Graded cubes of opposition and possibility theory with fuzzy events ☆

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A B S T R A C T

The paper discusses graded extensions of the cube of opposition, a structure that naturally emerges from the square of opposition in philosophical logic. These extensions of the cube of opposition agree with possibility theory and its four set functions. This extended cube then provides a synthetic and unified view of possibility theory. This is an opportunity to revisit basic notions of possibility theory, in particular regarding the handling of fuzzy events. It turns out that in possibility theory, two extensions of the four basic set functions to fuzzy events exist, which are needed for serving different purposes. The expressions of these extensions involve many-valued conjunction and implication operators that are related either via semi-duality or via residuation.

1. Introduction

The square of opposition is a logical structure introduced in Ancient Greek logic, in Aristotle times, in relation with the study of syllogisms [31]. It fell into oblivion with the advent of modern logic at the beginning of XXth century. Still studied in philosophical logic in the 1960's [6], its interest was rediscovered at the beginning of the new century [2–5]. The cube of opposition, of which two facets are squares of opposition, while the other facets exhibit other noticeable structures, appeared once in 1952, in a thorough discussion of syllogisms [36], but has remained largely ignored since that time. It has however been revived a few years ago [20] when trying to relate possibility theory [41,14] with the square of opposition. A graded extension that applies when the items in opposition are a matter of degree, rather than being binary-valued, has been recently proposed and applied to fuzzy rough sets [8] and more, generally, to the calculus of fuzzy relations [9]. This paper focuses on the application of the gradual cube to possibility theory.

Possibility theory, a framework for uncertainty modeling, offers a rich setting for expressing graded modalities. Indeed it is based on a body of four set functions, which respectively model strong (guaranteed) possibility, weak possibility viewed as evaluating plain consistency, and their duals that correspond to weak and strong necessity. From the beginning, the (weak) possibility measure has been extended to fuzzy events [41]. By duality, (weak) possibility measures immediately lead to (strong) necessity measures [11,43] for fuzzy events. Then the (weak) possibility of a fuzzy event is 0 only if there is absolutely no overlap between the fuzzy event and the possibility distribution. Conversely, the (strong) necessity of a fuzzy event is 1 only if the whole possibility distribution (including all the values with low but non-zero possibility) is included

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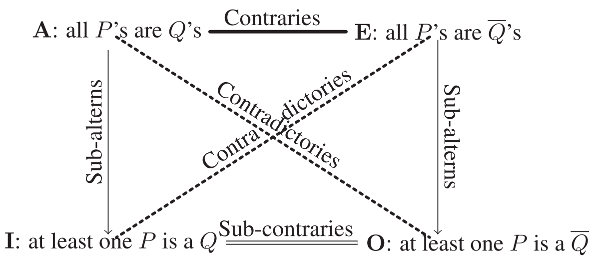

Fig. 1. Square of opposition.

in the core of the fuzzy event (corresponding to the values with maximal membership 1). These two measures of fuzzy events have been at the basis of fuzzy pattern matching [7] for evaluating how much a flexible requirement is fulfilled in presence of uncertain information. They also respectively correspond to optimistic and pessimistic decision criteria in qualitative decision theory (the pessimistic criterion is 1 only if the decision yields a fully satisfactory output, even when applied in a situation having a low (non-zero) possibility) [19].

However, it has been also noticed that this measure of (strong) necessity of a fuzzy event, together with a similar measure of (strong) possibility are no longer suitable for properly handling (guaranteed) possibility and necessity qualification of fuzzy events, i.e., for representing statements of the form "it is possible / certain at level α that x is A" (where A is a fuzzy event). This problem has led to suggest other measures of strong possibility and necessity for fuzzy events [17], that have been used in fuzzy logic programming (for instance Alsinet and Godo [1]).

Definitions and results regarding these extensions of possibility theory to fuzzy events, in the spirit of Zadeh's original intuitions,[1] have remained sparse and lacunary until now. The study of the graded cubes of opposition offers an opportunity to provide a complete view of the possibilistic evaluations of fuzzy events. The aim of this paper is twofold: on the one hand, to unify the different definitions of the four set functions of possibility theory for fuzzy events and to lay bare a minimal algebraic structure that can accommodate them; on the other hand, to show that this algebraic structure is capable of accounting for the graded cube of opposition and its possibility theory version, involving eight measures of fuzzy events and of their complements. These results also systematize preliminary proposals in [8,9] regarding conjunctions and implications needed to build a graded square of opposition.

We start by a reminder on the square and the cube of opposition in Section 2, and then in Section 3, after recalling the two-valued setting for possibility theory and its cube of opposition, we investigate the different extensions of the four basic measures of possibility theory to the case of fuzzy events, and lay bare a minimal algebraic structure involving two conjunctions and two implications related via suitable transformations, that supports a qualitative approach to possibility theory for fuzzy events. Finally, in Section 4 we organize these fuzzy set functions in a graded square of opposition.

2. The square and cube of opposition: Boolean case

In this section, we briefly recall what a square of opposition is, and how it can be characterized, before also introducing the cube of opposition, a related abstract structure which is an extension of the square.

2.1. The square of opposition

The traditional square of opposition [31] is built with universally and existentially quantified statements in the following way. Consider a statement (**A**) of the form "all P's are Q's", which is negated by the statement (**O**) "at least one P is not a Q", together with the statement (**E**) "no P is a Q", which is clearly in even stronger opposition to the first statement (**A**). These three statements, together with the negation of the last statement, namely (**I**) "at least one P is a Q" can be displayed on a square whose vertices are traditionally denoted by the letters **A**, **I** (affirmative half) and **E**, **O** (negative half) from the Latin "AffIrmo nEgO", as pictured in Fig. 1 (where $\overline{Q}$ stands for "not Q").

It is assumed that the set of P's is not empty for avoiding existential import problems, otherwise the statement **A** (resp. **E**) may be true while **I** (resp. **O**) is false [32]. As can be checked, noticeable logical relations hold in the square. They are the basis of the following abstract definition of a square of opposition.

Definition 1. In a square of opposition **AIEO**, the following relations are supposed to hold (**A**, **I**, **E**, **O** are viewed as propositional logic variables):

(a) **A** and **O** (resp. **E** and **I**) are the negation of each other, i.e., $\mathbf{A} \equiv \neg\mathbf{O}$ and $\mathbf{E} \equiv \neg\mathbf{I}$;
(b) **A** entails **I**, and **E** entails **O**, i.e., $\neg\mathbf{A} \vee \mathbf{I}$ and $\neg\mathbf{E} \vee \mathbf{O}$;

[1] We do not consider here definitions of the possibility and necessity of fuzzy events in the setting of Choquet integrals, e.g. Smets [38], Dubois and Prade [13] or more recently [24].

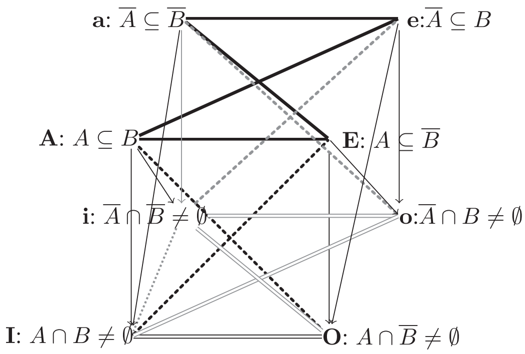

Fig. 2. Cube of opposition of set inclusion and non-empty intersection relations.

(c) together **A** and **E** cannot be true, but may be false, i.e., $\neg\mathbf{A} \vee \neg\mathbf{E}$;
(d) together **I** and **O** cannot be false, but may be true, i.e., $\mathbf{I} \vee \mathbf{O}$.

The statements (a), (b), (c) and (d) are not independent. Indeed there are three options for defining a square of opposition with independent conditions [9]. We can take

- either (a) and (b),
- or (a) and (c),
- or (a) and (d).

In particular, taking (a) and (c), we can express the content of a square of opposition in propositional logic using two propositional variables (say **A** and **E**) and a knowledge base containing only the axiom $\neg\mathbf{A} \vee \neg\mathbf{E}$.

Besides, it has been observed [34] that squares of opposition can be combined in the following way:

Proposition 1. *Given two squares of opposition* $\boldsymbol{AIEO}$ *and* $\boldsymbol{A'I'E'O'}$*, then the 4-tuple* $(\boldsymbol{A} \wedge \boldsymbol{A'})(\boldsymbol{I} \vee \boldsymbol{I'})(\boldsymbol{E} \wedge \boldsymbol{E'})(\boldsymbol{O} \vee \boldsymbol{O'})$ *forms another square of opposition satisfying the four characteristic conditions.*

Proof. It is straightforward to check that (a) and (b) still hold, due to De Morgan laws and to the monotony properties of implication. $\square$

2.2. The cube of opposition

Changing P and Q into their negations, $\overline{P}$ and $\overline{Q}$ respectively, leads to another similar square of opposition **aeoi**, provided that we also assume that the set of "not-P's" is non-empty. Then the 8 statements, **A**, **I**, **E**, **O**, **a**, **i**, **e**, **o** may be organized in what may be called a *cube of opposition*, as in Fig. 2, where, for clarity, the statements are replaced by their counterparts in terms of relations between sets expressing inclusions, or non-emptiness of intersections. In Fig. 2, we require $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A} \neq \emptyset$, $\overline{B} \neq \emptyset$. These conditions insure the consistency of the two squares of opposition (otherwise, e.g., $A \subseteq B$ would no longer entail $A \cap B \neq \emptyset$ when $A = \emptyset$).

This cube, rediscovered in [20], is rarely mentioned; it apparently appeared for the first time in Reichenbach's modern study of syllogisms [36] in the middle of last century. The front facet and the back facet of the cube are traditional squares of opposition, where the thick non-directed segments relate contraries, the double thin non-directed segments relate subcontraries, the diagonal dotted non-directed lines contradictories, and the vertical uni-directed segments point to subalterns, and express entailments.[2]

As a summary, a cube of opposition obeys the following requirements.

Definition 2. Let **A, I, E, O, a, i, e, o** be propositional variables. In a cube of opposition **AIEOaieo**, the following relations are supposed to hold:

Front and back facets (two squares of opposition):

(a) for the diagonals: $\mathbf{A} \equiv \neg\mathbf{O}$, $\mathbf{E} \equiv \neg\mathbf{I}$, $\mathbf{a} \equiv \neg\mathbf{o}$, $\mathbf{e} \equiv \neg\mathbf{i}$;
(b) for the vertical edges $\neg\mathbf{A} \vee \mathbf{I}$, $\neg\mathbf{E} \vee \mathbf{O}$, $\neg\mathbf{a} \vee \mathbf{i}$, $\neg\mathbf{e} \vee \mathbf{o}$;
(c) for the top edges $\neg\mathbf{A} \vee \neg\mathbf{E}$, $\neg\mathbf{a} \vee \neg\mathbf{e}$;

[2] This cube should not be confused with the so-called "logical cube" introduced by Moretti [29] and more thoroughly studied by Pellissier [33], where only edges bear constraints that have the same semantic interpretation in terms of entailments in a so-called 4-opposition structure.

(d) for the bottom edges $\mathbf{I} \vee \mathbf{O}$, $\mathbf{i} \vee \mathbf{o}$.

Side facets (entailments):

(e) $\neg\mathbf{A} \vee \mathbf{i}$;
(f) $\neg\mathbf{a} \vee \mathbf{I}$;
(g) $\neg\mathbf{e} \vee \mathbf{O}$;
(h) $\neg\mathbf{E} \vee \mathbf{o}$.

Top and bottom facets:

(i) $\mathbf{a}$ and $\mathbf{E}$ cannot be true together, but may be false together, i.e., $\neg\mathbf{a} \vee \neg\mathbf{E}$;
(j) the same for $\mathbf{A}$ and $\mathbf{e}$, i.e., $\neg\mathbf{A} \vee \neg\mathbf{e}$;
(k) $\mathbf{i}$ and $\mathbf{O}$ cannot be false together, but may be true together, i.e., $\mathbf{i} \vee \mathbf{O}$;
(l) the same for $\mathbf{I}$ and $\mathbf{o}$, i.e., $\mathbf{I} \vee \mathbf{o}$.

Just as in the square of opposition the properties that define it are not independent, likewise the conditions that define the cube are partially redundant. Let us study the relationships between conditions on the side facets, and conditions on the top and bottom facets in the cube of opposition.

Proposition 2. *In the Boolean case, for a cube of opposition,*

- *the properties on the front and back facets associated with the properties on the side facets entail the properties on the top and bottom facets;*
- *the properties on the front and back facets associated with the properties on the top (resp. bottom) facet entail the properties on the side facets and of the bottom (resp. top) facet.*

Proof.

- Indeed, one obtains $\neg\mathbf{a} \vee \neg\mathbf{E}$ from $\mathbf{E} \equiv \neg\mathbf{I}$ and $\neg\mathbf{a} \vee \mathbf{I}$; similarly $\mathbf{I} \vee \mathbf{o}$ from $\mathbf{a} \equiv \neg\mathbf{o}$ and $\neg\mathbf{a} \vee \mathbf{I}$.
- The logical relations of the front, back, top and bottom facets are respectively:

$$\mathbf{A} \equiv \neg\mathbf{O},\ \mathbf{E} \equiv \neg\mathbf{I},\ \mathbf{a} \equiv \neg\mathbf{o},\ \mathbf{e} \equiv \neg\mathbf{i},$$

$$\neg\mathbf{A} \vee \mathbf{I},\ \neg\mathbf{E} \vee \mathbf{O},\ \neg\mathbf{a} \vee \mathbf{i},\ \neg\mathbf{e} \vee \mathbf{o}$$

$$\neg\mathbf{A} \vee \neg\mathbf{E},\ \neg\mathbf{A} \vee \neg\mathbf{e},\ \neg\mathbf{a} \vee \neg\mathbf{E},\ \neg\mathbf{a} \vee \neg\mathbf{e}$$

$$\mathbf{I} \vee \mathbf{O},\ \mathbf{I} \vee \mathbf{o},\ \mathbf{i} \vee \mathbf{O}, \mathbf{i} \vee \mathbf{o}.$$

  Applying the four equivalences on the first line, the whole set of formulas above can be inferred from the properties of the top facet (line 3): $\neg\mathbf{A} \vee \neg\mathbf{E}$, $\neg\mathbf{a} \vee \neg\mathbf{e}$, $\neg\mathbf{A} \vee \neg\mathbf{e}$, $\neg\mathbf{a} \vee \neg\mathbf{E}$. This shows that these four equivalences together with the properties of the top facet entail the properties of the bottom facets. Moreover, from the properties of the top facet, we can also see that the entailments of the side facets hold (taking into account the four equivalences). □

We can even reduce further the number of conditions that define the cube of opposition:

Proposition 3. *In the Boolean case, the cube of opposition needs only the four variables* $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{a}$, $\boldsymbol{e}$, *together with the four mutual exclusiveness constraints (top facet):* $\neg\boldsymbol{A} \vee \neg\boldsymbol{E}$, $\neg\boldsymbol{A} \vee \neg\boldsymbol{e}$, $\neg\boldsymbol{a} \vee \neg\boldsymbol{E}$, $\neg\boldsymbol{a} \vee \neg\boldsymbol{e}$, *provided that we define* $\boldsymbol{O}$ *as* $\neg\boldsymbol{A}$, $\boldsymbol{I}$ *as* $\neg\boldsymbol{E}$, $\boldsymbol{o}$ *as* $\neg\boldsymbol{a}$, *and* $\boldsymbol{i}$ *as* $\neg\boldsymbol{e}$.

Proof. It is clear that 4 variables are enough due to constraints on the diagonals of the front and back facets. Under the four mutual exclusiveness constraints of the top facet, it is clear that the entailment relations on the side facets trivially follow. For instance, the mutual exclusion $\neg\mathbf{A} \vee \neg\mathbf{e}$ also reads $\mathbf{A} \models \neg\mathbf{e}$, which is one arrow on a side facet. We can also derive the fact that the bottom vertices on the front and the back, as well as on the diagonals are not mutually exclusive. For instance, the truth of $\neg\mathbf{A} \vee \neg\mathbf{E}$ is equivalent with the truth of $\mathbf{O} \vee \mathbf{I}$, etc. □

Using distributivity, the knowledge base $\{\neg\mathbf{A} \vee \neg\mathbf{E}, \neg\mathbf{A} \vee \neg\mathbf{e}, \neg\mathbf{a} \vee \neg\mathbf{E}, \neg\mathbf{a} \vee \neg\mathbf{e}\}$ is clearly equivalent to the single formula $(\neg\mathbf{A} \wedge \neg\mathbf{a}) \vee (\neg\mathbf{E} \wedge \neg\mathbf{e})$ which means that $\mathbf{A} \vee \mathbf{a}$ and $\mathbf{E} \vee \mathbf{e}$ cannot be true together. This condition also writes $\mathbf{A} \vee \mathbf{a} \models \neg\mathbf{E} \wedge \neg\mathbf{e}$, i.e., $\mathbf{A} \vee \mathbf{a} \models \mathbf{I} \wedge \mathbf{i}$.

Note that the above characterization leaves only seven possibilities in terms of truth-values of the variables ($\mathbf{A}$, $\mathbf{E}$, $\mathbf{a}$, $\mathbf{e}$) (namely three models of $\mathbf{A} \vee \mathbf{a}$, three models of $\mathbf{E} \vee \mathbf{e}$, and one interpretation where both are false). Interestingly, it corresponds, in agreement with the labeling of vertices of Fig. 2, to the seven possible relative positions of two non-empty proper subsets A and B of a set, a situation already described in [10][3] in terms of the four set-functions of possibility theory.

Lastly, the combination of squares of opposition extends to the cube:

Proposition 4. *Given two cubes of opposition associated to the variables $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{a}$, $\boldsymbol{e}$ and $\boldsymbol{A}'$, $\boldsymbol{E}'$, $\boldsymbol{a}'$, $\boldsymbol{e}'$ then the variables $\boldsymbol{A} \wedge \boldsymbol{A}$, $\boldsymbol{E} \wedge \boldsymbol{E}'$, $\boldsymbol{a} \wedge \boldsymbol{a}'$, $\boldsymbol{e} \wedge \boldsymbol{e}'$ associated with the four mutual exclusiveness constraints (top facet) form another square of opposition satisfying all the characteristic conditions.*

Proof. Clearly, the square of opposition properties of the front and the back facets are preserved. It is straightforward to check that the entailment properties of the side facets hold. Then by Proposition 2, the properties of the top and bottom facets hold. $\square$

Note that one may also think of combining the front facet of the first cube with the bottom facet of the second cube. This leads to the cube $(\mathbf{A} \wedge \mathbf{a}')(\mathbf{I} \vee \mathbf{i}')(\mathbf{E} \wedge \mathbf{e}')(\mathbf{O} \vee \mathbf{o}')(\mathbf{a} \wedge \mathbf{A}')(\mathbf{i} \vee \mathbf{I}')(\mathbf{e} \wedge \mathbf{E}')(\mathbf{o} \vee \mathbf{O}')$, which also satisfies all the expected properties. In case the two cubes are identical, with the instantiation of Fig. 2, the cube reduces to a square where the top vertices are respectively $A = B$ and $A = \overline{B}$, and the bottom vertices are $A \neq \overline{B}$ and $A \neq B$ respectively.

Cubes of oppositions have been applied to rough set theory [8] where various upper and lower approximations can be attached to vertices of the cube, and horizontal vertices on the top facet relate orthopairs of disjoint sets. More generally, logical modalities as induced by relations can also be organized in a cube of opposition [9]. In this paper, we focus on the basic set functions of possibility theory.

3. Possibility theory and fuzzy events

The idea of possibility theory as a non-additive modeling of uncertainty departing from probability theory was initially proposed by the English economist G.L.S. Shackle [39]. It was independently rediscovered by L.A. Zadeh [41] in order to account for incomplete information expressed by linguistic statements, modeled by means of fuzzy sets.

In this section, we provide a reminder on qualitative possibility theory, starting with the particular case of Boolean possibility theory, so as to lay bare the relation with the cube of opposition of Fig. 2, as first pointed out in [20]. We then recall the four set functions of possibility theory in case of classical (i.e., crisp) events, along with their standard max–min extensions to fuzzy events. Finally, we show that the proper certainty qualification of fuzzy statements in the framework of possibility theory requires an alternative extension of the four set functions to fuzzy events. We present the notions in the minimal algebraic structures needed to define them.

3.1. Boolean possibility theory

We first introduce the four set functions of possibility theory in the Boolean case, and then we point out the fact that the structure of the cube of opposition naturally captures them.

3.1.1. The four Boolean indicators

Let $E \subset X$ be a *non-empty* proper subset of a set of situations or states X, representing the available evidence about the value of an unknown entity x. In other words, all that is known is that $x \in E$. Let A be another subset representing an event ($A \subseteq X$). Then the four following Boolean indicator functions can be defined:

- **potential possibility** $\Pi(A) = \begin{cases} 1 & \text{if } A \cap E \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$
  Indeed, if $x \in E$ is known, then x is possibly in A.
- **actual possibility** if $\Delta(A) = \begin{cases} 1 & \text{if } A \subseteq E; \\ 0 & \text{otherwise.} \end{cases}$
  It is enough that $x \in A$ to be sure that x is possible.
- **actual necessity** if $N(A) = \begin{cases} 1 & \text{if } E \subseteq A; \\ 0 & \text{otherwise.} \end{cases}$
  If $x \in E$ then x is surely in A.
- **potential necessity** if $\nabla(A) = \begin{cases} 1 & \text{if } A \cup E \neq X; \\ 0 & \text{otherwise.} \end{cases}$
  Then, there is an impossible value for x outside A.

[3] As, here, we exclude the cases when A is empty or the whole set.

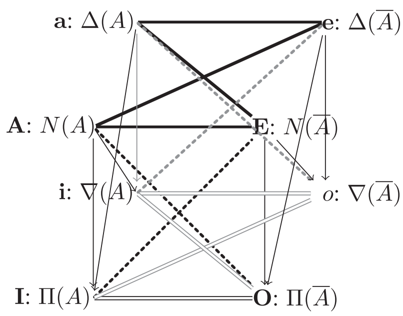

Fig. 3. Cube of opposition of possibility theory.

Under the assumption that $A \neq \emptyset$; $A \neq X$ (the event is not trivial) and $E \neq \emptyset$; $E \neq X$ (the evidence is consistent and brings information), it can be checked that one has (see, e.g., [20])

$$\max(N(A), \Delta(A)) \leq \min(\Pi(A), \nabla(A)). \tag{1}$$

This just expresses that $A \subseteq E$ (or $E \subseteq A$) entails $A \cap E \neq \emptyset$ and $\overline{A} \cap \overline{E} \neq \emptyset$ (under the assumption that the two subsets and their complements are not empty).

3.1.2. Boolean possibility theory and cube of opposition

An easy possibilistic counterpart of the cube in Fig. 2 is given in Fig. 3, which immediately follows from the definition of the four set functions of possibility theory in the Boolean case.

The inequality (1) suggests the following result that holds for any Boolean cube.

Proposition 5. *In the Boolean context, consider eight Boolean propositions labeling a cube* **AIOEaioe** *(as the one in Fig. 3). It is a cube of opposition if and only if the following properties are valid:*

- *property* (a) *namely:* ***A*** *and* ***O***, *as well as* ***E*** *and* ***I***, ***a*** *and* ***o***, *and* ***e*** *and* ***i*** *are the negation of each other;*
- *the entailment* $\boldsymbol{A} \vee \boldsymbol{a} \models \boldsymbol{I} \wedge \boldsymbol{i}$ *holds.*

Proof. Under the hypothesis $\mathbf{A} \vee \mathbf{a} \models \mathbf{I} \wedge \mathbf{i}$ it is easy to check that the properties (b), (e) and (f) are satisfied. Conversely, let us suppose that **AIOEaioe** is a cube of opposition and that $\mathbf{A} \vee \mathbf{a}$ is true. If **A** is true then **I** is true, **e** is false so **i** is true. If **a** is true then **I** is true, **e** is false, i.e., **i** is true. So $\mathbf{A} \vee \mathbf{a} \models \mathbf{I} \wedge \mathbf{i}$. □

Note that in the cube of opposition of possibility theory in Fig. 3, there is an internal negation that is applied when moving from the right side to the left side. This relation is a particular instance of the properties (c) and (d) of the cube.

We now move from the Boolean possibility theory to its many-valued counterpart, when the subset E representing what is known is replaced by a fuzzy set understood as a graded possibility distribution restricting the more or less possible states of the world [41], with a view to extend the Boolean cube in Fig. 3 to a *graded* cube for non-Boolean possibility theory.

3.2. Crisp events

Consider a possibility distribution π, defined from a set X of possible states to a totally ordered scale $(L, \leq)$ with top and bottom respectively denoted by 1 and 0 (for instance, $L = [0, 1]$, viewed as an ordinal scale). Moreover, we assume L is equipped with a *negation function*, i.e., a decreasing bijection from L to itself an order reversing function that is involutive, denoted by $1 - \cdot$. We denote this structure by $(L, \leq, 1 - .0, 1)$. Operations min, max are defined as usual, as $\min(a, b) = a$ if and only if $\max(a, b) = b$ if and only if $a \leq b$. We do have $\min(a, 1 - a) \leq \max(b, 1 - b)$ so that it is a Kleene algebra. This is the minimal structure we need to define an elementary form of possibility and necessity of fuzzy events.

The possibility distribution π restricts the more or less plausible states of the world according to the available information. On this basis, one can estimate the degrees of (weak) possibility and (strong) necessity of an event $A \subseteq X$, respectively as:

$$\Pi(A) = \sup_{x \in A} \pi(x); \;\; N(A) = \inf_{x \notin A} 1 - \pi(x).$$

$\Pi(A)$ evaluates to what extent A is consistent with π, while $N(A)$ evaluates to what extent A is certainly implied by π. The possibility-necessity duality is expressed by

$$N(A) = 1 - \Pi(\overline{A}),$$

where $\overline{A}$ is the complement of A. Generally, $\Pi(X) = N(X) = 1$ and $\Pi(\emptyset) = N(\emptyset) = 0$. Possibility measures satisfy the basic "maxitivity" property $\forall A, B \subseteq X$,

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

Necessity measures satisfy an axiom dual to the one of possibility measures, namely

$$N(A \cap B) = \min(N(A), N(B)).$$

On infinite spaces, these axioms must hold for union and intersection of infinite families of sets.

Apart from Π and N, a measure of guaranteed possibility or (strong) possibility can be defined [18]:

$$\Delta(A) = \inf_{x \in A} \pi(x).$$

It estimates to what extent *all* states in A are actually possible according to evidence represented by π. In contrast, Π appears to be a measure of *potential* possibility, while $\Delta(A)$ can be used as a degree of evidential support for A. By duality, a measure of (weak) necessity can be defined:

$$\nabla(A) = 1 - \Delta(\overline{A}) = \sup_{x \notin A} 1 - \pi(x);$$

$\nabla(A) = 1$ as soon as some state outside A is impossible. These two measures respectively satisfy the characteristic properties $\Delta(A \cup B) = \min(\Delta(A), \Delta(B)$ and $\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$, while $\Delta(X) = \nabla(X) = 0$ and $\Delta(\emptyset) = \nabla(\emptyset) = 1$. Δ and ∇ are decreasing set functions, while Π and N are increasing.

It can be checked that the inequality (1) still holds provided that the following normalizations from above and from below hold: $\exists x, \pi(x) = 1$; $\exists x', \pi(x') = 0$, to avoid again existential import problems; see [10].

Besides, the only possible way to aggregate in an event-wise manner a set of (weak) possibility measures $\Pi_1, \cdots, \Pi_k$ and obtain a (weak) possibility measure again is via an aggregation function of the form

$$\max(f_1(\Pi_1(A)), \cdots, f_k(\Pi_k(A)))$$

where for each $i = 1, \ldots, n$, f_i is a non-decreasing mapping $L \to L$ such that $f_i(0) = 0$ and $f_i(1) = 1$ [16]. By duality, a similar result holds for strong necessity measures $N_1, \cdots, N_k$ aggregated as

$$\min(g_1(N_1(A)), \cdots, g_k(N_k(A)))$$

where for each $i = 1, \ldots, n$, g_i is of the form $g_i(a) = 1 - f_i(1 - a)$, i.e., it has the same properties as f_i.

For instance, we can use a weighted max (resp. weighted min) combination. Namely, let $w_1, \cdots, w_k$ a set of normalized weight such as $\forall 0 \leq i \leq k$, $w_i \in [0, 1]$, and $\exists i, w_i = 1$; then $\forall A$, we can use

$$\max(\min(\Pi_1(A), w_1), \cdots, \min(\Pi_k(A), w_k))$$

$$(\text{resp. } \min(\max(N_1(A), 1 - w_1), \cdots, \max(N_k(A), 1 - w_k))).$$

This result will fit with the combination of graded squares of opposition discussed in the next section.

3.3. *Possibility and necessity of fuzzy events*

Possibility measures extend to fuzzy events A. A is now represented by a mapping from X to the totally ordered Kleene algebra $(L, \leq, 1 - ., 0, 1)$, and we assume that A is normalized (i.e., $\exists x, A(x) = 1$), as well as $\overline{A}$. Zadeh's definition [41] is the following:

$$\Pi(A) = \sup_{x \in X} \min(A(x), \pi(x))$$

It can be checked that $\Pi(A) = \sup_{\alpha \in [0,1]} \min(\alpha, \Pi(A_\alpha))$, which shows its agreement with the α-level cuts view of a fuzzy set ($A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$).

By duality, maintaining $N(A) = 1 - \Pi(\overline{A})$ (with $\overline{A}(x) = 1 - A(x)$), we get

$$N(A) = \inf_{x \in X} \max(A(x), 1 - \pi(x)).$$

This extension preserves a very strong view of the idea of inclusion, since $N(A) = 1$ iff $\{x \in X \mid \pi(x) > 0\} \subseteq A_1 = \{x \in X \mid A(x) = 1\}$.

Similarly, Δ and ∇ can be extended to fuzzy events in the same style, namely

$$\Delta(A) = \inf_{x \in X} \max(1 - A(x), \pi(x))$$
$$\nabla(A) = \sup_{x \in X} \min(1 - A(x), 1 - \pi(x)) = 1 - \Delta(\overline{A})$$

These definitions stem from the property $\Delta_\pi(A) = N_{1-\pi}(\overline{A})$, where the notation Δ_π makes it clear that Δ is computed from π. This identity, valid in the Boolean case is carried over to the gradual case. Moreover, there is the duality property $\nabla(A) = 1 - \Delta(\overline{A})$.

These extensions coincide with the previous definitions in case of crisp events, and preserve the decomposability properties of the four set functions in case of fuzzy sets, with $A(x) \cap B(x) = \min(A(x), B(x))$ and $A(x) \cup B(x) = \max(A(x), B(x))$.

Several authors have proposed the following alternative extension of a possibility measure to a fuzzy event, using the product in place of min when $L = [0, 1]$ in the usual sense. The first one was Shilkret [40] in 1971 (and later A. Kaufmann [27] in 1979 under the name of "admissibility"):

$$\Pi(A) = \sup_{x \in X} A(x) \cdot \pi(x)$$

The decomposability property of Π with respect to max-based union is clearly preserved.

More generally, one could replace min (resp. max) in the expressions of Π, N, Δ, ∇ by any monotonically increasing aggregation function $*$ on L such that $a * 1 = 1 * a = a$ (a semi-copula when $L = [0, 1]$ [25]) (resp., the associated De Morgan dual $1 - (1 - a) * (1 - b)$). In practice, it is natural to require also the commutativity of $*$ (otherwise there would be two distinct notions of possibility of a fuzzy event, as $\Pi_*(A) = \sup_{x \in X} A(x) * \pi(x)$ would differ from $\sup_{x \in X} \pi(x) * A(x)$) and use for instance a triangular norm $*$ [28]. Define likewise $N_*(A) = 1 - \Pi_*(\overline{A})$, $\Delta_*(A) = \inf_{x \in X}(1 - A(x) * (1 - \pi(x)))$, $\nabla_*(A) = 1 - \Delta_*(\overline{A})$.

We can now establish the following result for such generalized possibility and necessity of fuzzy events, that express relations between top vertices to bottom ones in a cube of opposition extending the one of Fig. 3.

Proposition 6. *For any monotonically increasing aggregation function $*$, such that $1 * a = a = a * 1$,*

- *if $\exists x, \pi(x) = 1$ then $N_*(A) \le \Pi_*(A)$;*
- *if $\exists x, A(x) = 1$ then $\Delta_*(A) \le \Pi_*(A)$;*
- *if $\exists x, A(x) = 0$ then $N_*(A) \le \nabla_*(A)$;*
- *if $\exists x, \pi(x) = 0$ then $\Delta_*(A) \le \nabla_*(A)$.*

Proof. $N_* \le \Pi_*$: Let x_0 such that $\pi(x_0) = 1$.

$$\begin{aligned} N_*(A) &= \inf_{x \in X}(1 - (1 - A(x)) * \pi(x)) \le 1 - ((1 - A(x_0)) * \pi(x_0)) = A(x_0) \\ &= A(x_0) * \pi(x_0) \le \sup_{x \in X} A(x) * \pi(x) = \Pi_*(A). \end{aligned}$$

$\Delta_* \le \Pi_*$: Let x' such that $A(x') = 1$. We proceed likewise:

$$\begin{aligned} \Delta_*(A) &= \inf_{x \in X}(1 - A(x) * (1 - \pi(x))) = \pi(x') \\ &= A(x') * \pi(x') \le \sup_{x \in X} A(x) * \pi(x) = \Pi_*(A). \end{aligned}$$

The two other inequalities are proved likewise. $\square$

Note that we use all properties of operation $*$ in the proof. Then we get the graded extension of inequality (1), which, as Proposition 5 shows, is necessary to get a cube of opposition.

Corollary 1. *For any monotonically increasing aggregation $*$, such that $1 * a = a = a * 1$, if $\exists x, \pi(x) = 1$, $\exists x', A(x') = 1$, $\exists x'', A(x'') = 0$, $\exists x''', \pi(x''') = 0$, then* $\max(\Delta_*(A), N_*(A)) \le \min(\nabla_*(A), \Pi_*(A))$.

3.4. Representation of certainty-qualified and possibility-qualified statements

Human knowledge is often expressed in a declarative way using statements in natural language. It corresponds to expressing (flexible) constraints that the world is supposed to comply with. Sometimes these fuzzy statement are attached belief degrees. Such a qualification of fuzzy statements (in terms of truth, probability, or possibility) was first considered by Zadeh [42] (see also [37]), especially fuzzy statements such as "x is A is α-possible" (or "α-true", or "α-probable"). In the following, we first discuss necessity qualification, and then guaranteed possibility qualification, before going back to (weak) possibility qualification and (weak) necessity qualification.

Certainty-qualified statements. Certainty-qualified pieces of uncertain information of the form "A is certain to degree α" are modeled by the constraint $N(A) \geq \alpha$. In case of crisp events, the least restrictive possibility distribution reflecting this information is [14]:

$$\pi_{(A,\alpha)}(x) = \begin{cases} 1 & \text{if } x \in A \\ 1-\alpha & \text{otherwise.} \end{cases} \tag{2}$$

More precisely we have [14]:

Proposition 7. *$N(A) \geq \alpha$ if and only if $\pi \leq \pi_{(A,\alpha)}$.*

In the following, we stick to the qualitative setting, using only min and max on L. When A is a fuzzy set, Equation (2) is generalized [35] by

$$\forall x, \pi_{(A,\alpha)}(x) = \max(A(x), 1-\alpha),$$

which still represents the qualified fuzzy statement "x is A is α-certain". This equality can be justified in terms of the level cuts of A [15]. It is a form of discounting of the information item "A is certain", whereby states that were impossible according to this information item now become all the more possible as the level of certainty α is smaller. In particular, it can be seen that "x is A is certain" ($\alpha = 1$) is equivalent to $\pi_{(A,1)} = 1_A$ while "x is A is totally uncertain" ($\alpha = 0$) is equivalent to $\pi_{(A,\alpha)} = 1$ (total ignorance).

At this point, we need to find the expression of a degree of certainty of A in the form $\mathcal{N}(A) = \inf_{x\in X} \pi(x) \to A(x) \geq \alpha$ if and only if $\pi(x) \leq \max(A(x), 1-\alpha)$, using an *implication function* $\to$ defined as follows:

Definition 3. An implication $\to$ from L^2 to L is a function that is non-increasing in the first place and non-decreasing on the second place, and coincides with an implication on $\{0,1\}$ (in particular, $0 \to a = 1$ and $1 \to 0 = 0$).

Namely, we introduce the residuated implication $\to_G$ in L, associated to min, such that $\min(a,b) \leq c \Leftrightarrow b \leq a \to_G c, \forall a,b,c \in L$. This implication is Gödel implication,

$$a \to_G c = \begin{cases} c, & \text{if } a > c, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, we get, as presented in [17]:

Proposition 8. *Let A be a fuzzy set. Then $\mathcal{N}(A) \geq \alpha$ if and only if $\pi \leq \pi_{(A,\alpha)}$, provided that $\mathcal{N}(A) = \inf_x (1-A(x)) \to_G (1-\pi(x))$.*

Proof. $\forall x, \pi(x) \leq \max(A(x), 1-\alpha) \Leftrightarrow \forall x, 1-\pi(x) \geq \min(1-A(x), \alpha) \Leftrightarrow \forall x, (1-A(x)) \to_G (1-\pi(x)) \geq \alpha$, which is of the form $\mathcal{N}(A) \geq \alpha$ for $a \to b = (1-b) \to_G (1-a)$, which is the symmetric contrapositive of Gödel implication. □

So $\max(A(x), 1-\alpha)$ is still the least specific possibility distribution such that $\mathcal{N}(A) \geq \alpha$ under this choice of implication. It can be checked that $\mathcal{N}$ is still minitive: $\mathcal{N}(A \cap B) = \min(\mathcal{N}(A), \mathcal{N}(B))$, and that $\mathcal{N}(A)$ and $N(A)$ coincide if A is crisp. Moreover it can be checked that "x is A is certain" $\mathcal{N}(A) = 1$ is still equivalent to $\pi_{(A,1)} = \mu_A$, the membership function of A, while this is not true if we choose to interpret 'x is A is certain" as $N(A) = 1$ using the standard definition in the previous subsection.

Possibility-qualification. Consider uncertain statements of the form "A is possible to degree α". These statements often mean that all realizations of A are possible to degree α. They are modeled by the inequality $\Delta(A) \geq \alpha$. It corresponds to the idea of observed evidence. This type of information is better exploited by assuming an informational principle opposite to the one of minimal specificity, namely, any situation not yet observed is regarded as impossible. This is similar to the closed-world assumption. The most specific distribution $\delta_{(A,\alpha)}$ in agreement with this information is:

$$\delta_{(A,\alpha)}(x) = \begin{cases} \alpha, & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

More precisely it is straightforward to check that:

Proposition 9. *$\Delta(A) \geq \alpha$ if and only if $\pi \geq \delta_{(A,\alpha)}$.*

In the fuzzy case [17], the information item "x is A is α-guaranteed possible" is represented by the constraint $\mathcal{D}(A) \geq \alpha$ where $\mathcal{D}(A) = \inf_x A(x) \to \pi(x)$ for a suitable implication. As for certainty-qualification we wish to preserve Equation (3) in the form:

$$\forall x, \delta_{(A,\alpha)}(x) = \min(A(x), \alpha).$$

$$
\begin{tikzcd}
\min(a,b) \arrow[r, "Res"] \arrow[d, shift left, "\mathcal{S}"] & a \to_G b \arrow[d, shift left, "\mathcal{S}"] \\
\max(1-a,b) \arrow[u, shift left, "\mathcal{S}"] & a \otimes_G b \arrow[l, "Res"'] \arrow[u, shift left, "\mathcal{S}"]
\end{tikzcd}
$$

Fig. 4. Relations between minimum and associated implications and conjunction.

Then we can prove [17]:

Proposition 10. *Let A be a fuzzy set. Then $\mathcal{D}(A) \geq \alpha$ if and only if $\delta_{(A,\alpha)} \leq \pi$ where $\mathcal{D}(A) = \inf_x A(x) \to_G \pi(x)$.*

Proof. Since $\min(a,b) \leq c \Leftrightarrow b \leq a \to_G c$, we have, $\forall x, \min(A(x),\alpha) \leq \pi(x) \Leftrightarrow \forall x, A(x) \to_G \pi(x) \geq \alpha \Leftrightarrow \mathcal{D}(A) \geq \alpha$. $\square$

We still have $\mathcal{D}(A \cup B) = \min(\mathcal{D}(A), \mathcal{D}(B))$, while $\mathcal{D}(A)$ and $\Delta(A)$ coincide if A is crisp. Note that the property $\mathcal{D}_\pi(A) = \mathcal{N}_{1-\pi}(\overline{A})$ is again valid.

By duality, we can introduce the following extension of weak possibility $\mathcal{P}(A) = 1 - \mathcal{N}(\overline{A})$.

Proposition 11. *$\mathcal{P}(A) = \sup_x A(x) \otimes_G \pi(x)$ where $a \otimes_G b = b$ if $a > 1 - b$; $a \otimes_G b = 0$ otherwise.*

Proof. $\mathcal{P}(A) = 1 - \inf_x A(x) \to_G (1 - \pi(x)) = \sup_x 1 - (A(x) \to_G (1 - \pi(x)))$. Note that

$$
1 - (A(x) \to_G (1 - \pi(x))) = \begin{cases} 0 & \text{if } A(x) \leq 1 - \pi(x) \\ \pi(x) & \text{if } A(x) > 1 - \pi(x) \end{cases} = A(x) \otimes_G \pi(x). \quad \square
$$

This non-commutative operation related to minimum coincides with a conjunction on $\{0,1\}$. It was first introduced in [12], noticing that the diagram on Fig. 4 commutes, where Res and $\mathcal{S}$ means residuation and semi-duality respectively:

$$
aRes(\min)b = \sup\{c : \min(a,c) \leq b\} = a \to_G b \text{ and}
$$
$$
a\mathcal{S}(\min)b = 1 - \min(a, (1-b)) = \max(1-a, b).
$$

Note that the expression $\sup_x \pi(x) \otimes_G A(x)$, obtained by exchanging the arguments of $\otimes_G$ in $\mathcal{P}(A)$ does not reduce to the standard possibility measure $\Pi(A)$ for crisp events A. Likewise, we cannot define necessity measures for fuzzy events as $\inf_x \pi(x) \to_G A(x)$, as it does not reduce to the standard necessity measure $N(A)$ for crisp events A.

We can now address the problem of the possibility-qualification in the sense of set-function $\mathcal{P}$.

Proposition 12. *$\mathcal{P}(A) > \alpha \Leftrightarrow \exists x, \pi(x) > \max(1 - A(x), \alpha)$.*

Proof. $\forall a,b,c \in L, a \otimes_G b \leq c \Leftrightarrow b \leq \max(1-a,c)$ or equivalently $a \otimes_G b > c \Leftrightarrow b > \max(1-a,c)$, we can check that: $\mathcal{P}(A) > \alpha \Leftrightarrow \exists x, A(x) \otimes_G \pi(x) > \alpha \Leftrightarrow \exists x, \pi(x) > \max(1 - A(x), \alpha)$. $\square$

This agrees with intuition. Indeed x is A is consistently possible at a level greater than α, if there exists an element x with a possibility degree greater than α, which sufficiently belongs to A (x should belong all the more to A as the possibility of x is small).

Similarly, we can introduce by duality $\reflectbox{$\mathcal{D}$}(A) = 1 - \mathcal{D}(\overline{A})$.

Proposition 13. *$\reflectbox{$\mathcal{D}$}(A) = \sup_x (1 - A(x)) \otimes_G (1 - \pi(x))$.*

Proof. $\reflectbox{$\mathcal{D}$}(A) = 1 - \inf_x (1 - A(x)) \to_G \pi(x) = \sup_x 1 - ((1 - A(x)) \to_G \pi(x))$. $\square$

Then we also have the following equivalence:

Proposition 14. *$\reflectbox{$\mathcal{D}$}(A) > \alpha \Leftrightarrow \exists x, \pi(x) < \min(1 - A(x), 1 - \alpha)$.*

Proof. Since $(1-a) \otimes_G (1-b) > c \Leftrightarrow 1 - b > \max(a,c)$, we have,

$$
\begin{aligned}
\reflectbox{$\mathcal{D}$}(A) > \alpha &\Leftrightarrow \exists x, (1 - A(x)) \otimes_G (1 - \pi(x)) > \alpha \Leftrightarrow \exists x, (1 - \pi(x)) > \max(A(x), \alpha) \\
&\Leftrightarrow \exists x, \pi(x) < \min(1 - A(x), 1 - \alpha). \quad \square
\end{aligned}
$$

$$
\begin{tikzcd}
a * b \arrow[r, "Res"] \arrow[d, shift left, "\mathcal{S}"] & a \to_* b \arrow[d, shift left, "\mathcal{S}"] \\
1 - a * (1 - b) \arrow[u, shift left, "\mathcal{S}"] & a \otimes_* b \arrow[l, "Res"'] \arrow[u, shift left, "\mathcal{S}"]
\end{tikzcd}
$$

Fig. 5. Relations between conjunction and associated implications and conjunction.

It means that there exists x outside A with a low possibility, which agrees with the intuition of the ∇ operator.

We can prove that the same inequalities hold between these set functions as for Zadeh-inspired definitions of the extensions of possibility and necessity to fuzzy events shown in the previous subsection:

Proposition 15.

- *If* $\exists x, \pi(x) = 1$ *then* $\mathcal{N}(A) \le \mathcal{P}(A)$*;*
- *if* $\exists x, A(x) = 1$ *then* $\mathcal{D}(A) \le \mathcal{P}(A)$*;*
- *if* $\exists x, A(x) = 0$ *then* $\mathcal{N}(A) \le \mathcal{C}(A)$*;*
- *if* $\exists x, \pi(x) = 0$ *then* $\mathcal{D}(A) \le \mathcal{C}(A)$*.*

Proof. $\mathcal{N}(A) = \inf_x (1 - A(x)) \to_G (1 - \pi(x)) = 0$ if and only if $\exists x_0$ such that $\pi(x_0) = 1$ and $A(x_0) \ne 1$. Then $\mathcal{N}(A) > 0$ if and only if $\forall x_0$, whenever $\pi(x_0) = 1$ then $A(x_0) = 1$; this means that $\mathcal{N}(A) > 0$ only if $core(\pi) \subseteq core(A)$. In such a case $\mathcal{P}(A) = \sup_x A(x) \otimes_G \pi(x) = 1 \otimes_G 1 = 1$. Hence $\mathcal{N}(A) \le \mathcal{P}(A)$.

Let x' such that $A(x') = 1$. $\mathcal{D}(A) = \inf_x A(x) \to_G \pi(x) \le A(x') \to_G \pi(x') = \pi(x') = A(x') \otimes_G \pi(x') \le \sup_x A(x) \otimes_G \pi(x) = \mathcal{P}(A)$.

Taking an x'' such that $A(x'') = 0, \mathcal{N}(A) \le \mathcal{C}(A)$ can be similarly established.

$\mathcal{C}(A) = \sup_x (1 - A(x)) \otimes_G (1 - \pi(x)) = 1$ as soon as $\exists x'''$ such that $\pi(x''') = 0$ and $A(x''') \ne 1$; this means that $\mathcal{C}(A) < 1$ only if $\forall x'''$ such that $\pi(x''') = 0$ we have $A(x''') = 1$. Then $\mathcal{D}(A) = \inf_x A(x) \to_G \pi(x) \le A(x''') \to_G \pi(x''') = 0$. Hence $\mathcal{D}(A) \le \mathcal{C}(A)$. $\square$

Corollary 2. *If* $\exists x, \pi(x) = 1, \exists x', A(x') = 1, \exists x'', A(x'') = 0, \exists x''', \pi(x''') = 0$, *then* $\max(\mathcal{D}(A), \mathcal{N}(A)) \le \min(\mathcal{C}(A), \mathcal{P}(A))$.

So, we again get a graded extension of inequality (1), which, as Proposition 5 shows, is necessary to get a cube of opposition.

3.5. A generalized setting for the possibility and the necessity of fuzzy events

In the previous subsections we developed a purely qualitative view of the evaluations of fuzzy events using a bounded chain equipped with an involutive negation and a residuated implication. In this way, we obtained two pairs of implication and conjunction connectives related via semi-duality, due to the involutive negation, each pair being related to the other pair via residuation. We have shown that each pair of implication and conjunction connectives generates a 4-tuple of possibilistic set functions. One of the necessity functions is instrumental in comparing a fuzzy set A and a possibility distribution π ($N(A)$) and the other necessity function properly handles the possibilistic modeling of certainty qualified fuzzy propositions ($\mathcal{N}(A)$). This scheme can be extended to more general conjunction operations than the minimum.

It has been shown by Fodor [26] that the diagram of Fig. 5 still holds starting from any operation $*$ whenever the residuation operation is well-defined. So it makes sense to study the counterparts of the min-based qualification-based fuzzy set-functions of this subsection, replacing the minimum by a generalized conjunction $*$ already met in Subsection 3.3:

Definition 4. A conjunction $*$ from L^2 to L is a two-place function monotonically increasing in both places and that coincides with conjunction on $\{0, 1\}$.

In the following we also use the functions linked to conjunction by residuation and the semiduality:

Definition 5. The residuation of a operation $*$ is defined by $aRes(*)b = \sup\{c : a * c \le b\}$ and its semi-dual is defined by $a\mathcal{S}(*)b = 1 - a * (1 - b)$.

Namely, starting with the pair $(*, \mathcal{S}(*))$ made of a conjunction function $*$ and its associated semi-dual implication, we can also consider its residuated implication $Res(*) = \to_*$ and the conjunction $\mathcal{S}(\to_*) = \otimes_*$ that is the semi-dual of the latter. Fodor [26] has shown that if $Res(*)$ exists,[4] then, in the finite setting, $Res(\otimes_*) = \mathcal{S}(*)$. So we again are in the

[4] It requires that $*$ be left-continuous when $L = [0, 1]$.

presence of two pairs of conjunctions and implications. It is easy to see that $a \otimes_* b = 0$ whenever $a \leq 1 - b$ and $1 \otimes_* b = b$ if and only if $1 \to_* b = b$.

The pair $(*, \mathcal{S}(*))$ was studied in Subsection 3.3, assuming that $1 \otimes_* a = a * 1 = a$. Let us consider the pair $(\otimes_*, \to_*)$. We can generalize the definitions of $\mathcal{N}, \mathcal{P}, \mathcal{D}, \mathcal{C}$ as follows:

- $\mathcal{N}_*(A) = \inf_x (1 - A(x)) \to_* (1 - \pi(x)) = \inf_x 1 - (1 - A(x)) \otimes_* \pi(x)$
- $\mathcal{D}_*(A) = \inf_x A(x) \to_* \pi(x) = \inf_x 1 - A(x) \otimes_* (1 - \pi(x)) = 1 - \mathcal{N}_*(\overline{A})$
- $\mathcal{P}_*(A) = \sup_x A(x) \otimes_* \pi(x)$
- $\mathcal{C}_*(A) = \sup_x (1 - A(x)) \otimes_* (1 - \pi(x)) = 1 - \mathcal{D}_*(\overline{A})$

It can be checked that by construction:

Proposition 16. *The following equivalences hold:*

- $\mathcal{N}_*(A) \geq \alpha$ *if and only if* $\pi(x) \leq 1 - ((1 - A(x)) * \alpha)$
- $\mathcal{D}_*(A) \geq \alpha$ *if and only if* $\pi(x) \geq A(x) * \alpha$

Note that even if $1 * a = a * 1 = a$, we may not have that $1 \otimes_* a = a \otimes_* 1 = a$ as we can observe for $\otimes_G$.

A generalization of Proposition 6 can then considered for $\mathcal{N}_*, \mathcal{P}_*, \mathcal{D}_*, \mathcal{C}_*$, replacing $*$ by $\otimes_*$ in the definition of the possibility measure. It also generalizes Proposition 15 to other conjunctions than min.

Proposition 17. *Let $*$ be a conjunction and $\to_* = Res(*)$:*

- *If $\exists x, \pi(x) = 1$ and $(1 - a) \to_* 0 \leq 1 - (a \to_* 0)$ then $\mathcal{N}_*(A) \leq \mathcal{P}_*(A)$;*
- *if $\exists x, A(x) = 1$ and $1 \to_* (1 - a) \leq 1 - (1 \to_* a)$, then $\mathcal{D}_*(A) \leq \mathcal{P}_*(A)$;*
- *if $\exists x, A(x) = 0$ and $1 \to_* (1 - a) \leq 1 - (1 \to_* a)$, then $\mathcal{N}_*(A) \leq \mathcal{C}_*(A)$;*
- *if $\exists x, \pi(x) = 0$ and $(1 - a) \to_* 0 \leq 1 - (a \to_* 0)$, then $\mathcal{D}_*(A) \leq \mathcal{C}_*(A)$.*

Proof.

- $\mathcal{N}_* \leq \mathcal{P}_*$: we must prove $\inf_x (1 - A(x)) \to_* (1 - \pi(x)) \leq \sup_x A(x) \otimes_* \pi(x)$. It is sufficient to find x_0, x_1 such that $(1 - A(x_0)) \to_* (1 - \pi(x_0)) \leq A(x_1) \otimes_* \pi(x_1)$. Pick them such that $x_0 = x_1 = x$ and $\pi(x) = 1$. Then, it is enough that $(1 - A(x)) \to_* 0 \leq A(x) \otimes_* 1$, which holds by assumption, since $A(x) \otimes_* 1 = 1 - (A(x) \to_* 0)$.
- $\mathcal{D}_* \leq \mathcal{P}_*$: we must prove $\inf_x A(x) \to_* \pi(x) \leq \sup_x A(x) \otimes_* \pi(x)$. It is sufficient to find x_0, x_1 such that $A(x_0) \to_* \pi(x_0) \leq A(x_1) \otimes_* \pi(x_1)$. Pick them such that $x_0 = x_1 = x$ and $A(x) = 1$. Then, it is enough that $1 \to_* \pi(x) \leq 1 \otimes_* \pi(x)$, which holds by assumption, since $1 \otimes_* \pi(x) = 1 - (1 \to_* (1 - \pi(x)))$.
- $\mathcal{N}_* \leq \mathcal{C}_*$: $\inf_x (1 - A(x)) \to_* (1 - \pi(x)) \leq \sup_x (1 - A(x)) \otimes_* (1 - \pi(x))$. Choosing x with $A(x) = 0$, it is enough tho check that $1 \to_* (1 - \pi(x)) \leq 1 \otimes_* (1 - \pi(x))$, which holds by assumption, since $1 \otimes_* (1 - \pi(x)) = 1 - (1 \to_* \pi(x))$.
- $\mathcal{D}_* \leq \mathcal{C}_*$: $\inf_x A(x) \to_* \pi(x) \leq \sup_x (1 - A(x)) \otimes_* (1 - \pi(x))$. Again pick x with $\pi(x) = 0$. A sufficient condition is $A(x) \to_* 0 \leq (1 - A(x)) \otimes_* 1$, which holds by assumption, since $(1 - A(x)) \otimes_* 1 = 1 - (1 - A(x)) \to_* 0)$. □

The extra conditions on implication functions $\to_*$:

$$(1 - a) \to_* 0 \leq 1 - (a \to_* 0) \text{ and } 1 \to_* (1 - a) \leq 1 - (1 \to_* a) \tag{4}$$

seem to be new and deserve some comments. The former uses two ways of expressing a double negation. It holds whenever $a \to_* 0 = 1 - a$ but it is weaker. The second one is true if $1 \to_* a = a$, but is weaker as well. If $\to_* = \max(1 - a, b)$ or $\to_G$, then these conditions are satisfied. So the above proposition covers the results of both sections 3.3 and 3.4 in the case when $*$ is the minimum. Moreover, for any conjunction $*$ such that $1 \otimes_* a = a \otimes_* 1 = a$, the two conditions are verified by the semi-dual implication $1 - a * (1 - b)$. So, Proposition 6 is also a particular case of Proposition 17. The conditions (4) are also verified by residuated implications based on t-norms with continuous generators, especially the product and Łukasiewicz t-norms. In the case of Łukasiewicz t-norm, it is well-known that the diagram on Fig. 5 collapses, in the sense that $* = \otimes_*$ and $Res(*) = \mathcal{S}(*)$, so that $(N_*, \Pi_*, \Delta_*, \nabla_*) = (\mathcal{N}_*, \mathcal{P}_*, \mathcal{D}_*, \mathcal{C}_*)$.

In terms of conjunctions $\otimes_*$ semi-dual to $\to_*$, the inequalities (4) take a remarkable form as follows:

$$1 \otimes_* a \geq 1 - 1 \otimes_* (1 - a) \text{ and } a \otimes_* 1 \geq 1 - (1 - a) \otimes_* 1.$$

Under the conditions (4), by Proposition 17, we do preserve the inequality

$$\max(\mathcal{D}_*(A), \mathcal{N}_*(A)) \leq \min(\mathcal{C}_*(A), \mathcal{P}_*(A)),$$

so that we can be in a position to build graded cubes of opposition in possibility theory, within a rather general framework.

4. Graded cubes of opposition

In the previous section, we have proposed two versions of possibility theory extended to fuzzy events, motivated by different concerns, and we proved results that suggest an extension of the cube of opposition of Fig. 3 to many-valued propositions that could accommodate these generalized possibilistic set functions. In the following we define an abstract many-valued (graded) cube of opposition extending the one of Fig. 2 and prove that this is indeed the case. Graded squares and cubes of opposition were first defined in [21], and further studied in [8] in connection with fuzzy rough sets.

4.1. Definition of graded square and cube

In Section 2, the statements associated to the vertices of a square or of a cube of opposition are supposed to be true or false. Now we are going to consider that the validity of statements is a matter of a degree and is measured on an evaluation scale L, which is a bounded totally ordered scale with a top element 1 and a bottom element 0. Thus, we associate a degree in L to each vertex **A, E, O, I**, denoted by α, ϵ, o and ι respectively.

Let n be an involutive *negation function*, i.e., a decreasing bijection from L to itself such that $n(n(a)) = a$, $n(0) = 1, n(1) = 0$ (possibly different from $1-$). Consider also an implication function I, and a many-valued conjunction function C. A function $\oplus$ from L^2 to L defined by De Morgan duality as $D(a,b) = n(C(n(a), n(b)))$ is called a *disjunction function*. We introduce new notation, not to confuse with negation, implication and conjunction functions needed for defining possibility and necessity of fuzzy events.

The required properties needed to form a square of opposition should be now written as follows:

Definition 6. A graded square of opposition is defined by attaching L-valued variables α, ϵ, o and ι to each vertex **A, E, O, I** respectively, in such a way that:

(a) α and o are the negation of each other as well as ϵ and ι: $o = n(\alpha)$ and $\iota = n(\epsilon)$, for an involutive negation n.
(b) α entails ι and ϵ entails o; so, we need an implication I and write $I(\alpha,\iota) = 1$, and $I(\epsilon, o) = 1$.
(c) α and ϵ cannot be true together but may be false together; so, we need a conjunction C and write $C(\alpha,\epsilon) = 0$.
(d) ι and o cannot be false together but may be true together; so, we need to use a symmetrical disjunction D and write $D(\iota, o) = 1$.

In order to keep the other properties of the square: (a) (b) imply (c), (a) (b) imply (d), (a) (c) imply (b) and (d), (a) (d) imply (b) and (c); the negation, the implication, the conjunction and the disjunction must be linked. More precisely, under the condition: α and o are the negation each other as well as ϵ and ι, the following properties $I(\alpha,\iota) = 1$ and $I(\epsilon, o) = 1$, $C(\alpha,\epsilon) = 0$ and $D(\iota, o) = 1$ must be equivalent. One solution to this problem is as follows:

Proposition 18. *If the conjunction C and the implication I are semi-duals, and the disjunction D is the De-Morgan dual of C, then $\alpha\iota o\epsilon$ forms a graded square of opposition.*

Proof. By semi-duality, $I(\alpha,\iota) = 1$ is indeed equivalent to $n(C(\alpha, n(\iota))) = 1$, which is equivalent to $C(\alpha,\epsilon) = 0$, which is equivalent to $D(\iota, o) = 1$. Likewise, $I(\epsilon, o) = I(n(\iota), n(\alpha)) = n(C(n(\iota), \alpha)) = n(C(\epsilon, \alpha)) = n(0) = 1$. □

Note that we can also observe a graceful degradation of the square properties when $I(\alpha,\iota) < 1$, namely, it can be checked that the smaller $I(\alpha,\iota)$ (i.e. α increases while ι decreases), the greater $C(\alpha,\epsilon) = n(I(\alpha, n(\iota))$.

A particular case is proposed in [9] where the authors use a symmetrical conjunction (i.e., $C(a,b) = C(b,a)$).

Example 1 *([21])*. For instance, taking the min for the symmetric conjunction and the negation $1-$ on the scale $[0,1]$ one obtains the Kleene system leading to a square of opposition:

- $\alpha = 1 - o$, $\epsilon = 1 - \iota$
- $\max(1 - \alpha, \iota) = 1$, $\max(1 - \epsilon, o) = 1$ (using Kleene–Dienes implication).
- $\min(\alpha, \epsilon) = 0$
- $\max(\iota, o) = 1$

Other similar examples of graded squares are proposed in [9], especially one based on Łukasiewicz conjunction and implication.

Example 2. Let us consider $(L, 0, 1, \leq, n, \otimes)$ where n is an involutive negation, $\otimes$ a left conjunction, i.e., a conjunction such that $1 \otimes x = x$, $0 \otimes x = 0$.

We define the implication by semi-duality: $I(a,b) = n(a \otimes n(b))$.

We have $I(\alpha,\iota) = 1$ is equivalent to $\alpha \otimes n(\iota) = \alpha \otimes \epsilon = 0$.

We consider the dual $a \oplus b = n(n(a) \otimes n(b))$. We have that $I(\alpha,\iota) = 1$ entails $n(\alpha \otimes \epsilon) = 1$, i.e., $\iota \oplus o = 1$.

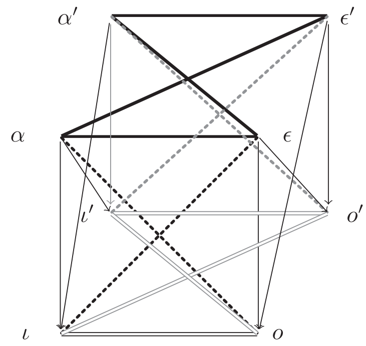

Fig. 6. Graded cube of opposition.

However, Proposition 6 shows that we do not need that C be symmetric. For instance, choosing $C(a,b) = \mathcal{S}(\rightarrow_*)$ where $\rightarrow_*$ is obtained from a conjunction $*$ by residuation also yields a graded square. The following example is new:

Example 3. For instance, taking the min for $*$ and the negation $n = 1 - (\cdot)$ on L, $\rightarrow_* = \rightarrow_G$, $C(a,b) = a \otimes_G b$, $D(a,b) = a \oplus_G b = 1 - (1-a) \otimes_G (1-b)$ one obtains the Gödel system leading to a square of opposition:

- $\alpha = 1 - o$, $\epsilon = 1 - \iota$
- $\alpha \rightarrow_G \iota = 1$, $\epsilon \rightarrow_G o = 1$ (using Gödel implication)
- $\alpha \otimes_G \epsilon = 0$
- $\iota \oplus_G o = 1$

Similarly the definition of the cube of opposition can be extended to the graded case using an involutive negation n and an implication I, and many-valued conjunction and disjunction C and D respectively:

Definition 7. A graded cube of opposition (Fig. 6) is defined by attaching L-valued variables α, ϵ, o, ι, α', ϵ', o' and ι' to each vertex **A, E, O, I, a, e, o, i** respectively, in such a way that:

- Front and back facets: $\alpha\iota\epsilon o$ and $\alpha'\iota'\epsilon' o'$ are squares of opposition in the sense of Definition 6,
- Side facets (entailments):
  (e) $I(\alpha, \iota') = 1$;
  (f) $I(\alpha'\iota) = 1$;
  (g) $I(\epsilon', o) = 1$;
  (h) $I(\epsilon, o') = 1$.
- Top and bottom facets:
  (i) α' and ϵ cannot be both true but may be false together, i.e., $C(\alpha', \epsilon) = 0$;
  (j) the same for α and ϵ', i.e., $C(\alpha, \epsilon') = 0$;
  (k) ι' and o cannot be false together but may be true together, i.e., $D(\iota', o) = 1$;
  (l) the same for ι and o', i.e., $D(\iota, o') = 1$.

Now we are in a position to prove the following results:

Proposition 19. *Let $\alpha, \iota, o, \epsilon, \alpha', \iota', o', \epsilon'$ be eight L-valued variables. Let n be an involutive negation, and I, C, D be many-valued implication, conjunction and disjunction respectively, such that*

1. *I and C are semi-dual to each other;*
2. $D(a,b) = n(C(n(a), n(b)))$.

Then $\alpha\iota o\epsilon\alpha'\iota' o'\epsilon'$ is a cube of opposition as soon as we have

- *the property (a) namely: $\alpha = n(o)$, $\epsilon = n(\iota)$, $\alpha' = n(o')$, $\epsilon' = n(\iota')$;*
- $I(\max(\alpha, \alpha'), \min(\iota, \iota')) = 1$.

Proof. The property $I(\max(\alpha, \alpha'), \min(\iota, \iota')) = 1$ implies $I(\alpha, \iota) = 1$, $I(\alpha, \iota') = 1$, $I(\alpha', \iota) = 1$ and $I(\alpha', \iota') = 1$. The entailment properties of the side facets of the cube entail the mutual exclusiveness properties of the top facet by semi-duality, as for the proof of Proposition 6. For instance, $I(\alpha, \iota') = 1$ implies $C(\alpha, \epsilon') = 0$. These properties on the top fact imply the properties of the bottom facet, using assumption 2. $\square$

The setting of Proposition 19 includes the case when the implication I is residuated with respect to some other conjunction $*$ and C is the semi-dual of the former. For instance, we can choose again $I = \to_G$, and $C(a, b) = \otimes_G$ and extend Example 3 to a cube. In this case we have that $I(a, b) = 1$ is equivalent to $a \le b$. We can also extend Example 1 to a cube as done in [8], and we still have that $I(a, b) = 1$ implies $a \le b$.

A particular system of implication and conjunction that satisfies Proposition 19 is Łukasiewicz system, where $C(a, b) = \max(a + b - 1, 0)$ and $I(a, b) = \min(1, 1 - a + b)$, with of course $n(a) = 1 - a$. See [9] for more details. In fact we can use any t-norm and its semi-dual implication, or any implication residuated from a t-norm, and its semi-dual implication.

In contrast, to get a graded square of opposition, one cannot use a structure of the form $(L, \ge, I, n, C)$ where I is the residuated implication of a strict triangular norm, $n(a) = I(a, 0)$, $C(a, b) = n(I(a, n(b)))$. Indeed, it is clear that $n(a) = 0$ as soon as $a > 0$, and $C(a, b) = n(I(a, n(b))) = n(I(a, 0)) = 1$ as soon as $b > 0$, and 0 otherwise. It means that if $\alpha > 0$ then one should have $\epsilon = o = 0$, which means that half of the square should be Boolean, even if the other half is not.

Besides, the associative combination of two squares or two cubes of opposition can be extended for graded squares and cubes.

Proposition 20. *Given two graded squares of opposition $\alpha_1\iota_1\epsilon_1 o_1$ and $\alpha_2\iota_2\epsilon_2 o_2$ then the functions* $\min(\alpha_1, \alpha_2)$, $\max(\iota_1, \iota_2)$, $\min(\epsilon_1, \epsilon_2)$ *and* $\max(o_1, o_2)$ *make another graded square of opposition.*

Proof. Property (a): $\max(o_1, o_2) = \max(n(\alpha_1), n(\alpha_2)) = n(\min(\alpha_1, \alpha_2))$ and $\max(\iota_1, \iota_2) = \max(n(\epsilon_1), n(\epsilon_2)) = n(\min(\epsilon_1, \epsilon_2))$.

Side edges: $I(\min(\alpha_1, \alpha_2), \max(\iota_1, \iota_2)) \ge I(\alpha_1, \iota_1) = 1$;

$$I(\min(\epsilon_1, \epsilon_2), \max(o_1, o_2)) \ge I(\epsilon_1, o_1) = 1.$$

Top edge: $C(\min(\alpha_1, \alpha_2), \min(\epsilon_1, \epsilon_2)) \le C(\alpha_1, \epsilon_1) = 1$.

Bottom edge: $D(\max(\iota_1, \iota_2), \max(o_1, o_2)) \ge D(\iota_1, o_1) = 0$. $\square$

Proposition 21. *Given two graded cubes of opposition of the form $\alpha_1\iota_1\epsilon_1 o_1\alpha'_1\iota'_1\epsilon'_1 o'_1$ and $\alpha_2\iota_2\epsilon_2 o_2\alpha'_2\iota'_2\epsilon'_2 o'_2$ then the functions* $\min(\alpha_1, \alpha_2)$, $\max(\iota_1, \iota_2)$, $\min(\epsilon_1, \epsilon_2)$, $\max(o_1, o_2)$ $\min(\alpha'_1, \alpha'_2)$, $\max(\iota'_1, \iota'_2)$, $\min(\epsilon'_1, \epsilon'_2)$, $\max(o'_1, o'_2)$ *make another graded cube of opposition.*

Proof. From the previous proposition, the functions $\min(\alpha_1, \alpha_2)$, $\max(\iota_1, \iota_2)$, $\min(\epsilon_1, \epsilon_2)$ $\max(o_1, o_2)$ and $\min(\alpha'_1, \alpha'_2)$, $\max(\iota'_1, \iota'_2)$, $\min(\epsilon'_1, \epsilon'_2)$ $\max(o'_1, o'_2)$ form front and back facets of squares of opposition. Moreover,

$$I(D(\min(\alpha_1, \alpha_2), \min(\alpha'_1, \alpha'_2)), C(\max(\iota_1, \iota_2), \max(\iota'_1, \iota'_2))) \ge I(D(\alpha_1, \alpha'_1), C(\iota_1, \iota'_1)) = 1.$$

So due to Proposition 19, we have defined a new graded cube of opposition. $\square$

4.2. Cubes of opposition in graded possibility theory

Now we are going to consider the case when the statements associated to the vertices of a cube of opposition are the four set functions of possibility theory, for crisp events, as per subsection 3.2, and also extended to fuzzy events, in each of the two forms that make sense, as reviewed in the previous section.

In the following, we consider a set X, a scale L, a negation $1-$ on L, a possibility distribution $\pi : X \to L$.

Crisp events. Let us extend the cube of opposition for 0–1 possibility theory on Fig. 3 to graded qualitative possibility L-valued degrees. When we consider crisp events, the (internal) conjunction defining the possibility function is the minimum and the (internal) implication associated by semi-duality is Kleene–Dienes implication $a \to_{KD} b = \max(1 - a, b)$. If the distributions π and $\overline{\pi} = 1 - \pi$ are normalized (there exists x and x' such that $\pi(x) = 1$ and $\pi(x') = 0$) then we obtain a possibilistic cube of opposition in the sense of Proposition 19, using the following external connectives:

- $n(a) = 1 - a$.
- The entailment properties on the side facets are expressed by means of the Gödel implication, in the form $a \to_G b = 1$ thus expressing inequalities $a \le b$.
- The conjunction defining the constraints on the top facets is the semi-dual of Gödel implication, $\otimes_G$.
- The disjunction defining the constraints on the bottom facets is the De Morgan dual $\oplus_G$ of $\otimes_G$.

We cannot apply the conditions of Proposition 19 to Fig. 3, using Kleene–Dienes implication for external implication, because on the side facets, even if we have that $\max(1 - N(A), \Pi(A)) = 1$, we do not have that $\max(1 - \Delta(A), \Pi(A)) = 1$, since we only have $0 < \Delta(A) \le \Pi(A) < 1$, and likewise $0 < N(A) \le \nabla(A) < 1$.

Likewise, for the constraints on the top facet, we do have that $\min(N(A), N(\overline{A})) = 0$, but generally, for instance, $\min(N(A), \Delta(\overline{A})) \neq 0$. However we do have that $N(A) \le 1 - \Delta(\overline{A})$, so that $N(A) \otimes_G \Delta(\overline{A}) = N(A) \otimes_G N(\overline{A}) = 0$. By duality, we do get that $\Pi(A) \oplus_G \nabla(\overline{A}) = \Pi(A) \oplus_G \Pi(\overline{A}) = 1$ for the bottom facet.

We thus can state the following result.

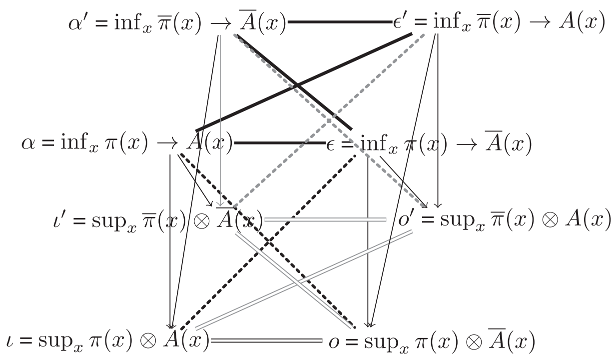

Fig. 7. Cube of weighted qualitative aggregations.

Proposition 22. *The qualitative graded version of the possibilistic cube of opposition with crisp events forms a graded cube of opposition in the sense of Definition 7 in agreement with Proposition 19, letting*

$$(\alpha, \epsilon, \iota, o, \alpha', \epsilon', \iota', o') = (N(A), N(\overline{A}), \Pi(A), \Pi(\overline{A}), \Delta(A), \Delta(\overline{A}), \nabla(A), \nabla(\overline{A}))$$

when we choose external connectives as $n = 1 - (\cdot)$, $I = \to_G$, $C = \otimes_G$, $D = \oplus_G$.

Note that the above cube is purely qualitative, and is valid on any bounded chain L equipped with residuation and an involutive negation. However, if we choose $L = [0, 1]$ and Łukasiewicz implication and conjunctions to define the cube, then we also obtain a cube of opposition in the sense of Proposition 19, a remark already done in [8]. Moreover this graded cube of opposition directly extends the classical cube of opposition with universal and existential quantifiers, for instance the degree $N(A)$ extends the classical logic statement $\forall x, Q(x) \to P(x)$, where the extension of Q is modeled by π and A is the extension of P. In consequence our graded cube differs from the spirit of the extended square of opposition with intermediate quantifiers, different from existential and universal ones, due to Murinová and Novák [30].

Fuzzy events. Consider a membership function $A : X \to L$ representing a fuzzy event, and we let $\overline{\pi} = 1 - \pi$, $\overline{A} = 1 - A$. We can define a cube with vertices attached to possibilistic set-functions based on an implication $\to$ and a conjunction $\otimes$ linked by semi-duality: $a \to b = 1 - a \otimes (1 - b)$. Namely, we let

$$(\alpha, \epsilon, \iota, o, \alpha', \epsilon', \iota', o') = (N_\otimes(A), N_\otimes(\overline{A}), \Pi_\otimes(A), \Pi_\otimes(\overline{A}), \Delta_\otimes(A), \Delta_\otimes(\overline{A}), \nabla_\otimes(A), \nabla_\otimes(\overline{A}))$$

where the expressions are given on Fig. 7. Clearly we can check property (*a*): $\alpha = n(o)$, $\epsilon = n(\iota)$, $\alpha' = n(o')$ and $\epsilon' = n(\iota')$ with $n(a) = 1 - a$, which expresses duality between $N_\otimes$ and $\Pi_\otimes$, $\Delta_\otimes$ and $\nabla_\otimes$. In order to ensure the inequality

$$\max[N_\otimes(A), \Delta_\otimes(A)] \le \min[\Pi_\otimes(A), \nabla_\otimes(A)],$$

we need to assume the normalization conditions: $\exists x \ \pi(x) = 1$, $\exists x \ \pi(x) = 0$, $\exists x \ A(x) = 1$, $\exists x \ A(x) = 0$, and conditions $1 \otimes a \ge 1 - 1 \otimes (1 - a)$ and $a \otimes 1 \ge 1 - (1 - a) \otimes 1$, as shown in Proposition 17. Provided that these conditions are fulfilled by the internal conjunction, it is obvious, due to Proposition 19, that $\max(\alpha, \alpha') \le \min(\iota, \iota') = 1$, so that the cube of Fig. 7 is a cube of opposition as soon as we choose external operations $(1-, I, C, D)$ such that if $a \le b$ then $I(a, b) = 1$, C is the semi-dual of I, and D the De Morgan dual of C for $n = 1-$.

We thus can state the following result.

Proposition 23. *The qualitative graded possibilistic cube with fuzzy events defined in Fig. 7 forms a graded cube of opposition in the sense of Definition 7 in agreement with Proposition 19, if the normalization of* $A, \overline{A}, \pi, 1 - \pi$ *holds, whenever we choose internal negation* $1-$, *internal conjunction* $\otimes$ *such that* $1 \otimes a \ge 1 - 1 \otimes (1 - a)$ *and* $a \otimes 1 \ge 1 - (1 - a) \otimes 1$, *internal implication* $\mathcal{S}(\otimes)$ *and external connectives* $n = 1 - (\cdot)$, $I = \to_* = Res(*)$, $C = \mathcal{S}(\to_*)$, *and* D *the De Morgan dual of* C *for* $n = 1-$, *where* $*$ *is any conjunction on* L *such that* $a * b \le \min(a, b)$.

Proof. Suffices to remark, that we do have $a \le b$ if and only if $a \to_* b = 1$. The remaining follows from Proposition 19. □

The cube of opposition in Fig. 7 subsumes the special frameworks for the possibility of fuzzy events in the previous section and accommodates the general setting of Proposition 17. Namely, Proposition 23 applies to the following cases encountered earlier:

- In the setting of Zadeh's max–min possibility theory, in the sense of Subsection 3.3, we use $a \to b = \max(1 - a, b)$ and the conjunction associated by semi-duality is: $a \otimes b = \min(a, b)$, and obtain a particular case of the cube of opposition in Fig. 7, in agreement with Proposition 19, for instance choosing for external connectives $I = \to_G = Res(*)$, $C = \otimes_G$ to remain qualitative.

- In the case of possibility functions induced by the qualification problem studied in Subsection 3.4, necessity measures are defined with the contrapositive symmetric of Gödel implication $a \to b = (1-b) \to_G (1-a)$ and the potential possibility measure with the conjunction associated by semi-duality $\otimes = \otimes_G$. The actual possibility measure Δ is defined using Gödel implication. If we have the normalization conditions: $\exists x\ \pi(x) = 1$, $\exists x\ \pi(x) = 0$, $\exists x\ A(x) = 1$, $\exists x\ A(x) = 0$, then we have a cube of opposition for the set functions $\mathcal{N}, \mathcal{P}, \mathcal{D}, \mathcal{C}$, another particular case of the cube in Fig. 7, for instance choosing for external connectives $I = \to_G$, $C = \otimes_G$ to remain qualitative.

Let us complete this subsection with a remark concerning the combination of graded cubes. According to Proposition 21 when we combine two cubes we consider the minimum for the vertices present in the top facet and the maximum for the vertices present in the bottom facets. This combination exactly fits with the aggregation of the four set functions of possibility theory as recalled at the end of subsection 3.2, since these aggregations preserve the nature of these set-functions.

5. Conclusion

The contributions of this paper are twofold. On the one hand, we have pointed out that the cube of opposition in the Boolean case is generated by four propositional variables leading to seven distinct worlds, just as the square of opposition is based on a tri-partition [20]. Moreover, we have established that a Boolean cube of opposition is entirely defined by the diagonal properties of the front and back facets, together with a condition summarizing the relations that hold in the side facets (equivalently, the mutual exclusion properties on the top facet). We have also defined conditions that define a general graded cube of opposition. It is important to notice the key-role of pairs of semi-dual implication and conjunction operations with respect to an involutive negation, that seem to be instrumental in order to keep the basic properties of squares and cubes of opposition. Namely, one could not get a proper opposition structure using a residuated implication such as Gödel implication along with its associated negation of the form $a \to_G 0$, as the latter is not involutive, which would destroy the symmetry of the square and cube structures.

On the other hand, we have provided a general algebraic setting that subsume two extensions of the four set functions of possibility theory in case of fuzzy events, and shown that they are compatible with a general graded extension of the cube of opposition, provided that some rather loose conditions on the conjunction defining the possibility of fuzzy events are satisfied. These conditions are satisfied by a large collection of possibly non-commutative conjunctions such as the semi-dual of the Gödel implication, and would deserve further study. This graded extension of the cube seems to include as well the one that is at work for Sugeno integrals, set functions of Shafer evidence theory (for crisp events) [22], or Choquet integrals [23] (using Łukasiewicz connectives). Besides, it should be clear that the algebraic structure laid bare in this paper for the graded cube of opposition and possibility theory for fuzzy events can be directly applied to existing preliminary results concerning the graded square of fuzzy rough sets [8] and the fuzzy relational calculus [9].

Acknowledgements

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