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Invariant manifolds of a non-autonomous quasi-bicircular problem computed via the parameterization method

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Abstract. The parameterization method (PM) has been used to compute high-order parameterizations of invariant manifolds of vector fields at fixed points. This paper extends such approach to invariant manifolds of periodically-perturbed vector fields about a periodic orbit with the same frequency, with a direct application on the libration points of the Sun-Earth-Moon system. The Sun-Earth-Moon environment is modeled by the so-called Quasi-Bicircular Model (QBPM), which is a coherent restricted four-body model that describes the motion of a spacecraft under the simultaneous gravitational influences of the Earth, the Moon, and the Sun. The PM is adapted to account for the explicit time-dependency of the corresponding vector field. This new procedure yields high-order periodic semi-analytical approximations of the center manifolds about the libration points \( L_{1,2} \) of the periodically-perturbed Sun-(Earth+Moon) and Earth-Moon systems. These approximations are then used to initialize the computation of Poincaré maps, which allow to get a qualitative description of the non-autonomous dynamics near the equilibrium points. It is shown that, with this new approach, the semi-analytical description of the center manifolds in a coherent four-body environment is valid in a neighborhood significant enough to be used in practice. In particular, the well-known Halo orbit bifurcation is recovered in all cases.

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1. Introduction

For the last three decades, the dynamics about the libration points of the Sun-Earth (SEL) and Earth-Moon (EML) systems have been increasingly studied and used as the backbone of numerous space missions, both in terms of transfer trajectory determination and nominal orbit design. Famous successful applications include the ARTEMIS probe [?] in the Earth-Moon system as well as the SOHO [?], DSCOVR [?] and Gaia [?] spacecrafts at SEL1,2, to mention just a few.

Besides these practical examples in each system, the dynamics of both problems can be combined to produce efficient transfers in the extended Sun-Earth-Moon (SEM) system. Such feature first emerged with the concept of Weak Stability Boundary (WSB) used to perform low-energy, long-duration Earth-to-Moon transfers [?]. It relies on an extended Earth-Moon system that includes the perturbation of the Sun, providing a natural correction of the energy of the spacecraft. Later, the invariant manifold dynamics were used to perform similar transfers [?, ?] in a systematic way [?, ?]. In this second approach, the SEM system is seen as two coupled Circular Restricted Three Body Problems (CRTBP), the Sun-Earth and Earth-Moon systems, with their associated libration points. The invariant manifolds of the orbits about EML2 and SEL1,2 provide dynamical channels that can be suitably combined to produce efficient transfers. This so-called coupled CRTBP approximation paved the way for other combinations of invariant manifolds, for instance in the jovian system [?, ?, ?]. Since then, the coupled approximation has been also used to perform Earth-to-EML2 [?, ?] and SEL1,2-to-EML2 transfers [?, ?]. All these applications uncovered the low-energy network that interconnects the EML1,2 points, the Moon, the SEL1,2 points, and the Earth. Such links contribute to making the EML1,2 points potential spatial hubs [?, ?, ?].

This paper is part of a project which aims to provide a systematic or near-systematic preliminary analysis tool for the motion of a spacecraft in the SEM low-energy network, with a particular focus on connections between SEL1,2 and EML2. Such a purpose calls for a model sufficiently detailed to capture the intricate dynamics of this system, yet simple enough to allow systematic use. From that perspective, three models of increasing complexity emerge:

A. The coupled CRTBP approximation. As previously noticed, it uses purely three-body dynamics to model the motion of a spacecraft in the presence of the Sun, Earth, and Moon. Broadly speaking, when the spacecraft is near the EM system, its motion is modeled by the EM CRTBP. Otherwise, it is modeled by the SEM CRTBP. Such a configuration preserves many of the features of the CRTBP, while permitting a spacecraft to be affected by all three massive bodies, albeit only two of them at any given moment [?]. Such approximation relies on the fact that the dynamics associated with the EM and SEM subsystems are partially preserved in the four-body context. However, it requires arbitrary connections for every computed trajectory, which prevents the use of this model as a basis for a systematic tool.

B. The Bicircular problem (BCP). Widely used in the literature for the study of the
EM triangular libration points \([? ,?, ?,?]\) and for transfer optimization \([? ,?, ?]\), it considers the Earth and the Moon moving in circular orbits around their barycenter which is also moving in circular orbit around the Sun. Such a model is not coherent since the motion of the three primaries is not a solution of the corresponding three-body problem.

C. The Quasi-Bicircular problem (QBCP). It is a bicircular coherent periodic model for the SEM problem that has been developed in \([?]\). In this framework, the motion of the three primaries is a planar self-consistent solution of the three-body problem along two quasi-circular orbits. This model acts as an intermediate step between the CRTBP and the restricted \(n\)-body model. Note that, despite being coherent, the QBCP does not describe the true motion of the Earth, the Moon and the Sun in the Solar System. In particular, the motion of the three primaries is supposed planar. Nevertheless, it has proven to be useful for the study of the libration point \(L_2\), in particular for the characterization of resonant families of tori with horizontal frequency equal to \(2\omega_s\), where \(\omega_s\) is the external frequency due to the perturbation of the Sun \([?]\). Moreover, it provides better initial conditions than the BCP for refinement procedures to higher-fidelity models, especially in the case of libration point orbits of the Earth-Moon system \([?]\). Finally, both the BCP and the QBCP are Hamiltonian systems with three degrees of freedom which depend periodically on time, so the use of the QBCP, instead of the BCP, does not produce any relevant increase of the complexity of the model. To the best of the authors’ knowledge, the QBCP has never been used to model transfers in the SEM system.

The long-term aim of this project is to assess the advantages of the BCP/QBCP over the coupled three-body approximation in the context of preliminary mission analysis. Indeed, the four-body models provide an all-in-one solution for the study of the dynamics in the SEM system. Yet, they represent a simple extension to the CRTBP, with only one additional frequency \([? ,?]\). In that sense, they can be seen as periodically-perturbed three-body problems. In particular, the essential dynamics of the CRTBP are largely preserved, which allows procedures developed in the CRTBP to be extended to the four-body problem. Indeed, previous efforts on the collinear libration points show that similar methods were used in both the CRTBP and the four-body models:

- Purely numerical techniques in the CRTBP \([?]\) and the BCP \([?]\).
- Computation of semi-analytical normal forms of the Hamiltonian, either in the CRTBP \([?]\), the BCP \([? ,?]\), or the QBCP \([?]\). They were also used in other periodically-perturbed three-body problems \([?]\).
- Linstedt-Poincaré procedures for some specific families of orbit in both the CRTBP and the QBCP \([? ,?]\).
- Parameterization method (PM) in the CRTBP \([? ,?]\). Originally developed to prove existence and regularity of invariant manifolds of dynamical systems, the PM can also be used to compute high-order power series expansions of parameterizations of invariant manifolds at fixed points of vector fields \([? ,?,?]\).
As the reader may know, the domain of the semi-analytical approaches are inherently limited since the resulting expansions are not convergent in any open set. However, there usually exists a domain of convergence within which the series are seemingly convergent for numerical purposes [?]. It is sometimes referred as the domain of practical convergence [?]. In such a domain, they can provide a very compact tool to describe the phase space. In this perspective, semi-analytical tools applied to SEM four-body model may yield a significant step for the systematic analysis of the SEM low-energy network.

In this paper, the parameterization method is extended for the first time to the non-autonomous qbcp framework in order to compute semi-analytical approximations of the center manifolds of the libration points EML 1,2 and SEL 1,2 in a SEM four-body environment. It is shown that these approximations yield an accurate description of the center manifolds in a domain big enough to be used in practice. Although the problem of non-autonomous PM has been previously addressed [?, ?], this paper yields the first practical application on a realistic model of a non-autonomous PM in the context of celestial mechanics.

The paper is organized as follows: in Section 2, the QBCP is introduced along with its equations of motion. In Section 3, some preliminary computations are performed on the corresponding vector field to account for its explicit 2π-periodic time-dependence. In Section 4, the parameterization method for invariant manifold is presented and extended to 2π-periodic center manifolds. In Sections 5 and 6, applications to the center manifold of EML 1,2 and SEL 1,2 are presented and discussed. In particular, in Section 5, the problem of low-order resonances around EML 2 is addressed. It is shown that the new approach developed here allows to better handle these resonances and extend the domain of practical convergence of the semi-analytical center manifold, compared to previous efforts [?].

2. The Quasi-Bicircular Problem (QBCP)

2.1. Definition

The Quasi-Bicircular Problem (QBCP) is a restricted four-body problem introduced in [?] to describe the motion of a massless particle subject to the gravitational influence of the Earth, the Moon, and the Sun, whose own motion is a quasi-bicircular solution of the Three-Body Problem. The resulting system is a Hamiltonian with three degrees of freedom and depending periodically on time. This Hamiltonian was originally derived in the EM synodical frame, which is a rotating-pulsating frame centered at the Earth-Moon barycenter B_{em} in such a way that the Earth and the Moon are located at fixed positions, as in the usual CRTCJP synodical frame (Figure 1a).

Denoting the state in this frame by \( \mathbf{Z} = (X \ Y \ Z \ P_X \ P_Y \ P_Z)^T \), the Hamiltonian of
the QBCP takes the form:

\[
H_{em}(Z, \theta) = \frac{1}{2} \alpha_1(\theta)(P_X^2 + P_Y^2 + P_Z^2) + \alpha_2(\theta)(P_X X + P_Y Y + P_Z Z) \\
+ \alpha_3(\theta)(P_X Y - P_Y X) \\
+ \alpha_4(\theta)X + \alpha_5(\theta)Y - \alpha_6(\theta) \left( \frac{1 - \mu}{Q_{pe}} + \frac{\mu}{Q_{pm}} + \frac{m_s}{Q_{ps}}, \right)
\]

(1)

where:

\[
Q^2_{pe} = (X - \mu)^2 + Y^2 + Z^2,
\]

\[
Q^2_{pm} = (X - \mu + 1)^2 + Y^2 + Z^2,
\]

\[
Q^2_{ps} = (X - \alpha_7(\theta))^2 + (Y - \alpha_8(\theta))^2 + Z^2,
\]

and with \( \mu \) the Earth-Moon mass ratio and \( m_s \) the mass of the Sun. Moreover, the \( \alpha_k \) functions are of the form:

\[
\alpha_k(\theta) = \alpha_{k0} + \sum_{j \geq 1} \alpha_{kj} \cos(j\theta), \text{ for } k = 1, 3, 4, 6, 7
\]

\[
\alpha_k(\theta) = + \sum_{j \geq 1} \alpha_{kj} \sin(j\theta), \text{ for } k = 2, 5, 8
\]

where \( \theta \) is the phase of the Sun, used to parameterize the time-dependency of the dynamics. The angle \( \theta \) is directly proportional to the time \( t \): \( \theta = \omega_s t \), where \( \omega_s \) is the pulsation of the Sun. The first coefficients \( \alpha_{kj} \) are available in [?] for \( k \in [1, 8] \). The Hamiltonian (1) is 2\( \pi \)-periodic with respect to \( \theta \) and \( T \)-periodic with respect to \( t \), with \( T = 2\pi/\omega_s \). Note that, using such a frame, the usual Earth-Moon CRTBP units are still relevant: the unit of length is the distance between the Earth and Moon; the unit of time is chosen so that the sidereal mean motion of the Earth-Moon system is one; the unit of mass is the sum of the Earth’s and Moon’s masses. Using these units, the gravitational constant is \( G = 1 \). Finally, the numerical values used in this paper are similar to [?]:

\[
\mu = 0.0121505816, \\
m_s = 328900.54, \\
\omega_s = 0.925195985520347,
\]

2.2. Equations in the Sun-Earth frame

The Sun-Earth (SE) synodical frame (see Figure 1b) is a rotating-pulsating frame centered at the barycenter \( B \) of the system in such a way that the Sun and the Earth-Moon barycenter \( B_{em} \) are located at fixed positions. Moreover, the unit system is adapted to the Sun-Earth framework: the unit of length is the distance between the Sun and \( B_{em} \); the unit of time is chosen so that the sidereal mean motion of the Sun and \( B_{em} \) is one; the unit of mass is the sum of the three primaries’ masses. In this
Invariant manifolds of a non-autonomous problem via the parameterization method

framework, denoting again the state by \( Z = (X Y Z P_X P_Y P_Z)^T \), the Hamiltonian of the QBCP takes the form:

\[
H_{se}(Z, \tau) = \frac{1}{2} \delta_1(\tau)(P_X^2 + P_Y^2 + P_Z^2) + \delta_2(\tau)(P_XX + P_YY + PZZ) \\
+ \delta_3(\tau)(P_XY - YX) \\
- \delta_6(\tau) \left( \frac{\bar{m}_e}{\bar{R}_{pe}} + \frac{\bar{m}_m}{\bar{R}_{pm}} + \frac{\bar{m}_s}{\bar{R}_{ps}} \right),
\]

where:

\[
\bar{R}_{pe}^2 = (X - \delta_7(\theta))^2 + (Y - \delta_8(\theta))^2 + Z^2,
\]

\[
\bar{R}_{pm}^2 = (X - \delta_9(\theta))^2 + (Y - \delta_{10}(\theta))^2 + Z^2,
\]

\[
\bar{R}_{ps}^2 = (X - \mu_{sem})^2 + Y^2 + Z^2,
\]

and \( \mu_{sem} = 1/(1 + m_s) \) is the mass ratio of the Sun-(Earth+Moon) system. The masses of the primaries are given in Sun-Earth units and can be computed from the Earth-Moon values via the following equations:

\[
\bar{m}_e = \frac{1 - \mu}{1 + m_s}, \quad \bar{m}_m = \frac{\mu}{1 + m_s}, \quad \bar{m}_s = \frac{m_s}{1 + m_s}.
\]

Finally, the \( \delta_k \) functions are of the form:

\[
\delta_k(\tau) = \delta_{k0} + \sum_{j \geq 1} \delta_{kj} \cos(j\tau), \quad \text{for } k = 1, 3, 6, 7, 9
\]

\[
\delta_k(\tau) = + \sum_{j \geq 1} \delta_{kj} \sin(j\tau), \quad \text{for } k = 2, 8, 10
\]

where \( \tau = \tilde{\omega}_s t \) is the phase of the Earth-Moon system and \( \tilde{\omega}_s = \omega_s/(1 - \omega_s) \) is the pulsation of the Earth-Moon system in the SE synodical frame. The first coefficients of the \( \delta_k \) functions are given in Table 1. Note that the Hamiltonian (2) is formally equivalent to (1) with \( \alpha_4 = \alpha_5 = 0. \)
Table 1: Main Fourier coefficients of the $\delta_k$ functions.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\delta_1$ (cos)</th>
<th>$\delta_2$ (sin)</th>
<th>$\delta_3$ (cos)</th>
<th>$\delta_6$ (cos)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.99999962e-01</td>
<td>1.00000000e+00</td>
<td>9.9999981e-01</td>
<td>2.07685015e-11</td>
</tr>
<tr>
<td>1</td>
<td>6.56940309e-12</td>
<td>-4.0626122e-11</td>
<td>3.28470146e-12</td>
<td>3.28470146e-12</td>
</tr>
<tr>
<td>3</td>
<td>-3.09507325e-05</td>
<td>3.14937800e-05</td>
<td>-2.51631272e-03</td>
<td>-2.51631272e-03</td>
</tr>
<tr>
<td>4</td>
<td>1.22792984e-11</td>
<td>9.66363988e-12</td>
<td>3.28470146e-12</td>
<td>3.28470146e-12</td>
</tr>
<tr>
<td>5</td>
<td>-9.99996973e-01</td>
<td>-9.99995820e-01</td>
<td>3.06255907e-12</td>
<td>3.06255907e-12</td>
</tr>
<tr>
<td>6</td>
<td>3.09507325e-05</td>
<td>3.14937800e-05</td>
<td>-2.51631272e-03</td>
<td>-2.51631272e-03</td>
</tr>
<tr>
<td>7</td>
<td>-1.22792984e-11</td>
<td>9.66363988e-12</td>
<td>3.28470146e-12</td>
<td>3.28470146e-12</td>
</tr>
</tbody>
</table>

All the coefficients not printed here are less than $10^{-12}$.

### 2.3. Translation to a collinear libration point

For the study of the collinear libration points, it is usual to translate these equations in a suitable framework, centered on the CRTBP libration point $L_i$, $i \in [1, 3]$, and normalized by the distance $\gamma$ between $L_i$ and the smallest primary [?]. Note that the distance $\gamma$ is computed in the CRTBP (see e.g. [?] for details).

Such a framework is called Normalized-Centered (NC) in the present paper, and the NC state is denoted by $z = (x, y, z, p_x, p_y, p_z)^T$. Coming from the EM synodical frame, the corresponding change of coordinates is given by:

$$Z = C(z) = \Gamma(z - z_b),$$

where:

$$\Gamma = \gamma \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z_b = c_1 \begin{pmatrix} 1 & 0 & 0 & -\frac{\alpha_2}{\alpha_1} & \frac{\alpha_3}{\alpha_1} & 0 \end{pmatrix}^T,$$

and:

$$c_1 = \frac{\mu - 1 \pm \gamma}{\gamma},$$

with the upper sign for EML$_1$ and the lower sign for EML$_2$.

With such a change of coordinates, the system can still be written in Hamiltonian form. For this purpose, let $S = \{e, m, s\}$ be the set of subscripts denoting the initials of the primaries involved (Earth, Moon and Sun). Let $(x_c, y_c, z_c)$ be the position of the primary $c \in S$ in NC coordinates, and $m_c$ its mass in Earth-Moon units – the
Table 2: Masses and position of the primaries in the NC reference frame and in EM units. The upper (resp. lower) signs stands for the EML₁ (resp. EML₂) case.

<table>
<thead>
<tr>
<th>Primary</th>
<th>( m_c )</th>
<th>( x_c )</th>
<th>( y_c )</th>
<th>( z_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>( 1 - \mu )</td>
<td>((-1 \pm \gamma)^{-1})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Moon</td>
<td>( \mu )</td>
<td>±1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sun</td>
<td>( m_s )</td>
<td>((\mu - 1 \pm \gamma - \alpha_7)^{-1})</td>
<td>(-\alpha_8 \gamma^{-1})</td>
<td>0</td>
</tr>
</tbody>
</table>

corresponding values are given in Table 2. Inserting this change of coordinates in (1), the NC Hamiltonian can be written as:

\[
H_{em}(z, \theta) = \frac{1}{2} \alpha_1 \left( p_x^2 + p_y^2 + p_z^2 \right) + \alpha_2 \left( p_x x + p_y y + p_z z \right) + \alpha_3 \left( p_x y - p_y x \right) - \alpha_{24} x - \alpha_{26} y - \frac{\alpha_6}{\gamma^3} \sum_{c \in S} \frac{m_c}{q_{pc}},
\]

where:

\[
\alpha_{24} = \frac{\alpha_4}{\gamma} - \left[ \frac{\dot{\alpha}_2}{\alpha_1} + \frac{\alpha_2^2 + \alpha_3^2}{\alpha_1} \right] c_1,
\]

\[
\alpha_{26} = \frac{\alpha_5}{\gamma} + \frac{\dot{\alpha}_3}{\alpha_1} c_1,
\]

and \( \dot{\alpha} = \frac{\partial \alpha}{\partial t} \) denotes the derivative with respect to time. Finally:

\[
q_{pc}^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2.
\]

Note that this change of coordinates redefines the unit of distance as the distance from the equilibrium point to the closest primary. A similar procedure can be applied to get a Normalized-Centered version of the dynamics from the SE synodical frame. Denoting again the NC state by \( z = (x \ y \ z \ p_x \ p_y \ p_z)^T \), the corresponding Hamiltonian is:

\[
H_{se}(z, \theta) = \frac{1}{2} \delta_1 \left( p_x^2 + p_y^2 + p_z^2 \right) + \delta_2 \left( p_x x + p_y y + p_z z \right) + \delta_3 \left( p_x y - p_y x \right) - \delta_{24} x - \delta_{26} y - \frac{\delta_6}{\gamma^3} \sum_{c \in S} \frac{m_c}{r_{pc}},
\]

where:

\[
\delta_{24} = - \left[ \frac{\dot{\delta}_2}{\delta_1} + \frac{\delta_2^2 + \delta_3^2}{\delta_1} \right] c_1,
\]

\[
\delta_{26} = \left( \frac{\dot{\delta}_3}{\delta_1} \right) c_1.
\]

Note that the NC framework is centered at the CRTBP libration points although, in the QBCP, due to the effect of the Sun, they are no longer equilibrium points. Assuming that (i) there is no resonance between \( \omega_s \) and the linear frequencies around the points and (ii) the perturbation of the Sun is sufficiently small, the equilibrium points of the
crtbp are replaced by periodic orbits with the same frequency as the perturbation (see [?, ?] for an explanation in the $L_{4,5}$ BCP case). The so-called dynamical equivalents of the Earth-Moon libration points $L_{1,2}$ are shown in Figure 2, along with the position of their geometric CRTBP counterparts.

![Figure 2: T-periodic orbits that act as dynamical equivalents of the EML$_{1,2}$ libration points, computed in NC frame and plotted in EM coordinates. Symbols: $\bigcirc$ is the CRTBP geometric position of the libration point; $\bullet$ is the starting point at $t = 0$; $\▲$ gives the direction of motion for $t > 0$. On these plots, one unit is equivalent to 384400 km.](image)

Using the adapted NC coordinates in the SE case, the dynamical equivalents of SEL$_{1,2}$ can be computed. The corresponding results are given on Figure 3. Contrary to the EM case, the CRTBP positions of the libration points are shifted outside of the scope of these plots, due to the relative positions of the perturbing body — the Moon — and the libration point.

**Notations.** From now on, and if not stated otherwise, the present paper makes use of the Earth-Moon NC formulation (3) and the corresponding vector field as the default examples for all computations. However, as much as possible, the numerical results are displayed in the widely used EM and SE normalized coordinates and units.

### 3. A suitable form for the linearized vector field

The semi-analytical methods used to compute high-order approximations of invariant manifolds about the libration points usually rely on the fact that the latter are fixed points of the vector field. The extension of such procedures to periodic objects is possible but requires a suitable transformation of the state. This is the subject of this section.
Invariant manifolds of a non-autonomous problem via the parameterization method

Figure 3: $T$-periodic orbits that act as dynamical equivalents of the SEL$_{1,2}$ libration points, computed in NC frame and plotted in SE coordinates. Symbols: • is the starting point at $t = 0$; ▲ gives the direction of motion for $t > 0$. On these plots, one unit is equivalent to $1.496 \times 10^8$ km.

In the current case, high-order approximations of invariant manifolds about the $T$-periodic dynamical equivalents of the libration points $L_{1,2}$ are sought via the parameterization method. The first step towards this computation is to put the linearized vector field into a form equivalent to the autonomous CRTBP case, addressed for example in [?]. Namely, this requires the origin to be a fixed point and the linearized vector field to be both diagonal and autonomous at the origin. For a system described by the Hamiltonian (3), such features can be obtained following a change of coordinates thoroughly described in [?] in the Earth-Moon $L_2$ case, and in [?] in the $L_{3,4,5}$ case. Its major steps are recalled hereafter:

(i) Artificially autonomize the Hamiltonian by introducing the couple $(\theta, y_\theta)$ so that the time is managed as a new variable. This is done by simply adding the term $\omega_s y_\theta$ to the initial Hamiltonian, forming an extended autonomous Hamiltonian system of degree four (see e.g. [?]):

$$H(z, y_\theta, \theta) = \omega_s y_\theta + \frac{1}{2} \alpha_1 \left(p_x^2 + p_y^2 + p_\theta^2\right) + \alpha_2 (p_x x + p_y y + p_\theta z) + \alpha_3 (p_x y - p_y x)$$
$$- \alpha_{24} x - \alpha_{26} y - \frac{\alpha_6}{\gamma^3} \sum_{c \in S} m_c q_{pc},$$

(ii) Cancel the terms of order one or, equivalently, translate the origin of coordinates to the periodic orbit which substitutes $L_{1,2}$ so that the origin becomes a fixed point.

(iii) Diagonalize the terms of order two. In particular, the differential of the vector field evaluated at the origin will be time-independent. This step requires a complexification of the variables.

Formally, the composition of these operations defines a change of coordinates
Invariant manifolds of a non-autonomous problem via the parameterization method

\[
z = C_0(\dot{z}, t) \text{ as well as a new complex Hamiltonian } \hat{H} \text{ of the form:}
\]

\[
\hat{H}(\hat{z}, \hat{\theta}, \hat{y}_0) = \omega_3 \hat{y}_0 + i \omega_1 \hat{x}_1 \hat{y}_1 + \omega_2 \hat{x}_2 \hat{y}_2 + i \omega_3 \hat{x}_3 \hat{y}_3 + \sum_{k \geq 3} \hat{H}_k(\hat{z}, \hat{\theta}),
\]

(5)

where \(i\) is the complex number of modulus one, and \(\hat{H}_k(\hat{z}, \hat{\theta})\) are homogeneous Fourier-Taylor polynomials of degree \(k\) in the complex variable \(\hat{z} = (\hat{x}_1 \ \hat{x}_2 \ \hat{x}_3 \ \hat{y}_1 \ \hat{y}_2 \ \hat{y}_3)^T\).

For the sake of completeness, each of these steps is developed in more details in the following subsections.

3.1. Expanding the Hamiltonian.

Prior to any development, it is convenient to formally expand the Hamiltonian (3). This operation can be performed up to an arbitrary order in the variable \(z\) (see e.g. [?]), although only the first orders are needed here. First, the terms of order zero can be discarded since they will not appear in the vector field. Then, let \(z = (x^T \ y^T)^T\) be the canonical form of the \(nc\) coordinates. With a simple expansion in power series, it becomes:

\[
H(x, y, \theta) = H_1(x, y, \theta) + H_2(x, y, \theta) + \sum_{n=3}^{\infty} H_n(x, y, \theta),
\]

with:

\[
H_1(x, y, \theta) = -V_1^T x,
\]

\[
H_2(x, y, \theta) = x^T Q_1 x + x^T Q_2 y + y^T Q_3 y,
\]

and where:

- \(V_1\) is a \(3 \times 1\), \(2\pi\)-periodic vector in the variable \(\theta\),
- \(Q_i(\theta)\) are \(3 \times 3\), \(2\pi\)-periodic matrices,
- \(Q_1\) and \(Q_3\) are symmetric.

More precisely, for the second order:

\[
Q_1 = \frac{\alpha_6}{2\gamma^3} \begin{pmatrix}
q_{11} & -3q_{12} & 0 \\
-3q_{12} & q_{22} & 0 \\
0 & 0 & q_{33}
\end{pmatrix}, \quad Q_2 = \begin{pmatrix}
\alpha_2 & -\alpha_3 & 0 \\
\alpha_3 & \alpha_2 & 0 \\
0 & 0 & \alpha_2
\end{pmatrix}, \quad Q_3 = \frac{1}{2} \alpha_1 I_3,
\]

where the \(q_{ij}\) coefficients are functions of the masses and positions of the three primaries:

\[
q_{11} = \sum_{c \in S} \frac{m_c}{\rho_c^6} (\rho_c^2 - 3x_c^2),
\]

\[
q_{22} = \sum_{c \in S} \frac{m_c}{\rho_c^6} (\rho_c^2 - 3y_c^2),
\]

\[
q_{33} = \sum_{c \in S} \frac{m_c}{\rho_c^6} (\rho_c^2 - 3z_c^2),
\]

\[
q_{12} = \sum_{c \in S} \frac{m_c}{\rho_c^6} x_c y_c,
\]

(6)

and:

\[
\rho_c^2 = x_c^2 + y_c^2 + z_c^2, \quad \forall c \in S.
\]
3.2. The complete change of coordinates

3.2.1. Canceling the terms of order one. As stated in [?], a fast and simple method consists in computing the periodic solution which substitutes the equilibrium point (see Figure 2), and then perform a time-dependent translation of the coordinates. After a Fourier analysis, the state along these solutions, generically denoted $V(\theta)$, is obtained in the form:

$$V(\theta) = \begin{pmatrix} g_1(\theta) & g_3(\theta) & 0 & g_2(\theta) & g_4(\theta) & 0 \end{pmatrix}^T,$$

where the functions $g_i(\theta)$ are $2\pi$-periodic Fourier series.

Denoting the new state by $(\tilde{z}, \tilde{\theta}, \tilde{y}_\theta)$, the corresponding change of coordinates is:

$$\begin{cases}
    z = \tilde{z} + V(\theta), \\
    \theta = \tilde{\theta}, \\
    y_\theta = \tilde{y}_\theta - \frac{1}{2}(g_2'g_1 - g_1'g_2 + g_4'g_3 - g_3'g_4) - G_1',
\end{cases}$$

where $s' = \frac{\partial s}{\partial \theta}$ and $G_1$ is the Lie generating function [?] given by

$$G_1 = -g_2(\theta)\tilde{x}_1 + g_1(\theta)\tilde{y}_1 - g_4(\theta)\tilde{x}_2 + g_3(\theta)\tilde{y}_2.$$

Such change of coordinates is canonical. Inserting this change of variables into the expansion of the Hamiltonian, it comes:

$$\tilde{H}(\tilde{x}, \tilde{y}, \theta) = \tilde{H}_2(\tilde{x}, \tilde{y}, \tilde{\theta}, \tilde{y}_\theta) + \sum_{n=3}^{\infty} \tilde{H}_n(\tilde{x}, \tilde{y}, \tilde{\theta}, \tilde{y}_\theta),$$

with

$$\tilde{H}_2(\tilde{x}, \tilde{y}, \tilde{\theta}, \tilde{y}_\theta) = \omega_s \tilde{y}_\theta + \frac{1}{2} \omega_1 (\tilde{x}_2^2 + \tilde{y}_2^2) + \omega_2 \tilde{x}_2 \tilde{y}_2 + \frac{1}{2} \omega_3 (\tilde{x}_3^2 + \tilde{y}_3^2),$$

and with the new translated form:

$$\tilde{Q}_1 = \frac{\alpha_6}{2\gamma^3} \begin{pmatrix}
    \tilde{q}_{11} & -3\tilde{q}_{12} & 0 \\
    -3\tilde{q}_{12} & \tilde{q}_{22} & 0 \\
    0 & 0 & \tilde{q}_{33}
\end{pmatrix}, \quad \tilde{Q}_2 = Q_2, \quad \tilde{Q}_3 = Q_3,$$

where

$$\tilde{q}_{ij}(m_c, x_c, y_c, z_c) = q_{ij}(m_c, x_c - g_1, y_c - g_2, z_c), \quad \text{with } c \in \{e, m, s\},$$

and the coefficients $q_{ij}$ have been defined in equation (6). In simple terms, the new coefficients $\tilde{q}_{ij}$ are formally equivalent to the $q_{ij}$, except that the position of the primaries take into account the translation to the periodic orbit.

3.2.2. Normal form of the second order. The idea of this section is to use the Floquet theorem to perform a change of coordinates that puts $\tilde{H}_2$ into the following real normal form:

$$\tilde{H}_2(\tilde{z}, \tilde{\theta}, \tilde{y}_\theta) = \omega_s \tilde{y}_\theta + \frac{1}{2} \omega_1 (\tilde{x}_1^2 + \tilde{y}_1^2) + \omega_2 \tilde{x}_2 \tilde{y}_2 + \frac{1}{2} \omega_3 (\tilde{x}_3^2 + \tilde{y}_3^2),$$
Invariant manifolds of a non-autonomous problem via the parameterization method

where the tilde denotes the new set of coordinates. This form allows to get rid of both the
time dependency and the coupling terms up to order 2. In the following developments,
the linearized system will be alternatively seen as a 2π-periodic system in the variable
θ = ωs t, or a T-periodic system in the variable t.

From [?], the following version of the Floquet theorem is available:

**Proposition 1** Consider the second order Hamiltonian (8) of the translated system
(the check notation * is discarded within this proposition, for the sake of clarity).
The Hamiltonian equations associated with the angular variables x and their respective
momenta y are:

$$
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
D_y H \\
-D_x H
\end{pmatrix} =
\begin{pmatrix}
Q_2^T \\
-2Q_1 \\
-Q_2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}.
$$

Knowing that $z = (x^T, y^T)^T$, this system of equations is denoted by $\dot{z} = Q(\theta)z$. The
matrix Q is 2π-periodic with respect to $\theta = \omega_s t$.

Using the Floquet theorem, there exists a 2π-periodic matrix $P(\theta)$ such that the
change of variables

$$
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix},
$$

which is denoted $z = P(\theta)\hat{z}$, where $\hat{z} = (\tilde{x}, \tilde{y})^T$, transforms $\dot{z} = Q(\theta)z$ into $\dot{\hat{z}} = B\hat{z}$, where B is a constant matrix.

Consider the complete change of variables:

$$
\begin{pmatrix}
x \\
y \\
\theta \\
y_\theta
\end{pmatrix} =
\begin{pmatrix}
P(\theta) \\
0 \\
0 & 1 \\
F_1^T & F_2^T
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\tilde{y} \\
\tilde{\theta} \\
\tilde{y_\theta}
\end{pmatrix},
$$

with $F_1 = F_{11} x + F_{12} y$, $F_2 = F_{21} x + F_{22} y$, being $F_{11}$, $F_{12}$, $F_{21}$, and $F_{22}$ 2π-periodic
matrices, functions of $P(\theta)$, $Q(\theta)$ and B (the exact expressions of these four periodic
matrices are not explicitly written here but can be found in [?]).

Then, the following results hold:

i) The matrix B can be chosen such that it is symplectic, $B_{22} = -B_{11}^T$, $B_{21} = B_{21}^T$ and $B_{12} = B_{12}^T$.

ii) The complete change of coordinates is symplectic.

iii) The initial Hamiltonian $H_2(z, \theta, y_\theta)$ is transformed, under the complete change of
variables, into

$$
\tilde{H}_2(\tilde{z}, \tilde{\theta}, \tilde{y_\theta}) = \omega_s \tilde{y_\theta} - \frac{1}{2} x^T B_{21} x - x^T B_{22} y + \frac{1}{2} y^T B_{12} y.
$$

The complete proof of Proposition 1 is available in [?]. The following paragraphs
just recall some key elements of this proof along with its numerical implementation.
First, the linear system associated with (8), $\dot{\mathbf{z}} = \mathbf{Q}(t)\mathbf{z}$, complies with the following conditions: $\mathbf{Q}(t)$ is $T$-periodic and Lebesgue integrable. Then, according to the Floquet theorem, every fundamental matrix solution $\mathbf{Z}(t)$ of this system has the form

$$\mathbf{Z}(t) = \mathbf{P}(t)e^{\mathbf{B}t},$$

with $\mathbf{P}(t)$ and $\mathbf{B}$ already defined in Proposition 1. Choosing the fundamental matrix $\mathbf{Z}(t)$ in such a way that $\mathbf{Z}(0) = \mathbf{I}$, the identity matrix, it comes $\mathbf{P}(0) = \mathbf{P}(T) = \mathbf{I}$ and $\mathbf{M} = \mathbf{Z}(T) = e^{\mathbf{B}T}$, where $\mathbf{M}$ is the monodromy matrix.

Taking the derivative of (10), it is possible to show that $\mathbf{P}$ can be computed from the differential equation

$$\dot{\mathbf{P}} = \mathbf{Q}\mathbf{P} - \mathbf{P}\mathbf{B},$$

while $\mathbf{B}$ is given by the logarithm of $\mathbf{M}$ divided by $T$. Note that the spectrum of $\mathbf{M}$ contains complex eigenvalues. As a consequence, its logarithm is not uniquely defined. This issue is addressed in the sequel, when $\mathbf{B}$ is explicitly computed.

These last results establish the first part of the proof of Proposition 1. From a practical point of view, they directly give the step-by-step recipe to numerically compute the desired change of variables. The case example of EML$_{1,2}$ is detailed hereafter, along with additional remarks on the numerical implementation.

(i) First, the variational equations $\dot{\mathbf{z}} = \mathbf{Q}(t)\mathbf{z}$ of the linearized system, are integrated during a time span equal to $T = \frac{2\pi}{\omega}$, starting with $\mathbf{P}(0) = \mathbf{I}_6$. The result of this integration is the monodromy matrix $\mathbf{M}$.

Once the monodromy matrix is obtained, $\mathbf{B}$ can be computed as the logarithm of $\mathbf{M}$ divided by $T$. However, in the real case, $\mathbf{B}$ is sought in its real form. So, even if many of the following computations are performed in complex arithmetic, the final result is real.

(ii) The eigenvalues of $\mathbf{M}$ are computed. As in the three-body case, the linear dynamics about the collinear points $L_{1,2}$ is a cross product of two centers — associated with two pairs of complex eigenvalues, $(\lambda_1, \lambda_1^{-1})$ and $(\lambda_3, \lambda_3^{-1})$, of modulus one— and one saddle — associated with a pair of real positive eigenvalues $(\lambda_2, \lambda_2^{-1})$, $\lambda_2 > 1$. Due to the large unstable eigenvalue $\lambda_2$, it is not possible to perform a direct and precise computation of the eigenvalues of $\mathbf{M}$. To tackle this difficulty, the monodromy matrix $\mathbf{M}$ is computed as a product of $N$ matrices: $\mathbf{M} = \mathbf{M}_1 \times \mathbf{M}_2 \times \ldots \times \mathbf{M}_N$, with $N = 20$. Then a power (resp. inverse power) method is used to compute the unstable (resp. stable) eigenvalue. Finally, Wielandt’s deflation is used to reduce the order of the matrix and get rid of the unstable eigenvector.

(iii) Once the diagonal form $\mathbf{D}_M$ of $\mathbf{M}$ is known, $\mathbf{B}$ can be computed as $\mathbf{B} = \mathbf{SD}_BS^{-1}$, with

$$\mathbf{D}_B = \frac{1}{T} \ln(\mathbf{D}_M), \quad \mathbf{D}_M = \begin{pmatrix} \mathbf{D}_m & 0 \\ 0 & \mathbf{D}_m^{-1} \end{pmatrix}, \quad \mathbf{D}_m = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$
Table 3: The coefficients $\omega_i$ that appear in the matrix $D_b$ in equation (12), in the Earth-Moon $L_{1,2}$ case. All values are given with the same precision as in Andreu [?].

<table>
<thead>
<tr>
<th>$\omega_i$</th>
<th>$L_1$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>-4.38968496e+01</td>
<td>1.34709423e-02</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>2.93720564e+00</td>
<td>2.16306748e+00</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>4.22768254e-01</td>
<td>-6.02217885e-02</td>
</tr>
</tbody>
</table>

Consequently, $D_B$ is given by:

$$D_B = \begin{pmatrix} D_b & 0 \\ 0 & -D_b \end{pmatrix} \quad \text{with} \quad D_b = \begin{pmatrix} i\omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & i\omega_3 \end{pmatrix}.$$  (12)

Since $\lambda_1$ and $\lambda_3$ are complex eigenvalues, the coefficients $\omega_1$ and $\omega_3$ are not uniquely defined: for a value $\omega_{1,3}$ any value of the form $\omega_{1,3} + k\omega_s$, $k \in \mathbb{Z}$ is suitable. Table 3 gives the numerical values of the coefficients $\omega_i$ used throughout this paper, in the Earth-Moon $L_{1,2}$ case. In particular, the EML$_2$ values are similar to the ones used in [?, ?].

(iv) Once the matrix $B$ is known, it is possible to compute the matrix $P(t)$ of the Floquet theorem by means of the differential equation (11). At the same time, the rest of the change of coordinates in (9) is also computed.

(v) A Fourier analysis is performed to obtain all the coefficients of the matrices $P$, $F_{11}$, $F_{12}$, $F_{21}$, $F_{22}$ as truncated Fourier series.

Finally, the following symplectic change of coordinates is obtained:

$$\begin{cases} \dot{z} &= P(\theta)\dot{z}, \\ \dot{\theta} &= \dot{\theta}, \\ \dot{y}_\theta &= \dot{y}_\theta + \tilde{x}^T F_{11}^T \tilde{x} + \tilde{y}^T F_{12}^T \tilde{x} + \tilde{x}^T F_{21}^T \tilde{y} + \tilde{y}^T F_{22}^T \tilde{y}, \end{cases}$$  (13)

which allows to transform the Hamiltonian $H_2$ into the real normal form:

$$\tilde{H}_2(\tilde{z}, \tilde{\theta}, \tilde{y}_\theta) = \omega_s \tilde{y}_\theta + \frac{1}{2} \omega_1 (\tilde{x}_1^2 + \tilde{y}_1^2) + \omega_2 \tilde{x}_2 \tilde{y}_2 + \frac{1}{2} \omega_3 (\tilde{x}_3^2 + \tilde{y}_3^2),$$

where $\tilde{z} = (\tilde{x}^T \tilde{y}^T)^T = (\tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3 \ \tilde{y}_1 \ \tilde{y}_2 \ \tilde{y}_3)^T$.

3.2.3. Complexification In order to get a diagonal form, a canonical change of variables is introduced on the first and third variables:

$$\tilde{x}_j = \frac{\tilde{x}_j + \tilde{y}_j i}{\sqrt{2}}, \quad \tilde{y}_j = \frac{\tilde{x}_j i + \tilde{y}_j}{\sqrt{2}}, \quad \text{for } j = 1, 3$$

This leads to the change:

$$\tilde{z} = C \tilde{z}$$  (14)
with:
\[
C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{11} \end{pmatrix}, \quad C_{11} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

This complexification finally gives the desired diagonal form for the order 2 of the Hamiltonian:
\[
\hat{H}_2(\hat{z}, \hat{\theta}, \hat{\gamma}_0) = \omega_s \hat{\gamma}_\theta + i \omega_1 \hat{x}_1 \hat{y}_1 + \omega_2 \hat{x}_2 \hat{y}_2 + i \omega_3 \hat{x}_3 \hat{y}_3.
\] (15)

3.2.4. Applying the complete change of coordinates In order to get a Hamiltonian of the form (15), the three canonical change of variables previously defined are composed: namely, the translation (7), the Floquet change of coordinates (13) and the complexification of the variables related to the central part (14), in that order. The final coordinates \( \hat{z} \) are denoted Translated-Floquet-Complexified (TFC). The real coordinates \( \hat{z} \) are called Translated-Floquet (TF).

The complete change of coordinates, denoted \( \text{coc} \), is of the form:
\[
\begin{align*}
\{ z &= P_c(\theta) \hat{z} + V(\theta), \\
\theta &= \hat{\theta}, \\
y_\theta &= \hat{\gamma}_\theta + p_0(\theta) + p_1(\hat{z}, \theta) + p_2(\hat{z}, \theta),
\end{align*}
\] (16)

with:
\[
P_c(\theta) = P(\theta) \text{C},
\]
and:
\[
p_0(\theta) = -\frac{1}{2} (g_2' g_1 - g_1' g_2 + g_4' g_3 - g_3' g_4),
\]
\[
p_1(\hat{z}, \theta) = -\frac{\partial G_1}{\partial \theta} = -U^T P(\theta) C \hat{z} \quad \text{and} \quad U = (-g_2' - g_4' 0 g_1' g_3' 0)^T,
\]
\[
p_2(\hat{z}, \theta) = \hat{x}^T \hat{F}_{11} \hat{x} + \hat{y}^T \hat{F}_{12} \hat{x} + \hat{x}^T \hat{F}_{21} \hat{y} + \hat{y}^T \hat{F}_{22} \hat{y},
\]
and:
\[
\begin{align*}
\hat{F}_{11}^T &= C_{11} F_{11}^T C_{11} + C_{12} F_{12}^T C_{11} + C_{11} F_{21}^T C_{12} + C_{12} F_{22}^T C_{12}, \\
\hat{F}_{12}^T &= C_{11} F_{11}^T C_{12} + C_{12} F_{12}^T C_{12} + C_{11} F_{21}^T C_{11} + C_{12} F_{22}^T C_{11}, \\
\hat{F}_{21}^T &= C_{12} F_{11}^T C_{11} + C_{11} F_{12}^T C_{11} + C_{12} F_{21}^T C_{12} + C_{11} F_{22}^T C_{12}, \\
\hat{F}_{22}^T &= C_{12} F_{11}^T C_{12} + C_{11} F_{12}^T C_{12} + C_{12} F_{21}^T C_{11} + C_{11} F_{22}^T C_{11},
\end{align*}
\]
and where \( \hat{z} = (\hat{x}^T \hat{y}^T)^T \) is the canonical form of the coordinates \( \hat{z} \).

Written in this form, the \( \text{coc} \) is symplectic [7]. In practice, only the first line of equation (16) is incorporated into the numerical implementation.
Invarinat manifolds of a non-autonomous problem via the parameterization method

3.3. Corresponding vector field.

The change of variables (16) provides a Hamiltonian \( \hat{H} \) in autonomous diagonal form at order 2. The corresponding vector field \( \mathcal{F} \) is required, since the PM is based on manipulations on the latter. From equation (15), getting rid of the additional couple \((\theta, \tilde{y}_\theta)\), the vector field can be derived and plugged into the parameterization method:

\[
\dot{\hat{z}} = \mathcal{F}(\hat{z}, \theta) = J \nabla \hat{H}(\hat{z}, \theta) = \left( \begin{array}{cc} \Omega & 0 \\ 0 & -\Omega \end{array} \right) \left( \begin{array}{c} \dot{\hat{x}} \\ \dot{\hat{y}} \end{array} \right) + \sum_{k=3}^{\infty} J \nabla \hat{H}_k(\hat{z}, \theta),
\]

(17)

where \( J \) be the \( 6 \times 6 \) symplectic matrix defined as:

\[
J = \left( \begin{array}{cc} 0 & I_3 \\ -I_3 & 0 \end{array} \right),
\]

with:

\[
\Omega = \left( \begin{array}{ccc} i \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & i \omega_3 \end{array} \right),
\]

and the \( \omega_i \)'s are given in Table 3 for the EML\(_{1,2} \) case.

The linearized part of (17) has the desired autonomous diagonal form for the initialization of the parameterization method. However, at each order \( k \geq 3 \), the term \( \hat{H}_k(\hat{z}, \theta) \) is not explicitly given in (17). In order to compute it, the vector field \( \mathcal{F} \) in nc coordinates is used. The latter is obtained from its Hamiltonian (3):

\[
\dot{z} = \mathcal{F}(z, \theta) = J \nabla H(z, \theta) = \left( \begin{array}{cc} Q_2^T & 2Q_3 \\ -2Q_1 & -Q_2 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \sum_{k=3}^{\infty} \frac{\partial H_k}{\partial x} (x, \theta) \right),
\]

(18)

with \( z = (x^T \ y^T)^T \). Then, the time derivative of the first line of (16) is computed:

\[
\dot{z} = \dot{P}_c(\theta) \dot{z} + P_c(\theta) \ddot{z} + \dot{V}(\theta) = \dot{P}_c(\theta) \dot{z} + P_c(\theta) \mathcal{F}(\dot{z}, \theta) + \dot{V}(\theta).
\]

Inverting this equation, the vector field in TFC coordinates is obtained from the one in nc through the following equality:

\[
\dot{\mathcal{F}}(\mathcal{z}, \theta) = \dot{z} = P_c(\theta)^{-1} \left( \mathcal{F}(P_c(\theta) \dot{z} + V(\theta), \theta) - \dot{P}_c(\theta) \dot{z} - \dot{V}(\theta) \right),
\]

(19)

providing that the matrix \( P_c(\theta) \) is invertible for all values of \( \theta \), which is the case in practice.

The idea is then to find the order \( k \) of (17) by injecting the change of coordinates (16) into (18), and injecting again the result into (19). To do so, the matrix \( P_c \) is divided into four \( 3 \times 3 \) submatrices:

\[
P_c = \left( \begin{array}{cc} P_{c11} & P_{c12} \\ P_{c21} & P_{c22} \end{array} \right).
\]

Then, injecting the COC (16) in (18), we have:

\[
\mathcal{F}(P_c \dot{z} + V, \theta) = \left( \begin{array}{cc} Q_2^T & 2Q_3 \\ -2Q_1 & -Q_2 \end{array} \right) \left( \begin{array}{c} P_{c11} \dot{x} + P_{c12} \dot{y} + V_x \\ P_{c21} \dot{x} + P_{c22} \dot{y} + V_y \end{array} \right) - \left( \sum_{k=3}^{\infty} J_k(\dot{x}, \dot{y}, V, \theta) \right),
\]

(20)
Invariant manifolds of a non-autonomous problem via the parameterization method

with

$$\hat{J}_k(\hat{x}, \hat{y}, V, \theta) = \frac{\partial H_k}{\partial \hat{x}}^T \left( P_{c_{11}} \hat{x} + P_{c_{12}} \hat{y} + V_x, P_{c_{12}} \hat{x} + P_{c_{22}} \hat{y} + V_y, \theta \right).$$

Finally, the vector field $\hat{\mathcal{F}}(\hat{z}, \theta)$ is obtained by injecting (20) into (19). Isolating the term of order $k$ yields the term $J \nabla \hat{H}_k(\hat{z}, \theta)$.

3.4. Precision of the complete change of coordinates

At this point, one can see that the overall accuracy of the change of coordinates directly depends on the precision with which the periodic coefficients in $P(\theta)$ and $V(\theta)$ are integrated along the periodic orbit, on a full period. This precision is limited by the magnitude of the hyperbolic unstable direction associated to this orbit which naturally perturbs any integration along its trajectory. Recall that $\lambda_2$ is the eigenvalue associated with the unstable direction. Then the initial error made along the unstable direction, denoted here $\epsilon(0)$, leads, after a time $t$, to an error:

$$\epsilon(t) \sim \epsilon(0) e^{T \ln(\lambda_2)} = \epsilon(0) (\lambda_2)^{\frac{t}{T}}.$$

This error is doomed to grow fast: in the case of EML2, the initial error is multiplied by a factor roughly equal to 1500 after half a period, $2 \times 10^6$ after a full period. The EML4 case is even worse: the error takes a factor about 21000 (resp. $5 \times 10^8$) after half a period (resp. a full period). On the contrary, in the SE case, the unstable eigenvalues are quite small: the error is roughly multiplied by 2 after half a period and by 4 after a full period, both in the $L_1$ and $L_2$ cases. This hyperbolic behavior is of paramount importance for the overall the accuracy of the COC and eventually for the parameterization of the invariant manifolds about the libration points (see section 5).

Conclusion on section 3. This section has shown that, in the QBCP, the linearized motion around $L_{1,2}$ is of type center $\times$ saddle $\times$ center, as in the CRTBP case (see [3]). In particular, the product of the two centers is a normally hyperbolic invariant manifold (NHIM), associated to a stable and an unstable manifolds. In a small neighborhood of the libration points, these results can be extended to the nonlinear dynamics since NHIMs persist under perturbations [3]. One can also see the invariant manifolds as quasi-periodically perturbed versions of the nonlinear invariant manifolds around the libration points of the CRTBP. The persistence of the center manifolds about $L_{1,2}$ and their stable and unstable manifolds has already been addressed in both the BCP [7] and the QBCP [3]. These results hold in the present context. The rest of this paper is dedicated to the computation of these objects with a recent approach, the parameterization method.

4. The parameterization method in the QBCP

The parameterization method (PM) has been used to compute high order power series expansions of parameterizations of invariant manifolds of vector fields at fixed points [7].
This section extends such approach to invariant manifolds of periodically-perturbed vector fields about a periodic orbit with the same frequency.

Note that the parameterization method has already been used in a more general quasi-periodic context for the computation of invariant tori [?, ?]. Although these previous efforts were focused on quasi-periodic maps, the authors indicate that the adaptation to NHIMs of quasi-periodic flows is possible, and the present work can be seen as a purely periodic example of this statement. Moreover, the case of the one-dimensional stable directions about periodic orbits has been also addressed in [?], in the case of autonomous vector fields, and from the theoretical standpoint. In particular, no practical assumption is made on the time-dependency of the expansions. Subsections 4.1 and 4.3 heavily rely on the in-depth work on the CRTBP autonomous case in [?, ?]. The reader is kindly referred to either of these references for details.

4.1. Formulation in the 2π-periodic case

Let \( z_p(\theta) : T \to \mathbb{R}^n \) be a 2π-periodic reference orbit of a 2π-periodic \( n \)-dimensional vector field:
\[
\dot{z} = \mathcal{F}(z, \theta),
\]
where the dot denotes derivation with respect to time. Let \( z = C_o(\hat{z}, \theta) \) be a linear change of coordinates that defines a new vector field \( \dot{\hat{z}} = \hat{\mathcal{F}}(\hat{z}, \theta) \) which satisfies the following conditions:
\begin{itemize}
    \item \( C_o(0, \theta) = z_p(\theta) \),
    \item \( \hat{\mathcal{F}}(0, \theta) = 0, \forall \theta \),
    \item \( D_z\hat{\mathcal{F}}(0, \theta) \) is diagonal and does not depend on time.
\end{itemize}

In the case of the QBCP dynamics about the libration points EML\(_{1,2}\), such change of coordinates is given by equation (16), with \( n = 6 \).

Let \( L \in \mathbb{C}^{n \times d} \) be a matrix formed by a subset of \( d \leq n \) column vectors of \( D_z\hat{\mathcal{F}}(0, \theta) \), and let \( N \in \mathbb{C}^{n \times (n-d)} \) be the matrix formed by the subset of column vectors of \( D_z\hat{\mathcal{F}}(0, \theta) \) complement of \( L \). Therefore, the matrix \( H := (L \ N) \) is a permutation of the diagonal matrix \( D_z\hat{\mathcal{F}}(0, \theta) \) and is invertible. The matrix \( L \) spans a \( d \)-dimensional invariant manifold \( V^L \subset \mathbb{C}^n \) for the linearization \( \dot{\hat{z}} = D_z\hat{\mathcal{F}}(0, \theta)\hat{z} \) around the translated origin. The goal is to compute a high order approximation of the invariant manifold \( W \) for the nonlinear dynamics, tangent to \( V^L \) at the origin.

4.2. Form of the parameterization

The idea of the parameterization method is to approximate the \((d + 1)\)-dimensional invariant manifold \( W \) by a map \( \tilde{W} : (s, \theta) \mapsto \tilde{z} = \tilde{W}(s, \theta) \), where \( s \in \mathbb{C}^d \) are the reduced coordinates and \( \theta \in T \) parameterizes the time-dependency. Moreover, the motion on
the manifold is described by a vector field $f : \mathbb{C}^d \times T \to \mathbb{C}^d$ such that $\dot{s} = f(s, \theta)$. Then, applying the equations of motion on $\hat{W}$, it comes:

$$\mathcal{F}(\hat{W}(s, \theta), \theta) = \frac{d\hat{W}(s, \theta)}{dt} = D_s\hat{W}(s, \theta)\dot{s} + \frac{\partial \hat{W}}{\partial t}(s, \theta).$$

Injecting the reduced vector field $f$ in the former equation allows to derive the following invariance equation:

$$\mathcal{F}(\hat{W}(s, \theta), \theta) = D_s\hat{W}(s, \theta)f(s, \theta) + \frac{\partial \hat{W}}{\partial t}(s, \theta),$$

Equation (22) expresses that at points in the range of $\hat{W}$, the vector field $\hat{F}$ is tangent to the range of $\hat{W}$. The latter is therefore invariant under the flow of $\hat{F}$. The presence of the term $\frac{\partial \hat{W}}{\partial t}$ is due to the explicit time-dependency of the dynamics. The map $\hat{W}(s, \theta)$ is denoted the parameterization of $W$. Moreover, $f(s, \theta)$ is denoted the reduced vector field. Both $\hat{W}$ and $f$ are unknowns and are computed so as to satisfy the invariance equation.

From a practical standpoint, $\hat{W}$ and $f$ are sought in the form of a Fourier-Taylor ($ft$) expansion, i.e. a power series in the variable $s$ whose coefficients are $T$-periodic Fourier series. See e.g. [?] for a discussion of the advantages of the $ft$ formalism in an equivalent context. In the case of $\hat{W}$, the $ft$ series take the form:

$$\hat{W}(s, \theta) = \sum_{k \geq 1} \hat{W}_k(s, \theta),$$

with:

$$\hat{W}_k(s, \theta) = \left( \hat{W}_k^1(s, \theta) \quad \cdots \quad \hat{W}_k^p(s, \theta) \quad \cdots \quad \hat{W}_k^n(s, \theta) \right)^T,$$

$$\hat{W}_k^p(s, \theta) = \sum_{r \in R^k} w_p^r(\theta)s^r, \ \forall p \in [1, n],$$

and where $R^k = \{ r \in \mathbb{N}^d, |r| = r_1 + \ldots + r_d = k \}$, $s^r = s_1^{r_1} \ldots s_d^{r_d}$, and the coefficients $w_p^r(\theta)$ are trigonometric functions of the form:

$$w_p^r(\theta) = \sum_j w_{r,j}^p e^{ij\theta}.$$

The reduced vector field $f$ is sought in a similar form.

At this point, the first order parameterization $(\hat{W}_1, f_1)$ can be obtained by solving the invariance equation (22) at order one. To this end, let $\Lambda \in \mathbb{C}^{n \times n}$ be the matrix defined as follows:

$$\Lambda := H^{-1}D_x\mathcal{F}(0, \theta)H = \begin{pmatrix} \Lambda_L & O \\ O & \Lambda_N \end{pmatrix},$$

with $\Lambda_L \in \mathbb{C}^{d \times d}$ and $\Lambda_N \in \mathbb{C}^{(n-d) \times (n-d)}$ both diagonal under the current hypotheses. Since $\hat{W}$ is tangent to $V^L$ at the origin, it directly imposes $\hat{W}_1(s, \theta) = Ls$. Then,
searching \( f_1 \) in the form \( f_1(s, \theta) = As \), \( A \in \mathbb{C}^{d \times d} \), the order one of the invariance equation is reduced to:

\[
D_z \mathcal{F}(0, \theta)L = LA.
\]

Given the definition of \( \Lambda \), we simply have \( D_z \hat{\mathcal{F}}(0, \theta)L = L\Lambda L \) and \( f_1(s, \theta) = \Lambda LS \) satisfies the order one of (22). The couple \((\hat{W}, f)\) is then, at order one:

\[
\begin{aligned}
\hat{W}_1(s, \theta) &= Ls, \\
f_1(s, \theta) &= \Lambda LS
\end{aligned}
\]  

(24)

From there, the standard procedure of the parameterization is to formally solve the invariance equation (22), starting from (24), by substituting the expansions of \( \hat{W} \) and \( f \) in (22), and find homogeneous terms in increasing order.

At this point, the reader can already notice that all the following computations will be performed on \((d < n)\)-dimensional spaces. This is a major difference with respect to Hamiltonian-based algorithms, where the full \( n \)-dimensional space is used.

4.3. The homological equations

Isolating the \( k \)-order terms in the invariance equation (22), \( k > 1 \), allows to get the \( k \)-order homological equation for \( \hat{W}_k(s, \theta) \) and \( f_k(s, \theta) \):

\[
D_z \mathcal{F}(0, \theta)\hat{W}_k - D_s \hat{W}_k \Lambda LS - Lf_k - \frac{\partial \hat{W}_k}{\partial t} = \left[ D_s \hat{W}_{<k} f_{<k} \right]_k - \left[ \mathcal{F} \left( \hat{W}_{<k} \right) \right]_k,
\]

(25)

where the dependency in \((s, \theta)\) has been skipped for the sake of clarity. The terms \( \hat{W}_{<k} \), \( f_{<k} \), and \( \left[ \mathcal{F} \left( \hat{W}_{<k} \right) \right]_{<k} \) are assumed to have been obtained in previous steps. The goal is to compute \( \hat{W}_k \) and \( f_k \) in order to satisfy (25).

To this end, the right-hand side of (25) is computed. Such computation requires operations over the Fourier-Taylor space, such as products of FT series as well as composition of FT series with algebraic functions. All these operations have been implemented as part of a complete Fourier-Taylor algebra and written in C++ by the authors. This algebra is based on automatic differentiation [?] which allows to reduce both the time and the memory required for the computation of the operations on the Fourier-Taylor objects. Moreover, note that the computation of the order \( k \) of the vector field \( \left[ \mathcal{F} \left( \hat{W}_{<k} \right) \right]_{<k} \) requires the use of the algebraic manipulations detailed in subsection 3.3.

Then, following the example of [?], the normal part of (25) is separated from its tangent part by multiplying by \( H^{-1} \). The following functions are introduced:

\[
\begin{aligned}
\upsilon_k &= H^{-1} \hat{W}_k, \\
\eta_k &= H^{-1} \left( \left[ D_s \hat{W}_{<k} f_{<k} \right]_k - \left[ \mathcal{F} \left( \hat{W}_{<k} \right) \right]_k \right)
\end{aligned}
\]

The homological equations then take the form:

\[
\Lambda \upsilon_k(s, \theta) - D_s \upsilon_k(s, \theta) \Lambda LS - \begin{pmatrix}
I_d \\
0
\end{pmatrix} f_k(s, \theta) - \frac{\partial \upsilon_k}{\partial t}(s, \theta) = \eta_k(s, \theta).
\]

(26)
Invariant manifolds of a non-autonomous problem via the parameterization method

Under the current assumption, the matrices $\Lambda_L$ and $\Lambda_N$ are diagonal: $\Lambda_L = \text{diag}(\lambda_1, \ldots, \lambda_d)$ and $\Lambda_N = \text{diag}(\lambda_{d+1}, \ldots, \lambda_n)$. This simple form allows to solve the homological equations separately on the subspaces tangent and normal to $V^L$, as it is detailed hereafter.

4.3.1. Normal homological equations. Defining $\nu^N_k = (0 \ I_{n-d}) \nu_k$, equation (26) is projected on the normal space:

$$\Lambda \nu^N_k(s, \theta) - D_s \nu^N_k(s, \theta) \Lambda L - \frac{\partial \nu^N_k(s, \theta)}{\partial t} = \eta^N_k(s, \theta).$$

Thus, for all $p = d+1, \ldots, n$, it comes:

$$\lambda_p \nu^p_k(s, \theta) - D_s \nu^p_k(s, \theta) \Lambda L - \frac{\partial \nu^p_k(s, \theta)}{\partial t} = \eta^p_k(s, \theta).$$

Thanks to the diagonal forms of $\Lambda$ and $\Lambda_L$, these equations are also diagonal in the coefficients $\nu^p$ of the homogeneous polynomials $\nu^p_k$. In particular, for $p = d+1, \ldots, n$, $|r| = k$:

$$\lambda_p \nu^p_r(\theta) - \lambda_L r \nu^p_r(\theta) - \dot{\nu}^p_r(\theta) = \eta^p_r(\theta),$$

where $\lambda_L r = \lambda_1 r_1 + \ldots + \lambda_d r_d$. Noticing that $\dot{\nu}^p_r(\theta) = \omega_s \frac{\partial \nu^p_r}{\partial \theta}$, the function $\nu^p_r(\theta)$ is governed by a linear differential equation:

$$\omega_s \frac{\partial \nu^p_r}{\partial \theta} - (\lambda_p - \lambda_L r) \nu^p_r(\theta) = -\eta^p_r(\theta).$$

Let $J_{r,p} = \{ j \in \mathbb{Z}, j \omega_s i - (\lambda_p - \lambda_L r) = 0 \}$ be the set of cross resonances associated to $(r, p)$. Providing that $J_{r,p} = \emptyset$, a solution of the linear differential equation is:

$$\nu^p_r(\theta) = \sum_{j \in \mathbb{Z}} \frac{-\eta^p_{r,j}}{j \omega_s i - (\lambda_p - \lambda_L r)} e^{ij \theta}.$$

The pairs $(r, p) \in \mathbb{N}^d \times \{d+1, \ldots, n\}$ such that $J_{r,p} \neq \emptyset$ are an obstruction to the resolution of the normal homological equations. Fortunately, the most usual examples of invariant manifolds — stable, unstable, center, center-stable, center-unstable — are free of cross resonances, as in the autonomous case [?]. For example, if $W$ is the center manifold, the following conditions hold:

$$\begin{cases}
\text{Re}(\lambda_p) \neq 0, & \forall p \in \{d+1, \ldots, n\} \\
\text{Re}(\lambda_L r) = 0, & \forall r \in \mathbb{N}^d
\end{cases}$$

So that, finally, it simply comes $\text{Re}(j \omega_s i - (\lambda_p - \lambda_L r)) = \text{Re}(\lambda_p) \neq 0$. Similar arguments can be made in the other common cases.

One can notice that the Fourier-Taylor form is required to reduce the problem down to simple linear differential equations. This justifies a posteriori the choice of the FT formulation over a pure Taylor scheme where the $\theta$ would be treated as another state variable.
4.3.2. Tangent homological equations. Defining \( \mathbf{v}_k^L = (I_d \ 0) \mathbf{v}_k \), equation (26) is projected on the tangent space:

\[
\Lambda \mathbf{v}_k^L(s, \theta) - D_k \mathbf{v}_k^L(s, \theta) \Lambda_L s - \mathbf{f}_k(s, \theta) - \frac{\partial \mathbf{v}_k^L}{\partial t}(s, \theta) = \eta_k^L(s, \theta).
\]

Thus, for all \( p = 1, \ldots, d \), we have:

\[
\lambda_p \mathbf{v}_p^r(s, \theta) - D_k \mathbf{v}_k^p(s, \theta) \Lambda_L s - f_k^p(s, \theta) - \frac{\partial \mathbf{v}_p^r}{\partial t}(s, \theta) = \eta_p^r(s, \theta).
\]

Once again, these equations are diagonal in the coefficients \( \mathbf{v}_p^r \) of the homogeneous polynomials \( \mathbf{v}_k^p \). In particular, for \( p = 1, \ldots, d, |\mathbf{r}| = k \):

\[
\lambda_p \mathbf{v}_p^r(\theta) - \lambda_L \mathbf{r} \mathbf{v}_p^r(\theta) - g_k^p(\theta) - \dot{\mathbf{v}}_p^r(\theta) = \eta_p^r(\theta).
\]

Such equality can be written in the form of a linear differential equation for the function \( \mathbf{v}_p^r(\theta) \):

\[
\omega_s \frac{\partial \mathbf{v}_p^r}{\partial \theta} - (\lambda_p - \lambda_L \mathbf{r}) \mathbf{v}_p^r(\theta) = -\eta_p^r(\theta) - f_k^p(\theta).
\] (27)

Let \( \mathcal{J}_{r,p} = \{ j \in \mathbb{Z}, j \omega_s - (\lambda_p - \lambda_L \mathbf{r}) = 0 \} \) be the set of internal resonances associated to \( (r, p) \). Providing that \( \mathcal{J}_{r,p} = \emptyset \), a solution of this equation is:

\[
\mathbf{v}_p^r(\theta) = \sum_{j \in \mathbb{Z}} \frac{-\eta_p^{r,j} - f_k^{p,j}}{j \omega_s - (\lambda_p - \lambda_L \mathbf{r})} e^{i j \theta}.
\]

As in the normal case, the pairs \( (r, p) \in \mathbb{N}^d \times \{1, \ldots, d\} \) such that \( \mathcal{J}_{r,p} \neq \emptyset \) are an obstruction to the resolution of the tangent homological equations. But, contrary to the set of cross resonances, the set of internal resonances is not empty. Fortunately, equation (27) can be solved even in the presence of resonances by adjusting the coefficients \( f_k^{r,j} \). Several strategies available in the autonomous case, denoted styles [?], are still relevant in the current context. Both styles used in this paper are detailed below: the graph style and the normal form style.

**The graph style:** It consists in simplifying the parameterization of the manifold \( \mathbf{v} \), by taking \( \mathbf{v}_k^L(s, \theta) = 0 \) and \( \mathbf{f}_k(s, \theta) = -\eta_k^L(s, \theta) \) at each step. That is, for \( p = 1, \ldots, d \) and \( |\mathbf{r}| = k \):

\[
f_k^p = -\eta_p^L, \quad \mathbf{v}_p^r = 0.
\]

Such a parameterization is suitable for all manifolds and is particularly adequate for center manifolds for which there exists an infinite number of internal resonances. However, it leads to a full time-dependent Fourier-Taylor form for the reduced vector field \( \mathbf{f} \).
The normal form style: This style consists in simplifying the equations of the dynamics on the manifold, finding a normal form for $f$. That is, for example, for $p = 1, \ldots, d$, $|r| = k$, and $j \in \mathbb{Z}$:

$$f^p_{r,j} = 0, \quad v^p_{r,j} = -\eta^p_{r,j} \frac{j\omega s i - (\lambda_p - \lambda_L r)}{j\omega s i - (\lambda_p - \lambda_L r) \neq 0}, \quad (28a)$$

$$f^p_{r,j} = -\eta^p_{r,j}, \quad v^p_{r,j} = 0, \quad \text{if } j\omega s i - (\lambda_p - \lambda_L r) = 0. \quad (28b)$$

From a numerical standpoint, the case $(28b)$ may be extended to near zero values of the divisor in order to limit the divergence of the coefficients $v^p_{r,j}$. In practice, a threshold $\epsilon$ can be implemented in order to select $(28a)$ only when the norm of the corresponding divisor is greater than $\epsilon$. With such method, the normal form style is equivalent to the Hamiltonian normal form approach developed in [?, ?], within which a simple autonomous form for the reduced Hamiltonian was sought, with $\epsilon = 0.05$.

In that perspective, the normal form theory can be viewed as a subclass of parameterization method, at least in its usual implementation about libration points, for which simple reduced dynamics are usually sought. With its unified styles formalism, the parameterization method brings flexibility to the problem of semi-analytical approximation of resonant invariant manifolds. In particular, the graph style avoids to use an arbitrary threshold $\epsilon$. The price to pay is the non-autonomous nature of the reduced vector field. Note however that equivalent styles can be formalized for Hamiltonian-based algorithms.

5. Application to the EML$_{1,2}$ center manifold

In this section, the parameterization method is applied to the computation of the center manifold of the QBCP EML$_{1,2}$ points. If not stated otherwise:

- The graph style is used throughout the section.
- The order of the parameterization method, denoted $N$, corresponds to the order of the 4-variable Fourier-Taylor expansions in the coordinates $s$ that describe the center manifold. The parameterization of the center manifolds has been performed up to order $N = 30$.
- Each coefficient of such expansions are Fourier series of order $J$. For all subsequent applications, $J$ is arbitrarily taken equal to 30 in the non-autonomous case, which is big enough to comprise all coefficients whose relative contribution is more than $10^{-16}$, the truncation error of the computer double precision arithmetic.

With $N = 30$ and $J = 30$, each Fourier-Taylor expansion contains 2,828,936 monomials. The computation of the center manifold up to these orders takes about 24 minutes on a 2.3 GHz AMD Opteron Processor.
5.1. Initialization of the process

First, the order one of the parameterization is expressed. It comes from equation (17) that:

\[ D_z \hat{\mathcal{F}}(0, \theta) = \text{diag}(i\omega_1, \omega_2, i\omega_3, -i\omega_1, -\omega_2, -i\omega_3). \]

To approximate the center manifold, the matrix \( H \) should then take the form:

\[ H = \begin{pmatrix}
i\omega_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega_2 & 0 \\
0 & i\omega_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -i\omega_1 & 0 & 0 & -\omega_2 \\
0 & 0 & 0 & 0 & 0 & -i\omega_3 \\
0 & 0 & 0 & -i\omega_3 & 0 & 0 \\
\end{pmatrix}. \]

It directly comes:

\[ \Lambda = H^{-1} D_z \hat{\mathcal{F}}(0, \theta) H = \begin{pmatrix}
i\omega_1 & 0 & 0 & 0 & 0 & 0 \\
0 & i\omega_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -i\omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i\omega_3 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -\omega_2 \\
\end{pmatrix}. \]

Finally, the expansions are initialized by:

\[ \hat{\mathbf{W}}_1(s, \theta) = Ls = \begin{pmatrix}
i\omega_1 s_1 \\
0 \\
i\omega_3 s_2 \\
-i\omega_1 s_3 \\
0 \\
i\omega_3 s_4 \\
\end{pmatrix}, \quad f_1(s, \theta) = \Lambda_L s = \begin{pmatrix}
i\omega_1 s_1 \\
i\omega_3 s_2 \\
-i\omega_1 s_3 \\
-i\omega_3 s_4 \\
\end{pmatrix}. \quad (29) \]

5.2. Realification of the center manifold

Initialized with (29), the parameterization (23) of the center manifold \( \hat{\mathbf{W}}(s, \theta) \) is obtained at a given order \( n \) in the TFC coordinates, using the procedure described in the previous section. Its counterpart \( \mathbf{W}(s, \theta) \) in NC coordinates is obtained by applying the change of coordinates (16):

\[ \mathbf{W}(s, \theta) = P_c(\theta) \hat{\mathbf{W}}(s, \theta) + \mathbf{V}(\theta). \]

Given the complexification performed in section 3.2.4, the vector \( s \) is in \( \mathbb{C}^4 \), which does not guarantee that \( \mathbf{W}(s, \theta) \) is real. It is then necessary to perform a realification,
Invariant manifolds of a non-autonomous problem via the parameterization method

i.e. a new change of coordinates that will ensure that the final result, in physical coordinates, is real. This realification is given by the following change of variables:

\[ \tilde{s} = \mathcal{C} s, \]  

with

\[ \mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}, \]

and where \( s = (s_1, s_2, s_3, s_4)^T \) are the complex reduced coordinates of the center manifold (CCM), and \( \tilde{s} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4)^T \) are the real ones (RCM).

Injecting (30) in the parameterization, the following real function is obtained:

\[ W : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^6 \]

\[ (\tilde{s}, \theta) \mapsto W(\tilde{s}, \theta). \]

Notations. For the sake of simplicity, and since there is no further use of the complex parameterization (CCM coordinates), the tilde notation is skipped when referring to the RCM coordinates for the rest of this paper.

5.3. Comments on the energy

In the QBCP, the characteristic ranges of energy are very different from the energies of the CRTBP. In the CRTBP, EML\(_1\) has an energy close to \( H_{L_1} \approx -1.594 \), whereas, in the case of EML\(_2\), \( H_{L_2} \approx -1.586 \), in EM units. Moreover, the energy of the family of orbits around these points are relatively close to these values. As an example, the family of planar Lyapunov orbits around EML\(_1\) loses its stability and give rise to the halo families for an energy between \(-1.590\) and \(-1.580\).

In the QBCP, the huge potential well of the Sun leads to a severe drop of the energy. As an example, the energy of the libration orbit around EML\(_1,2\) at time \( t \) is always close to \( H_{L_i}(t) \approx -847.5 \). To address this issue, a relative estimate of the energy is used throughout this paper. The energy of any object is given with respect to the initial energy \( H_{L_i}(0) \) of its associated libration periodic orbit: it is measured by the variable \( \delta H_t \) so that the true energy \( H(t) \) satisfies \( H(t) = H_{L_i}(0) + \delta H_t \).

5.4. Accuracy of the center manifold

5.4.1. Orbital error. In the context of semi-analytical expansions around libration points, a usual precision test is the computation of the orbital error \([?], [?]\), defined hereafter. For given reduced initial conditions \( s_0 \in \mathbb{R}^4 \), let \( z(t) \) and \( s(t) \) be the solutions of the Cauchy problems \( \dot{z} = \mathcal{F}(z, t), \; z(t_0) = W(s_0) \) and \( \dot{s} = \mathcal{F}(s), \; s(t_0) = s_0 \), respectively. Then, the orbital error is defined as:

\[ e_O(t, s_0, t_0) = \|W(s(t)) - z(t)\|_2. \]
Moreover, let $E_O$ be its counterpart computed in the usual Earth-Moon coordinates:

$$E_O(t, s_0, t_0) = \|C(W(s(t)) - C(z(t))\|_2.$$ 

The orbital error has to be measured on small time spans, since the hyperbolic directions produce exponentially growing errors (see for example subsection 3.4).

As a first approach, the orbital error is computed on a simple one-dimensional subspace defined as follows: the initial reduced vector $s_0$ satisfies $s_1 = s_2 = s_3 = s_4$, and the initial time is taken equal to $t_0 = 0$. Moreover, the orbital error $E_O$ is estimated after a time $t = 1 \approx 0.147T$, as it has been done in [?]. The results are displayed on Figure 4, as a function of the initial energy. The following remarks can be made:

- On both plots, at high energy, there is a turning point beyond which the precision does not improve when increasing the order. This behavior is always expected since the expansions involved in the parameterization are not convergent in any open set.

- At small relative energy, the precision of the EML1 case appears to be shifted up by roughly two orders of magnitude when compared to its EML2 equivalent. One has to recall that EML1 is linearly more unstable than EML2: for the considered time span, any initial error on the initial conditions about EML1 (resp. EML2) is amplified by a factor about 20 (resp. 10). However, such difference is not big enough to explain the discrepancy displayed on Figure 4.

Nevertheless, it has been seen in section 3 that the accuracy of the change of coordinates (16) (COC) is limited because it involves the integration of the unstable motion on a full period of the system. The greater instability of EML1 then leads to a greater error on the COC: quantitatively, the difference between the two libration points is of two orders of magnitude. This difference is expected to be found again in the parameterization because the COC is involved in all the operations on the vector field via equation (19).

In practice, the limit of the domain of practical convergence of the semi-analytical approximations is often defined by a given threshold on the orbital error. For example, to ensure an initial error equivalent to 10 meters, the final orbital error $E_O^1$ should be around $5 \times 10^{-7}$ (resp. $2.5 \times 10^{-7}$) for the EML1 (resp. EML2) case. For the results at order 26 displayed on Figure 4, this limit is reached for $\delta H_0 \approx 0.018$ in the EML1 case, which corresponds to an initial distance from the libration point of about 25000 km. Equivalently, this limit is attained for $\delta H_0 \approx 0.026$ in the EML2 case which corresponds to an initial distance of about 16000 km.

The same estimations can be made for an initial error equivalent to 1 km. In this case, the final orbital error $E_O^1$ should be around $5 \times 10^{-5}$ (resp. $2.5 \times 10^{-5}$) for the EML1 (resp. EML2) case. For the results displayed on Figure 4, this limit is reached for $\delta H_0 \approx 0.035$ in the EML1 case, which corresponds to an initial distance from the libration point of about 36000 km. Equivalently, this limit appears at $\delta H_0 \approx 0.034$ in the EML2 case, which corresponds to an initial distance of about 19000 km.

Note that all these quantitative results are limited to the straight line defined by $s_1 = s_2 = s_3 = s_4$. Qualitatively however, all the previous tendencies remain true when
Invariant manifolds of a non-autonomous problem via the parameterization method

Figure 4: The orbital error \( E_O \) after \( t = 1 \) as a function of the initial energy \( \delta H_0 \), for initial conditions that satisfies \( s_1 = s_2 = s_3 = s_4, t_0 = 0 \), and for various orders of the parameterization of the center manifold. The initial energy is computed as the gap with respect to the energy of the associated libration point.

5.4.2. Invariance error on average. Beyond particular subspaces, a systematic quantitative estimation of the accuracy of the parameterizations would require to compute the orbital error on the complete (4+1)-dimensional phase space about each libration point. Fortunately, the parameterization method provides a faster, simpler, yet powerful estimate of the accuracy of its numerical implementation: namely, the precision with which the invariance error (22) is satisfied \( \mathcal{E} \). Let \( e_I \) be the invariance error, i.e. the error in the invariance equation along the solutions:

\[
e_I(t, s_0) = \left\| \mathcal{F}(s(t), t) - D_s W(s(t), t) f(s(t), t) - \frac{\partial W}{\partial t}(s, t) \right\|_2.
\] (31)

At starting time \( t_0 \), the state is initialized on the current approximation of the center manifold and the derivative of the orbital error is equivalent to the invariance error. In this sense, the invariance error gives an estimate of the initial behavior of the orbital error while avoiding the integration of the numerical error along the hyperbolic directions. Moreover, when estimated only at \( t_0 \), the invariance error is quite fast to compute, which allows to use it on spaces of higher dimensions than the orbital error. Finally, the estimation of \( e_I \) requires no additional computation over the Fourier-Taylor algebra since all terms that are involved in (31) are obtained as bypass products of the parameterization method.

Note that \( e_I \) provides an estimate in normalized-centered coordinates. As in the case of the orbital error, its counterpart \( E_I \) in Earth-Moon normalized coordinates is...
used. To estimate the invariance error on the whole phase space, the following method is implemented. First, a total of 20000 random initial conditions \( s_0 \) is taken in the neighborhood of \( \text{EML}_{1,2} \). For each of these seeds, the invariance error \( E_I \) is computed at various initial times \( t_0 \). Then, for a given invariance error bound \( E^b_I \), the mean energy \( \bar{\delta H}_0(E^b_I) \) over the set \( \{ s_0, E_I(t, s_0) = E^b_I \} \) is computed. This process is repeated for various orders \( N \) of the parameterization, producing isoprecision curves in the \( (\delta H_0, N) \)-plane. The corresponding results are given on Figure 5 for \( t_0 = 0.0 \). Using different starting times \( t_0 \) yields very similar results.

![Figure 5: Isoprecision curves in the \((\delta H_0, N)\)-plane, at \( t_0 = 0.0 \). \( N \) is the order of the parameterization, \( \delta H_0 \) is the mean energy at each level of error \( E_I \).](image)

For a given threshold on the invariance error, one can get an idea of the size of the domain of practical convergence by looking at Figure 5. For example, taking an error threshold of \( 10^{-7} \) — which, in the \( \text{EM} \) system, is roughly equivalent to \( 10^{-7} \) km/s —, the parameterization method provides, on average, a good approximation of the center manifold up to \( \delta H_0 \sim 0.017 \) (resp. 0.023) in the \( \text{EML}_1 \) (resp. \( \text{EML}_2 \)) case. Note that, with this threshold, the errors obtained are never higher than the inherent errors of the \( \text{QBCP} \), that is the inner discrepancy with respect to a higher-fidelity solar system model.

5.4.3. **Invariance error in the plane.** Looking at the planar case allows to draw additional conclusions on the shape of the domain of practical convergence. To this end, the procedure of the previous subsection is repeated, taking initial conditions in the \( xy \)-plane. Such condition is satisfied when \( s_2 = s_4 = 0 \). Fixing again the error threshold to \( 10^{-7} \), the limits of the domain of practical convergence are plotted on Figure 6, for various orders \( N \) of the parameterization. One can see that the corresponding domain around both points increases in size as the order goes up, with the following limitations:
- In the EML₁ case, the improvement stops between $N = 20$ and $N = 25$, much earlier than in the EML₂ case. As in the case of the orbital error, this is probably ought to the COC, which acts as a precision bottleneck.
- In the EML₂ case, the improvement is not isotropic: there is a slight decrease of the precision between the orders 15 and 20 along the $x$-axis.

![Figure 6: Limit of the domain $E_I < 10^{-7}$ in the XY-plane in EM coordinates, for various orders $N$ of the parameterization, and both in the EML₂ (left) and EML₁ (right) cases. The black circle • is the Moon, and its apparent radius is to scale.](image)

Figure 6: Limit of the domain $E_I < 10^{-7}$ in the XY-plane in EM coordinates, for various orders $N$ of the parameterization, and both in the EML₂ (left) and EML₁ (right) cases. The black circle • is the Moon, and its apparent radius is to scale.

To further investigate the planar case, the same test is performed using the normal form style to compute the parameterization of the center manifold. As in Andreu [?], the threshold in the normal form process is fixed to $\epsilon = 0.05$. The corresponding results are given on Figure 7. On this figure, the dashed curve shadows four close $T/2$-periodic orbits resonant with the Sun, denoted PO2a-d in [?], see Figure 2b therein). Low-order resonances tend to introduce small divisors early in the semi-analytical procedures and therefore act as a natural obstruction to the existence of a good parameterization of the center manifold. This is clearly visible on Figure 7, in the normal form case. However, the use of the graph style allows to keep the number of small divisors involved in the solving of the homological equations to a minimum (see subsection 4.3.2 of this paper). As a consequence the associated parameterization can partially cross these resonances.

The same procedure is applied on the EML₁ case, leading to results of Figure 8, for which the same remarks apply. On this figure, the dashed curve is a $2T$-periodic orbit resonant with the Sun, cited in [?] as the natural limit for the domain of practical convergence around EML₁.

As a conclusion to this subsection, the order $N = 20$ (resp. 30) yields the best results in terms of accuracy of the center manifold about EML₁ (resp. EML₂) and is
Invariant manifolds of a non-autonomous problem via the parameterization method

Figure 7: Limit of the domain $E_I < 10^{-7}$ in the $XY$-plane about EML$_2$, for various orders $N$ of the parameterization. The black circle indicates the position of the Moon, with an apparent radius to scale. The dashed curve shadows four close $T/2$-periodic orbits resonant with the Sun denoted PO2a-d in [?].

therefore selected for the rest of the computations.

5.5. Poincaré maps

Poincaré maps provide a qualitative insight into the dynamics inside a center manifold. In this paper, Poincaré maps with a Poincaré section of the form $z = 0$, $p_z > 0$ have been computed in the EML$_{1,2}$ case. Such sections are also usually used in the CRTBP autonomous case [?, ?]. In the latter context, an additional condition on the energy of the form $\delta H_0 = h$ allows to produce two-dimensional Poincaré maps.

In the QBCP case, the energy is no longer constant but its variations remain bounded for any solution in the center manifold: in the range of energy considered in this subsection, numerical integrations on long time scale show that the standard deviation $\sigma$ of the energy $\delta H_{kT}$, $k \in \mathbb{N}$ is never greater than $3 \times 10^{-4}$ (resp. $10^{-3}$) in the EML$_4$ (resp. EML$_2$) case. In this paper, the condition $\delta H_0 = h$ is still imposed to limit the amount of data displayed on the maps. As a consequence, looking at a given Poincaré map of initial energy $h$, one has to recall that (i) the energy is not constant but bounded along the trajectories, and (ii) the range of displayed solutions is a representative but non-exhaustive set of solutions inside the energy layer $[h - \sigma(h), h + \sigma(h)]$.
Invariant manifolds of a non-autonomous problem via the parameterization method

Figure 8: Limit of the domain $E_I < 10^{-7}$ in the $XY$-plane about EML$_1$, for various orders $N$ of the parameterization. The black circle indicates the position of the Moon, with an apparent radius to scale. The dashed curve is a $2T$-periodic orbit resonant with the Sun.

5.5.1. EML$_1$ case. Figure 9 shows Poincaré maps in the EML$_1$ case for some values of the initial energy $\delta H_0$. One can see that the bifurcation that give rise to the Halo orbit family [?] is obtained for an energy between $\delta H_0 = 0.005$ and $\delta H_0 = 0.0075$.

5.5.2. EML$_2$ case. Figure 10 does the same in the EML$_2$ case. The halo bifurcation happens for a relative energy $\delta H_0 \in [0.010, 0.011]$. Figure 11 shows an example of a quasi-halo orbit with initial energy $\delta H_0 = 0.012$. The orbital and invariance errors are computed on a period $T$. Moreover, the orbit is integrated using the reduced vector field on a time span of $50T$.

6. Application to the SEL$_{1,2}$ center manifold

In this section, the parameterization method is applied again to compute the center manifold of the SEL$_{1,2}$ points in the QBCP. As in the previous section, the graph style is used by default, and the parameterization is computed up to $N = 30$.

To this end, the Hamiltonian (4) is used. In this context, the dynamics about the SEL$_{1,2}$ libration points have a small time dependency compared to the Earth-Moon case. Indeed, the relative motion of the Sun and the Earth-Moon barycenter is very close to a circular motion (see for example Table 1). As a consequence, the time dependency
of the system only comes from the combined gravitational influence of the Earth and the Moon as seen from the Sun-Earth system (see Figure 1b). The positions of the Earth and the Moon vary with time, but their gravitational influence has small periodic variations as seen from the $SE_{L_{1,2}}$ libration points. Hence a reduced time dependency of the dynamics in the center manifolds around these points. As an example, for the energy range considered in this section, numerical integrations on long time scale show

Figure 9: Poincaré maps of the center manifold of $EML_{1}$. An arbitrary colormap is used to distinguish the solutions.
that the standard deviation of the energy $\delta H_{kT}, k \in \mathbb{N}$ is always smaller than 2% of the initial energy.

Moreover, the hyperbolic motion in the vicinity of $\text{SEL}_{1,2}$ is mildly unstable compared to its EM counterpart. Consequently, the operations detailed in section 3 are performed down to machine precision and do not act as a precision bottleneck, as they do in the EM case. Both the mild instability and the quasi-autonomous behavior
Figure 11: Example of quasi-halo orbit around EML$_2$, with initial energy $\delta H_0 = 0.012$. Top: errors on a full period $T$. Bottom: the same orbit integrated in the center manifold over a time span of $50T$.

Contribute to making the computation of the SE center manifolds almost equivalent to the CRTBP case, both in terms of precision and resulting dynamics.

As an example, Figure 12 shows Poincaré maps in the SEL$_2$ case for some values of the initial energy $\delta H_0$. One can see that there is almost no overlapping of the solutions, which is a direct consequence of the quasi-conservation of the energy. The SEL$_1$ exhibits very similar results and there is no need to present them in more detail.
Figure 12: Poincaré maps of the center manifold of \( \text{SEL}_2 \). An arbitrary colormap is used to distinguish the solutions.

7. Conclusion

In this paper, the parameterization method (PM) has been applied for the first time to obtain high-order semi-analytical approximations of center manifolds about the dynamical equivalents of the libration points \( \text{EML}_{1,2} \) and \( \text{SEL}_{1,2} \) in a single coherent framework, namely the Quasi-Bicircular Four-Body Problem (QBCP). The overall process can be seen as an application of previous work on quasi-periodic invariant
Invariant manifolds of a non-autonomous problem via the parameterization method

This new implementation confirms the advantages of the PM brought out in earlier effort on the autonomous case [?, ?]. Namely:

- The formalism of the PM in the form of styles brings flexibility and clarity to the user.
- The PM allows to work on a Fourier-Taylor space with a dimension \( d \) equals to the dimension of the invariant object that is approximated. On the contrary, equivalent normal form procedures are usually performed on the complete 6-dimensional phase space. Working on a lower-dimensional space allows to reduce both the dimension and number of coefficients of the Fourier-Taylor series, which can be critical at high order.
- More generally, working on the vector field rather than on the Hamiltonian should allow the PM implementation to be used outside the scope of Hamiltonian dynamics, providing that the linear invariant manifolds persist under nonlinear perturbation in the problem at hand.

This paper is part of a project which aims to study the motion of a spacecraft in the semi-low-energy network, with a particular focus on natural connections between \( \text{SEL}_{1,2} \) and \( \text{EML}_{2} \). In that perspective, the current implementation, using the graph style, allows to get the corresponding center manifolds in a neighborhood big enough to comprise the halo bifurcation. The advantages of this new approach are especially relevant in the Earth-Moon case, which was previously addressed e.g. in [?] using an equivalent of the normal form style. The comparison can be seen on Figures 7 and 8. The price to pay for such a large domain of practical convergence is the presence of time-dependent reduced dynamics, in the form of full Fourier-Taylor expansions. Hence, the study of the reduced phase space is harder but bigger orbits are available for practical use. This tradeoff is the backbone of a promising compact tool that will be used to build systematic connections between the center manifolds in a full Four-Body model.

Finally, from the point of view of the quality of the model, a natural extension of this work is to incorporate a second or more frequencies, as it as been done in the Sun-Jupiter-Saturn context [?]. In principle, such an extension would only require first to compute the dynamical equivalents of the libration points, then to modify the Fourier-Taylor algebra to account for the additional frequencies.

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