ... \cup \overline{\overline{X_j}}, \big((\underset{j\in J}{\quad} A_j)^{\Pi}\big)^{\mathrm{N}}\big)$$

$$\Longleftrightarrow \underset{j\in J}{\quad} \big(X_j, A_j\big) = \big(X_1 \cup ... \cup X_j, \big((\underset{j\in J}{\quad} A_j)^{\Pi}\big)^{\mathrm{N}}\big)$$

$$\Longleftrightarrow \underset{j\in J}{\quad} \big(X_j, A_j\big) = \big(\underset{j\in J}{\quad} X_j, \big((\underset{j\in J}{\quad} A_j)^{\Pi}\big)^{\mathrm{N}}\big)$$

The proof for the infimum operator is similarly obtained. □

The disjunctive attribute implications that hold in a formal context $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ can be obtained from the NΠ-lattice $\mathfrak{L}_{\mathrm{N}\Pi}$, as the following proposition illustrates.

PROPOSITION 6. *Given a formal context* $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, $\mathcal{K} \models a \longmapsto \bigvee B$ *iff* $(\{a\}^{\Pi}, (\{a\}^{\Pi})^{\mathrm{N}}) \preceq (B^{\Pi}, (B^{\Pi})^{\mathrm{N}})$.

Proof. (a) Necessary condition : $\mathcal{K} \models a \longmapsto \bigvee B$ implies $\{a\}^{\Pi} \subseteq B^{\Pi}$ (using Proposition 5), hence $(\{a\}^{\Pi})^{\mathrm{N}} \subseteq (B^{\Pi})^{\mathrm{N}}$ (using P_4). The latter is in turn equivalent to $(\{a\}^{\Pi}, (\{a\}^{\Pi})^{\mathrm{N}}) \preceq (B^{\Pi}, (B^{\Pi})^{\mathrm{N}})$.

(b) Sufficient condition: $(\{a\}^{\Pi}, (\{a\}^{\Pi})^{\mathrm{N}}) \preceq (B^{\Pi}, (B^{\Pi})^{\mathrm{N}})$ implies $(\{a\}^{\Pi})^{\mathrm{N}} \subseteq (B^{\Pi})^{\mathrm{N}}$, hence $\{a\} \subseteq (\{a\}^{\Pi})^{\mathrm{N}} \subseteq (B^{\Pi})^{\mathrm{N}}$ (using P_5). So, $\{a\} \subseteq (B^{\Pi})^{\mathrm{N}}$. □

This proposition may be used to extract a disjunctive attribute implication of the form $a \longmapsto \bigvee B$. Indeed, we have just to check whether the NΠ-pair associated with a is located above the supremum of all NΠ-pairs associated with b from B in the NΠ-lattice $\mathfrak{L}_{\mathrm{N}\Pi}$. For instance, $a_5 \longmapsto \{a_1, a_4\}$ is a disjunctive attribute implication satisfied by the context $\mathcal{K}_{\mathcal{S}}$ of Example 1 since $(\{a_5\}^{\Pi}, (\{a_5\}^{\Pi})^{\mathrm{N}}) \preceq (\{a_1, a_4\}^{\Pi}, (\{a_1, a_4\}^{\Pi})^{\mathrm{N}})$.

For the purpose of studying the direct derivation of minimal set of disjunctive attribute implications, we define the notion of "pseudo-NΠ-intent."

DEFINITION 11. *(Pseudo-*NΠ*-intent). Let* $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ *be a formal context. A set* $A \subseteq \mathcal{P}$ *is called a pseudo-*NΠ*-intent of* $\mathcal{K}$ *iff* $A \neq (A^{\Pi})^{\mathrm{N}}$, *and it contains the* NΠ*-closures of all pseudo-*NΠ*-intents which are its subsets.*

The following proposition establishes that applying the closure operator $((.)^{\Pi})^{\mathrm{N}}$ to pseudo-NΠ-intents yields a base of disjunctive attribute implications.

PROPOSITION 7. *The two following statements are equivalent:*

- *The set* $\mathcal{B}_{GD}^{\overline{\mathcal{K}}} = \{A^{\neg} \longmapsto ((A^{\neg})^{\Delta}_{\overline{\mathcal{K}}})^{\Delta}_{\overline{\mathcal{K}}} \mid A^{\neg}$ *is a pseudointent of* $\overline{\mathcal{K}}\}$ *is a conjunctive base of* $\overline{\mathcal{K}}$.
- *The set* $\mathcal{D}_{\mathrm{N}\Pi}^{\mathcal{K}} = \{\bigvee(A^{\Pi})^{\mathrm{N}} \longmapsto \bigvee A \mid A$ *is a pseudo-*NΠ*-intent of* $\mathcal{K}\}$ *is a disjunctive base of* $\mathcal{K}$.

Proof. Suppose $A^{\neg}$ is a pseudointent in $\overline{\mathcal{K}}$, it means A is a pseudo-NΠ-intent in $\mathcal{K}$, since $((Q^{\neg})^{\Delta}_{\overline{\mathcal{K}}})^{\Delta}_{\overline{\mathcal{K}}} \subseteq A)^{\neg}$ is equivalent to $(Q^{\Pi})^{\mathrm{N}} \subseteq A$. Now, $\mathcal{B}^{\overline{\mathcal{K}}}_{GD}$ is a conjunctive base is equivalent to $\mathcal{D}^{\mathcal{K}}_{\mathrm{N}\Pi}$ is a disjunctive base (by Theorem 1). □

In the remaining, we are interested to show the usefulness of $\mathcal{D}^{\mathcal{K}}_{\mathrm{N}\Pi} = \{\bigvee(A^{\Pi})^{\mathrm{N}} \longmapsto \bigvee A \mid A$ is a pseudo-NΠ-intent of $\mathcal{K}\}$. Indeed, it will be proved through Theorem 10 that $\mathcal{D}^{\mathcal{K}}_{\mathrm{N}\Pi}$ is a minimal base. The two following lemmas are given to achieve this important result.

LEMMA 4. *Let B be an* NΠ*-intent and Q be a pseudo*-NΠ*-intent, with $B \nsubseteq Q$ and $Q \nsubseteq B$, then $B \cap Q \in INT^{\mathcal{K}}_{\mathrm{N}\Pi}$.*

Proof. It is well established that for any pseudointent $Q^{\neg}$ of $\overline{\mathcal{K}}$ and any intent $B^{\neg}$ of $\overline{\mathcal{K}}$ such that $Q^{\neg} \not\subset B^{\neg}$ the intersection $B^{\neg} \cap Q^{\neg}$ is closed (it is an intent of $\overline{\mathcal{K}}$).[2] It follows that Q is a pseudo-NΠ-intent of $\mathcal{K}$, B is an NΠ-intent of $\mathcal{K}$, and $B \cap Q$ is a pseudo-NΠ-intent of $\mathcal{K}$, which means $B \cap Q \in INT^{\mathcal{K}}_{\mathrm{N}\Pi}$. □

The following lemma shows among other things that there can be no complete set that contains fewer disjunctive attribute implications than there are pseudo-intents.

LEMMA 5. *For each pseudo*-NΠ*-intent P, every base $\mathcal{D}^{\mathcal{K}}$ of disjunctive attribute implications contains an implication $\bigvee A \longmapsto \bigvee B$ s.t. $((B)^{\Pi})^{\mathrm{N}} = ((P)^{\Pi})^{\mathrm{N}}$.*

Proof. Let P be a pseudo-NΠ-intent. Then $P \neq ((P)^{\Pi})^{\mathrm{N}}$, i.e., $P \notin INT^{\mathcal{K}}_{\mathrm{N}\Pi}$ or again $P \notin INT^{\mathcal{K}}_{\mathrm{N}\Pi}$. So, P does not satisfy $\mathcal{D}^{\mathcal{K}}$ (by Theorem1), which reads

$$\exists \bigvee A \longmapsto \bigvee B \in \mathcal{D}^{\mathcal{K}} : P \text{ does not satisfy } \bigvee A \longmapsto \bigvee B$$

which means (a): $A \nsubseteq P$ and (b): $B \subseteq P$.

By Proposition 5, we have $A \subseteq ((B)^{\Pi})^{\mathrm{N}}$ (c)

From (a) and (c), we get $((B)^{\Pi})^{\mathrm{N}} \nsubseteq P$ (d)

As $((B)^{\Pi})^{\mathrm{N}} \cap P \subseteq INT^{\mathcal{K}}_{\mathrm{N}\Pi}$ (from Lemma 4) then

$$\begin{aligned} & ((B)^{\Pi})^{\mathrm{N}} \cap P \text{ satisfies } \mathcal{D}^{\mathcal{K}} \\ \Longrightarrow\ & ((B)^{\Pi})^{\mathrm{N}} \cap P \text{ satisfies } \bigvee A \longmapsto \bigvee B \\ \Longrightarrow\ & A \subseteq (((B)^{\Pi})^{\mathrm{N}} \cap P) \text{ or } B \nsubseteq (((B)^{\Pi})^{\mathrm{N}} \cap P) \end{aligned}$$

This statement is false (by (a) and (b)). Therefore, $((B)^{\Pi})^{\mathrm{N}} \cap P \nsubseteq INT^{\mathcal{K}}_{\mathrm{N}\Pi}$ then

$$\begin{aligned} & ((B)^{\Pi})^{\mathrm{N}} \subseteq P \text{ or } P \subseteq ((B)^{\Pi})^{\mathrm{N}} && \text{(using Lemma 4)} \\ \Longrightarrow\ & P \subseteq ((B)^{\Pi})^{\mathrm{N}} && \text{(by (d))} \\ \Longrightarrow\ & ((P)^{\Pi})^{\mathrm{N}} \subseteq ((((B)^{\Pi})^{\mathrm{N}})^{\Pi})^{\Pi} && \text{(using} (P_3) \text{ and } (P_4)) \\ \Longrightarrow\ & ((P)^{\Pi})^{\mathrm{N}} \subseteq ((B)^{\Pi})^{\mathrm{N}} \end{aligned}$$

From (b): $B \subseteq P$, by applying successively properties (P_3) and (P_4), we get: $((B)^{\Pi})^{\mathrm{N}} \subseteq ((P)^{\Pi})^{\mathrm{N}}$. This finally yields $((B)^{\Pi})^{\mathrm{N}} = ((P)^{\Pi})^{\mathrm{N}}$. □

THEOREM 10. *The disjunctive attribute implications base* $\mathcal{D}^{\mathcal{K}}_{\mathrm{N}\Pi}$ *is minimal for* $\mathcal{K}$.

Proof. Let $\mathcal{D}^{\mathcal{K}}$ be a base of disjunctive attribute implications for $\mathcal{K}$. Note that it is sufficient to show that $|\mathcal{D}^{\mathcal{K}}_{\mathrm{N}\Pi}| \leq |\mathcal{D}^{\mathcal{K}}|$.

Clearly, there is a bijection between the minimal base $\mathcal{D}^{\mathcal{K}}_{\mathrm{N}\Pi}$ and the set of all pseudo-NΠ-intents. Also, every base $\mathcal{D}^{\mathcal{K}}$ of disjunctive attribute implications must contain an implication $\bigvee A \longmapsto \bigvee B$ with $((B)^{\Pi})^{\mathrm{N}} = ((P)^{\Pi})^{\mathrm{N}}$ for every pseudo-NΠ-intent P by Lemma 5. Hence, $\mathcal{D}^{\mathcal{K}}_{\mathrm{N}\Pi}$ is a minimal base of disjunctive attribute implications. □