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Admissibility and unifiability in contact logics

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Abstract. Contact logics are logics for reasoning about the contact relations between regular subsets in a topological space. Admissible inference rules can be used to improve the performance of any algorithm that handles provability within the context of contact logics. The decision problem of unifiability can be seen as a special case of the decision problem of admissibility. In this paper, we examine the decidability of admissibility problems and unifiability problems in contact logics.

Keywords: Contact logics, admissibility, unifiability, decidability.

1 Introduction

The decision problem of unifiability in a logical system $L$ can be formulated as follows: given a formula $\phi(X_1, \ldots, X_n)$, determine whether there exists formulas $\psi_1, \ldots, \psi_n$ such that $\phi(\psi_1, \ldots, \psi_n) \in L$. The research on unifiability was motivated by a more general decision problem, the admissibility problem: given an inference rule “from $\{\phi_1(X_1, \ldots, X_n), \ldots, \phi_m(X_1, \ldots, X_n)\}$, infer $\psi(X_1, \ldots, X_n)$”, determine whether for all formulas $\chi_1, \ldots, \chi_n$, if $\{\phi_1(\chi_1, \ldots, \chi_n), \ldots, \phi_m(\chi_1, \ldots, \chi_n)\} \subseteq L$, then $\psi(\chi_1, \ldots, \chi_n) \in L$. In 1984, Rybakov [15] proved that there exists a decision procedure for determining whether a given inference rule is admissible in intuitionistic propositional logic. See also [16]. Later on, Ghilardi [11, 12] proved that intuitionistic propositional logic has a finitary unification type and extended this result to various extensions of $K4$. See also [9, 10] where decision procedures for unifiability in extensions of $K4$ are suggested.

Contact logics are logics for reasoning about the contact relations between regular subsets in a topological space [5, 17]. They are based on the primitive notion of regular regions and on the Boolean operations (empty region, complement of a region and union of two regions) that allow to obtain new regular regions from given ones. In contact logics, formulas are built from simple formulas of the form $C(a, b)$ and $a \equiv b$ — where $a$ and $b$ are terms in a Boolean language — using the Boolean constructs $\bot$, $\neg$ and $\lor$, the intuitive reading of $C(a, b)$ and $a \equiv b$ being “the regular regions denoted by $a$ and $b$ are in contact” and “the regular regions...
denoted by \( a \) and \( b \) are equal”. The main semantics of contact logics are the contact algebras of the regular subsets in a topological space [6–8]. But contact logics have also received a relational semantics that allow to use methods from modal logic for studying them [4].

In this setting, one important issue is the mechanization of reasoning in contact logics. Since admissible inference rules can be used to improve the performance of any algorithm that handles provability, it becomes natural to consider admissibility and unifiability within the context of contact logics. In this paper, we will examine variants of contact logics. The central result in this paper is the proof that the admissibility problem and the unifiability problem are decidable in these variants. In Section 2, we present the syntax and the semantics of these variants. Section 3 is about their axiomatization/completeness and their decidability/complexity. In Sections 4–6, we define the admissibility problem and we study its decidability. Section 7 is about the unifiability problem and its decidability. See [16] for details about admissibility and unifiability and [17] for details about contact logics.

2 Syntax and semantics of contact logics

In this section, we present the syntax and the semantics of contact logics. We adopt the standard rules for omission of the parentheses.

2.1 Syntax

To start with syntax, let us first consider a countable set \( AT \) of atomic terms (with typical members denoted \( x, y \), etc) and a countable set \( AF \) of atomic formulas (with typical members denoted \( X, Y \), etc). The terms (denoted \( a, b \), etc) are inductively defined as follows:

\[
\begin{align*}
-a & := x \mid 0 \mid -a \mid (a \sqcup b).
\end{align*}
\]

The other Boolean constructs for terms (1, \( \sqcap \), etc) are defined as usual. We will use the following notations:

\[
\begin{align*}
-\neg a & \text{ for } -a, \\
\neg a & \text{ for } a.
\end{align*}
\]

Reading terms as regions, the constructs 0, \(-\) and \( \sqcup \) should be regarded as the empty region, the complement operation and the union operation. For all positive integers \( n \) and for all \((\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n\), formulas of the form \( x_1^{\epsilon_1} \cap \ldots \cap x_n^{\epsilon_n} \) will be called monoms. In the sequel, we use \( a(x_1, \ldots, x_n) \) to denote a term \( a \) whose atomic terms form a subset of \( \{x_1, \ldots, x_n\} \). Considering \( a(x_1, \ldots, x_n) \) as a formula in classical propositional logic, let \( \text{mon}(a(x_1, \ldots, x_n)) \) be the set of all monoms of the form \( x_1^{\epsilon_1} \cap \ldots \cap x_n^{\epsilon_n} \) inconsistent with \( -a(x_1, \ldots, x_n) \), that is to say \( \text{mon}(a(x_1, \ldots, x_n)) = \{x_1^{\epsilon_1} \cap \ldots \cap x_n^{\epsilon_n} : (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n \text{ and } a(x_1, \ldots, x_n) \text{ is a tautological consequence of } x_1^{\epsilon_1} \cap \ldots \cap x_n^{\epsilon_n}\} \). The formulas (denoted \( \phi, \psi \), etc) are inductively defined as follows:
\(- \phi := X \mid \bot \mid \neg \phi \mid (\phi \lor \psi) \mid C(a,b) \mid a \equiv b.\)

The other Boolean constructs for formulas (\(\top, \wedge, \text{etc}\)) are defined as usual. We will use the following notations:

- \(\not C(a,b)\) for \(\neg C(a,b)\),
- \(a \not\equiv b\) for \(\neg a \equiv b\),
- \(a \leq b\) for \(a \sqcap b \equiv 0\).

Reading formulas as properties about regions, the constructs \(C\) and \(\equiv\) should be regarded as the contact relation and the equality relation. Sets of formulas will be denoted \(\Gamma, \Delta, \text{etc}\). Formulas and sets of formulas are also called “expressions” (denoted \(\alpha, \beta, \text{etc}\).)

We shall say that an expression \(\alpha\) is weak iff no atomic formula occurs in \(\alpha\). In the sequel, we use \(\alpha(x_1, \ldots, x_n)\) to denote a weak expression \(\alpha\) whose atomic terms form a subset of \(\{x_1, \ldots, x_n\}\). A substitution is a function \(s\) assigning to each atomic term \(x\) a term \(s(x)\) and to each atomic formula \(X\) a formula \(s(X)\). As usual, \(s\) induces a homomorphism \(s(\cdot)\) assigning to each term \(a\) a term \(s(a)\) and to each expression \(\alpha\) an expression \(s(\alpha)\).

### 2.2 Semantics

Now, for the semantics. In [5, 17], the language of contact logics is interpreted either in relational structures, or in topological structures. In both cases, terms are interpreted by sets of points. The main difference between the two kinds of structures is the following: in relational structures, two regions are in contact when at least one point of the first region is related to at least one point of the second region whereas in topological structures, two regions are in contact when their topological closures have a nonempty intersection. The two semantics have been proved to be equivalent [5, 17]. In this paper, we only consider the relational semantics. A frame is a relational structure \(F = (W, R)\) where \(W\) is a non-empty set of points and \(R\) is a binary relation on \(W\). A valuation based on \(F\) is a function \(V\) assigning to each atomic term \(x\) a subset \(V(x)\) of \(W\). \(V\) induces a function \((\cdot)^V\) assigning to each term \(a\) a subset \((a)^V\) of \(W\) such that

- \((x)^V = V(x),\)
- \((\emptyset)^V = \emptyset,\)
- \((-a)^V = W \setminus (a)^V,\)
- \((a \sqcup b)^V = (a)^V \cup (b)^V.\)

As a result,

- \((a^0)^V = W \setminus (a)^V,\)
- \((a^1)^V = (a)^V.\)

We shall say that \(V\) is balanced iff for all terms \(a\), either \((a)^V = \emptyset\), or \((a)^V = W\), or \((a)^V\) is infinite and cocointfinite. An interpretation is a subset \(I\) of \(AF\). A model is a structure \(M = (W, R, V, I)\) where \(F = (W, R)\) is a frame, \(V\) is a valuation based on \(F\) and \(I\) is an interpretation. The satisfiability of a formula \(\phi\) in \(M\), in symbols \(M \models \phi\), is defined as follows:
\[ M \models X \text{ iff } X \in I, \]
\[ M \not\models \perp, \]
\[ M \models \neg \phi \text{ iff } M \not\models \phi, \]
\[ M \models \phi \lor \psi \text{ iff either } M \models \phi, \text{ or } M \models \psi, \]
\[ M \models C(a, b) \text{ iff } ((a)^V \times (b)^V) \cap R \neq \emptyset, \]
\[ M \models a \equiv b \text{ iff } (a)^V = (b)^V. \]

As a result,
\[ M \models C(a, b) \text{ iff } ((a)^V \times (b)^V) \cap R \neq \emptyset, \]
\[ M \models a \neq b \text{ iff } (a)^V \neq (b)^V, \]
\[ M \models a \leq b \text{ iff } (a)^V \subseteq (b)^V. \]

Let \( \mathcal{F} \) be a frame. A formula \( \phi \) is valid in \( \mathcal{F} \), in symbols \( \mathcal{F} \models \phi \), iff for all models \( M \) based on \( \mathcal{F} \), \( M \models \phi \). A set \( \Gamma \) of formulas is valid in \( \mathcal{F} \), in symbols \( \mathcal{F} \models \Gamma \), iff for all formulas \( \phi \in \Gamma \), \( \mathcal{F} \models \phi \). Let \( CF \) be a class of frames. A formula \( \phi \) is valid in \( CF \), in symbols \( CF \models \phi \), iff for all frames \( \mathcal{F} \) in \( CF \), \( \mathcal{F} \models \phi \). Let \( CF_0 \) be the class of all frames. Obviously,

**Proposition 1.** The following formulas are valid in \( CF_0 \):
\[ - C(x, y) \rightarrow x \neq 0, \]
\[ - C(x, y) \rightarrow y \neq 0, \]
\[ - C(x, y) \land x \leq z \rightarrow C(z, y), \]
\[ - C(x, y) \land y \leq z \rightarrow C(x, z), \]
\[ - C(x \sqcup y, z) \rightarrow C(x, z) \lor C(y, z), \]
\[ - C(x, y \sqcup z) \rightarrow C(x, y) \lor C(x, z). \]

In this paper, we will consider the following classes of frames:
- the class \( CF_r \) of all reflexive frames,
- the class \( CF_s \) of all symmetrical frames.

Obviously,

**Proposition 2.** The following formula is valid in \( CF_r \):
\[ - x \neq 0 \rightarrow C(x, x). \]

The following formula is valid in \( CF_s \):
\[ - C(x, y) \rightarrow C(y, x). \]

### 3 Axiomatization and decidability of contact logics

In this section, we present the axiomatization and the decidability of contact logics. From now on, formulas will also be called “axioms” and pairs of the form \( (\Gamma, \phi) \) where \( \Gamma \) is a finite set of formulas and \( \phi \) is a formula will also be called “inference rules”. When an axiom or an inference rule contains no occurrence of atomic formulas, it is qualified as “weak”. An axiomatic system consists of a collection of axioms and a collection of inference rules. Let \( \lambda_0 \) be the axiomatic system consisting of
– a complete set of axioms for Classical Propositional Calculus (i.e. \( X \rightarrow (Y \rightarrow X) \), \( (X \rightarrow (Y \rightarrow Z)) \rightarrow ((X \rightarrow Y) \rightarrow (X \rightarrow Z)) \), etc),
– a complete set of axioms for non-degenerate Boolean algebras (i.e. \( x \sqcup (y \sqcup z) \equiv (x \sqcup y) \sqcup z, x \sqcup y \equiv y \sqcup x \), etc),
– the following axioms:
  • \( C(x, y) \rightarrow x \neq 0 \),
  • \( C(x, y) \rightarrow y \neq 0 \),
  • \( C(x, y) \land x \leq z \rightarrow C(z, y) \),
  • \( C(x, y) \land y \leq z \rightarrow C(x, z) \),
  • \( C(x \sqcup y, z) \rightarrow C(x, z) \lor C(y, z) \),
  • \( C(x, y \sqcup z) \rightarrow C(x, y) \lor C(x, z) \),
– the inference rule of modus ponens (i.e. \( \{ X, X \rightarrow Y \}, Y \)).

We will consider extensions of \( \lambda_0 \) — denoted \( \lambda, \mu \), etc — by either adding new axioms, or adding new inference rules. The extension of \( \lambda_0 \) with a set \( A \) of axioms will be denoted \( \lambda_0(A) \). The extension of \( \lambda_0 \) with a single axiom \( \phi \) will be denoted \( \lambda_0(\phi) \). In this paper, we will consider the following extensions of \( \lambda_0 \):

– \( \lambda_r = \lambda_0(x \neq 0 \rightarrow C(x, x)) \),
– \( \lambda_s = \lambda_0(C(x, y) \rightarrow C(y, x)) \).

The extension of \( \lambda_0 \) with a single inference rule \( (\Gamma, \phi) \) will be denoted \( \lambda_0 + (\Gamma, \phi) \). A formula \( \phi \) is said to be derivable in an extension \( \lambda \) of \( \lambda_0 \) from a finite set \( \Gamma \) of formulas, in symbols \( \Gamma \vdash_{\lambda} \phi \), if there exists a finite sequence \( \phi_0, \ldots, \phi_m \) of formulas such that \( \phi_m = \phi \) and for all nonnegative integers \( i \), if \( i \leq m \), then at least one of the following conditions holds:

– \( \phi_i \in \Gamma \),
– there exists an axiom \( \psi \) in \( \lambda \) and there exists a substitution \( s \) such that \( \phi_i = s(\psi) \),
– there exists an inference rule \( (\Delta, \psi) \) in \( \lambda \) and there exists a substitution \( s \) such that \( \phi_i = s(\psi) \) and \( \{ \phi_0, \ldots, \phi_{i-1} \} \supseteq s(\Delta) \).

The finite sequence \( \phi_0, \ldots, \phi_m \) is called “derivation of \( \phi \) in \( \lambda \) from \( \Gamma \)”. The propositions below contain facts which can be found in most elementary logic texts.

**Proposition 3.** Let \( \Gamma \) be a finite set of formulas and \( \phi \) be a formula. If \( \Gamma \vdash_{\lambda} \phi \), then for all substitutions \( s \), \( s(\Gamma) \vdash_{\lambda} s(\phi) \).

**Proposition 4.** Let \( \Gamma \) be a finite set of formulas and \( \phi, \psi \) be formulas. The following conditions are equivalent:

– \( \Gamma \cup \{ \phi \} \vdash_{\lambda} \psi \),
– \( \Gamma \vdash_{\lambda} \phi \rightarrow \psi \).

A formula \( \phi \) is said to be provable in \( \lambda \), in symbols \( \vdash_{\lambda} \phi \), iff \( \emptyset \vdash_{\lambda} \phi \). In this case, every derivation of \( \phi \) in \( \lambda \) from \( \emptyset \) is called “proof of \( \phi \) in \( \lambda \)”. The provable formulas of \( \lambda \) will be called “theorems of \( \lambda \)”. We will denote by \( Th(\lambda) \) the set of
all theorems of \( \lambda \). We shall say that \( \lambda \) is consistent iff \( \bot \not\in \text{Th}(\lambda) \). We will denote by \( CF(\lambda) \) the class of all frames \( \mathcal{F} \) such that \( \mathcal{F} \models \text{Th}(\lambda) \) and we will denote by \( CF_{\text{fin}}(\lambda) \) the class of all finite frames \( \mathcal{F} \) such that \( \mathcal{F} \models \text{Th}(\lambda) \). We shall say that \( \lambda \) is balanced iff for all formulas \( \phi \), the following conditions are equivalent:

- \( \phi \not\in \text{Th}(\lambda) \),
- there exists a countable frame \( \mathcal{F} \in CF(\lambda) \), there exists a balanced valuation \( V \) on \( \mathcal{F} \) and there exists an interpretation \( I \) such that \( (\mathcal{F}, V, I) \not\models \phi \).

\( \lambda_0 \) itself is balanced, but also most extensions of \( \lambda_0 \) considered in [5, 17] like \( \lambda_r \) and \( \lambda_s \) are balanced. In [5, 17], one can also find the facts contained in the following

**Proposition 5.** Let \( \phi \) be a formula. The following conditions are equivalent:

- \( \phi \in \text{Th}(\lambda_0) \),
- \( CF_0 \models \phi \).

**Proposition 6.** Let \( \phi \) be a formula. The following conditions are equivalent:

- \( \phi \in \text{Th}(\lambda_r) \),
- \( CF_r \models \phi \).

The following conditions are equivalent:

- \( \phi \in \text{Th}(\lambda_s) \),
- \( CF_s \models \phi \).

More generally,

**Proposition 7.** Let \( \phi \) be a formula. If there exists a finite set \( A \) of axioms such that \( \lambda = \lambda_0(A) \), then the following conditions are equivalent:

- \( \phi \in \text{Th}(\lambda) \),
- \( CF(\lambda) \models \phi \),
- \( CF_{\text{fin}}(\lambda) \models \phi \).

A consequence of Proposition 7 is the following

**Proposition 8.** If there exists a finite set \( A \) of axioms such that \( \lambda = \lambda_0(A) \), then \( \text{Th}(\lambda) \) is decidable.

Later on, we will use Propositions 3–8 without explicit reference.

### 4 Admissibility: definitions

Let \( \lambda \) be an extension of \( \lambda_0 \). An inference rule \( (\Gamma, \phi) \) is said to be admissible in \( \lambda \) iff for all substitutions \( s \), if \( s(\Gamma) \subseteq \text{Th}(\lambda) \), then \( s(\phi) \in \text{Th}(\lambda) \). The next proposition indicates that inference rules admissible in \( \lambda \) do not increase \( \text{Th}(\lambda) \) when added to \( \lambda \).
Proposition 9. Let \((\Gamma, \phi)\) be an inference rule. If \((\Gamma, \phi)\) is admissible in \(\lambda\), then \(Th(\lambda + (\Gamma, \phi)) = Th(\lambda)\).

Proof. Suppose \((\Gamma, \phi)\) is admissible in \(\lambda\). If \(Th(\lambda + (\Gamma, \phi)) \neq Th(\lambda)\), then obviously, there exists a formula \(\psi\) such that \(\psi \in Th(\lambda + (\Gamma, \phi))\) and \(\psi \notin Th(\lambda)\). Hence, there exists a proof \(\psi_0, \ldots, \psi_m\) of \(\psi\) in \(\lambda + (\Gamma, \phi)\). Since \((\Gamma, \phi)\) is admissible in \(\lambda\), each use of \((\Gamma, \phi)\) in \(\psi_0, \ldots, \psi_m\) can be replaced by a corresponding proof in \(\lambda\). Thus, there exists a proof of \(\psi\) in \(\lambda\). Therefore, \(\psi \in Th(\lambda)\): a contradiction.

Inference rules that are admissible in \(\lambda\) can be used to improve the performance of any algorithm that handles \(\lambda\)-provability. In this respect, the following decision problem, called “admissibility problem in \(\lambda^*\)”, in symbols \(ADM(\lambda)\), is of the utmost importance:

- input: an inference rule \((\Gamma, \phi)\),
- output: determine whether \((\Gamma, \phi)\) is admissible in \(\lambda\).

Applicability of inference rules that are admissible in \(\lambda\) to ameliorate algorithms for \(\lambda\)-provability incites us to study the decidability of \(ADM(\lambda)\). To start this study, let us first define the notion of derivability in \(\lambda\). We shall say that an inference rule \((\Gamma, \phi)\) is derivable in \(\lambda\) iff \(\Gamma^* \vdash_\lambda \phi\). It happens that derivability is a special case of admissibility.

Proposition 10. Let \((\Gamma, \phi)\) be an inference rule. If \((\Gamma, \phi)\) is derivable in \(\lambda\), then \((\Gamma', \phi')\) is admissible in \(\lambda\).

Proof. Suppose \((\Gamma', \phi')\) is derivable in \(\lambda\). If \((\Gamma', \phi')\) is not admissible in \(\lambda\), then there exists a substitution \(s\) such that \(s(\Gamma) \subseteq Th(\lambda)\) and \(s(\phi) \notin Th(\lambda)\). Since \((\Gamma', \phi')\) is derivable in \(\lambda\), \(\Gamma' \vdash_\lambda \phi\). Hence, \(s(\Gamma) \vdash_\lambda s(\phi)\). Thus, there exists a derivation \(\phi_0, \ldots, \phi_m\) of \(s(\phi)\) in \(\lambda\) from \(s(\Gamma)\). Since \(s(\Gamma) \subseteq Th(\lambda)\), each use of \(s(\Gamma)\) in \(\phi_0, \ldots, \phi_m\) can be replaced by a corresponding proof in \(\lambda\). Therefore, there exists a proof of \(s(\phi)\) in \(\lambda\). Consequently, \(s(\phi) \in Th(\lambda)\): a contradiction.

Nevertheless, in the general case, it may happen that derivability and admissibility in such-or-such contact logic do not coincide. It suffices, for instance, to consider the inference rule \(\{(C(x, y)), C(y, x)\}\). Since for all substitutions \(s\), \(s(C(x, y)) \notin Th(\lambda_0)\), \(\{(C(x, y)), C(y, x)\}\) is admissible in \(\lambda_0\). Since \(C(x, y) \rightarrow C(y, x) \notin Th(\lambda_0)\), \(\{(C(x, y)), C(y, x)\}\) is not derivable in \(\lambda_0\). As a result, the following decision problem, called “derivability problem in \(\lambda^*\)”, in symbols \(DER(\lambda)\), has its importance:

- input: an inference rule \((\Gamma, \phi)\),
- output: determine whether \((\Gamma, \phi)\) is derivable in \(\lambda\).

Obviously,

Proposition 11. If there exists a finite set \(A\) of axioms such that \(\lambda = \lambda_0(A)\), then \(DER(\lambda)\) is decidable.
We shall say that $\lambda$ is structurally complete iff for all inference rules $(\Gamma, \phi)$, if $(\Gamma, \phi)$ is admissible in $\lambda$, $(\Gamma, \phi)$ is derivable in $\lambda$. By Propositions 10 and 11,

**Proposition 12.** If $\lambda$ is structurally complete and there exists a finite set $A$ of axioms such that $\lambda = \lambda_0(A)$, then $\text{ADM}(\lambda)$ is decidable.

Now, we intend to extend Proposition 12 to structurally incomplete extensions of $\lambda_0$. However, in this paper, we will only be able to study the decidability of the following decision problem, called “weak admissibility problem in $\lambda$”, in symbols $\text{wADM}(\lambda)$:

- input: a weak inference rule $(\Gamma, \phi)$,
- output: determine whether $(\Gamma, \phi)$ is admissible in $\lambda$.

We end this section with the following

**Proposition 13.** Let $(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ be a weak inference rule. The following conditions are equivalent:

- $(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ is not admissible in $\lambda$,
- there exists terms $a_1, \ldots, a_n$ such that $\Gamma(a_1, \ldots, a_n) \subseteq \text{Th}(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin \text{Th}(\lambda)$.

**Proof.** $(\Rightarrow)$ Suppose $(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ is not admissible in $\lambda$. Hence, there exists a substitution $s$ such that $s(\Gamma(x_1, \ldots, x_n)) \subseteq \text{Th}(\lambda)$ and $s(\phi(x_1, \ldots, x_n)) \notin \text{Th}(\lambda)$. Let $a_1, \ldots, a_n$ be terms such that for all positive integers $i$, if $i \leq n$, then $a_i = s(x_i)$. Since $s(\Gamma(x_1, \ldots, x_n)) \subseteq \text{Th}(\lambda)$ and $s(\phi(x_1, \ldots, x_n)) \notin \text{Th}(\lambda)$, $\Gamma(a_1, \ldots, a_n) \subseteq \text{Th}(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin \text{Th}(\lambda)$.

$(\Leftarrow)$ Suppose there exists terms $a_1, \ldots, a_n$ such that $\Gamma(a_1, \ldots, a_n) \subseteq \text{Th}(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin \text{Th}(\lambda)$. Let $s$ be a substitution such that for all positive integers $i$, if $i \leq n$, then $s(x_i) = a_i$. Since $\Gamma(a_1, \ldots, a_n) \subseteq \text{Th}(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin \text{Th}(\lambda)$, $s(\Gamma(x_1, \ldots, x_n)) \subseteq \text{Th}(\lambda)$ and $s(\phi(x_1, \ldots, x_n)) \notin \text{Th}(\lambda)$. Hence, $(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ is not admissible in $\lambda$.

## 5 Admissibility: useful lemmas

Let $\lambda$ be an extension of $\lambda_0$. The decidability of $\text{wADM}(\lambda)$ is difficult to establish and we defer proving it till next section. In the meantime, we present useful lemmas. Let $n$ be a nonnegative integer. Let $\Phi_n$ be the set of all weak formulas with atomic terms in $x_1, \ldots, x_n$. We define on $\Phi_n$ the equivalence relation $\equiv^\lambda_n$ as follows:

- $\phi(x_1, \ldots, x_n) \equiv^\lambda_n \psi(x_1, \ldots, x_n)$ iff $\phi(x_1, \ldots, x_n) \iff \psi(x_1, \ldots, x_n) \in \text{Th}(\lambda)$.

Obviously, considered as formulas in classical propositional logic, the terms $a(x_1, \ldots, x_n)$ and $b(x_1, \ldots, x_n)$ are equivalent iff $\text{mon}(a(x_1, \ldots, x_n)) = \text{mon}(b(x_1, \ldots, x_n))$. Hence, there exists exactly $2^{2^n}$ pairwise non-equivalent terms in $x_1, \ldots, x_n$. Since each weak formula $\phi(x_1, \ldots, x_n)$ in $\Phi_n$ is a Boolean combination of elementary formulas of the form $C(a(x_1, \ldots, x_n), b(x_1, \ldots, x_n))$ or of the form $a(x_1, \ldots, x_n) \equiv b(x_1, \ldots, x_n)$. 
Lemma 1. $\equiv^\lambda_n$ has finitely many equivalence classes on $\Phi_n$.

Let $A_n$ be the set of all $n$-tuples of terms. Note that $n$-tuples of terms in $A_n$ may contain occurrences of atomic terms distinct from $x_1, \ldots, x_n$. Given $(a_1, \ldots, a_n) \in A_n$, a frame $F \in CF(\lambda)$ and a valuation $V$ on $F$, let

$\phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}$ be the set of all $C$-free weak formulas $\phi(x_1, \ldots, x_n)$ such that $(\mathcal{F}, V) \models \phi(a_1, \ldots, a_n)$.

Consider a complete list $\phi_1(x_1, \ldots, x_n), \ldots, \phi_k(x_1, \ldots, x_n)$ in $\Phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}$ of representatives for each equivalence class on $\Phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}$ modulo $\equiv^\lambda_n$ and define

$- \phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) = \phi_1(x_1, \ldots, x_n) \land \ldots \land \phi_k(x_1, \ldots, x_n)$.

Obviously,

**Lemma 2.** $(\mathcal{F}, V) \models \phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}(a_1, \ldots, a_n)$.

Hence, $\psi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n)$ is in $\Phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}$. Let

$- \Phi_{a_1, \ldots, a_n} = \{ \phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) : F \in CF(\lambda) \text{ and } V \text{ is a valuation on } F \}$.

Consider a complete list $\psi_1(x_1, \ldots, x_n), \ldots, \psi_l(x_1, \ldots, x_n)$ in $\Phi_{(a_1, \ldots, a_n)}$ of representatives for each equivalence class on $\Phi_{(a_1, \ldots, a_n)}$ modulo $\equiv^\lambda_n$ and define

$- \psi_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) = \psi_1(x_1, \ldots, x_n) \lor \ldots \lor \psi_l(x_1, \ldots, x_n)$.

We have the

**Lemma 3.** $\psi_{(a_1, \ldots, a_n)}(a_1, \ldots, a_n) \in Th(\lambda)$.

**Proof.** Suppose $\psi_{(a_1, \ldots, a_n)}(a_1, \ldots, a_n) \not\in Th(\lambda)$. Thus, there exists a frame $F \in CF(\lambda)$ and there exists a valuation $V$ on $F$ such that $(\mathcal{F}, V) \not\models \psi_{(a_1, \ldots, a_n)}(a_1, \ldots, a_n)$. Let $i$ be a positive integer such that $1 \leq i \leq l$ and $\phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n)$ is equivalent to $\psi_i(x_1, \ldots, x_n)$ modulo $\equiv^\lambda_n$. Since $(\mathcal{F}, V) \models Th(\lambda)$ and, by Lemma 2, $(\mathcal{F}, V) \models \phi^{\mathcal{F}, V}_{(a_1, \ldots, a_n)}(a_1, \ldots, a_n)$, $(\mathcal{F}, V) \not\models \psi_i(a_1, \ldots, a_n)$. Therefore, $(\mathcal{F}, V) \models \psi_{(a_1, \ldots, a_n)}(a_1, \ldots, a_n)$: a contradiction.

Moreover,

**Lemma 4.** For all $C$-free weak formulas $\phi(x_1, \ldots, x_n)$, the following conditions are equivalent:

$\phi(a_1, \ldots, a_n) \in Th(\lambda),$
$\psi_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) \rightarrow \phi(x_1, \ldots, x_n) \in Th(\lambda)$. 

Proof. (⇒) Suppose \( \phi(a_1, \ldots, a_n) \in Th(\lambda) \). If \( \psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \to \phi(x_1, \ldots, x_n) \) \( \notin Th(\lambda) \), then there exists a frame \( F \in CF(\lambda) \) and there exists a valuation \( V \) on \( F \) such that \((F, V) \not\models \psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \to \phi(x_1, \ldots, x_n)\). Hence, \((F, V) \models \psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \) and \((F, V) \not\models \phi(x_1, \ldots, x_n) \). Thus, there exists a positive integer \( i \) such that \( i \leq l \) and \((F, V) \models \psi_i(x_1, \ldots, x_n) \). Let \( F' \in CF(\lambda) \) be a frame and \( V' \) be a valuation on \( F' \) such that \( \phi_{\psi_i}(a_1, \ldots, a_n)(x_1, \ldots, x_n) \) is equivalent to \( \psi_i(x_1, \ldots, x_n) \) modulo \( \equiv_\lambda^{\psi_i} \). Since \((F, V) \models Th(\lambda) \) and \((F, V) \models \psi_i(x_1, \ldots, x_n), (F, V) \models \phi_{\psi_i}(a_1, \ldots, a_n)(x_1, \ldots, x_n) \). Therefore, for all \( C \) free weak formulas \( \theta(x_1, \ldots, x_n) \), if \((F', V') \models \theta(a_1, \ldots, a_n) \), then \((F, V) \models \theta(x_1, \ldots, x_n) \). Since \((F', V') \models Th(\lambda) \) and \( \phi(a_1, \ldots, a_n) \in Th(\lambda) \), \((F', V') \models \phi(a_1, \ldots, a_n) \). Since for all \( C \) free weak formulas \( \theta(x_1, \ldots, x_n) \), if \((F', V') \models \theta(a_1, \ldots, a_n) \), then \((F, V) \models \theta(x_1, \ldots, x_n), (F, V) \models \phi(x_1, \ldots, x_n) \): a contradiction.

(⇐) Suppose \( \psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \to \phi(x_1, \ldots, x_n) \in Th(\lambda) \). Consequently, \( \psi(a_1, \ldots, a_n)(a_1, \ldots, a_n) \to \phi(a_1, \ldots, a_n) \in Th(\lambda) \). By Lemma 3, \( \psi(a_1, \ldots, a_n)(a_1, \ldots, a_n) \in Th(\lambda) \). Since \( \psi(a_1, \ldots, a_n)(a_1, \ldots, a_n) \to \phi(a_1, \ldots, a_n) \in Th(\lambda) \), \( \phi(a_1, \ldots, a_n) \in Th(\lambda) \).

We define on \( A_n \) the equivalence relation \( \equiv_\lambda^{\psi} \) as follows:

- \( (a_1, \ldots, a_n) \equiv_\lambda^{\psi} (b_1, \ldots, b_n) \) iff for all weak formulas \( \phi(x_1, \ldots, x_n) \) in \( \Phi_n \), \( \phi(a_1, \ldots, a_n) \in Th(\lambda) \) iff \( \phi(b_1, \ldots, b_n) \in Th(\lambda) \).

By Lemma 1,

**Lemma 5.** \( \equiv_\lambda^{\psi} \) has finitely many equivalence classes on \( A_n \).

It is of interest to consider the equivalence relation \( \equiv_\lambda^{\psi} \), seeing that, according to our definitions,

**Lemma 6.** If \( (a_1, \ldots, a_n) \equiv_\lambda^{\psi} (b_1, \ldots, b_n) \), then for all weak inference rules \( (\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n)) \), the following conditions are equivalent:

- \( \Gamma(a_1, \ldots, a_n) \subseteq Th(\lambda) \) and \( \phi(a_1, \ldots, a_n) \notin Th(\lambda) \),
- \( \Gamma(b_1, \ldots, b_n) \subseteq Th(\lambda) \) and \( \phi(b_1, \ldots, b_n) \notin Th(\lambda) \).

Now, we define on \( A_n \) the equivalence relation \( \simeq_\lambda^{\psi} \) as follows:

- \( (a_1, \ldots, a_n) \simeq_\lambda^{\psi} (b_1, \ldots, b_n) \) iff for all \( C \) free weak formulas \( \phi(x_1, \ldots, x_n) \) in \( \Phi_n \), \( \phi(a_1, \ldots, a_n) \in Th(\lambda) \) iff \( \phi(b_1, \ldots, b_n) \in Th(\lambda) \).

Obviously,

**Lemma 7.** If \( (a_1, \ldots, a_n) \simeq_\lambda^{\psi} (b_1, \ldots, b_n) \), then \( (a_1, \ldots, a_n) \simeq_\lambda^{\psi} (b_1, \ldots, b_n) \).

Moreover, by Lemma 1,

**Lemma 8.** \( \simeq_\lambda^{\psi} \) has finitely many equivalence classes on \( A_n \).

The key things to note about the equivalence relations \( \equiv_\lambda^{\psi} \) and \( \simeq_\lambda^{\psi} \) are contained in the following lemmas.
Lemma 9. The following conditions are equivalent:

- $(a_1, \ldots, a_n) \cong^\lambda (b_1, \ldots, b_n)$,
- $\psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \equiv \psi(b_1, \ldots, b_n)(x_1, \ldots, x_n) \in Th(\lambda)$.

Proof. Suppose $(a_1, \ldots, a_n) \cong^\lambda (b_1, \ldots, b_n)$. If $\psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \not\equiv \psi(b_1, \ldots, b_n)(x_1, \ldots, x_n) \not\in Th(\lambda)$, then either $\psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \not\equiv \psi(b_1, \ldots, b_n)(x_1, \ldots, x_n) \not\in Th(\lambda)$ or $\psi(b_1, \ldots, b_n)(x_1, \ldots, x_n) \not\equiv \psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \not\in Th(\lambda)$. Without loss of generality, let us assume that $\psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \not\in Th(\lambda)$. By Lemma 4, $\psi(b_1, \ldots, b_n)(a_1, \ldots, a_n) \not\in Th(\lambda)$. Since $(a_1, \ldots, a_n) \cong^\lambda (b_1, \ldots, b_n)$, $\psi(b_1, \ldots, b_n)(a_1, \ldots, a_n) \not\in Th(\lambda)$. By Lemma 3, $\psi(b_1, \ldots, b_n)(a_1, \ldots, a_n) \not\in Th(\lambda)$: a contradiction.

$(\Leftarrow)$ Suppose $\psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \not\in Th(\lambda)$. Hence, for all $C$-free weak formulas $\phi(x_1, \ldots, x_n)$, $\psi(a_1, \ldots, a_n)(x_1, \ldots, x_n) \not\equiv \phi(x_1, \ldots, x_n) \not\in Th(\lambda)$ if $\psi(b_1, \ldots, b_n)(x_1, \ldots, x_n) \not\equiv \phi(x_1, \ldots, x_n) \not\in Th(\lambda)$. By Lemma 4, for all $C$-free weak formulas $\phi(x_1, \ldots, x_n)$, $\phi(a_1, \ldots, a_n) \not\in Th(\lambda)$ if $\phi(b_1, \ldots, b_n) \not\in Th(\lambda)$. Thus, $(a_1, \ldots, a_n) \cong^\lambda (b_1, \ldots, b_n)$.

Lemma 10. If $\lambda$ is balanced and $(a_1, \ldots, a_n) \cong^\lambda (b_1, \ldots, b_n)$, then $(a_1, \ldots, a_n) \cong^\lambda (b_1, \ldots, b_n)$.

Proof. Suppose $\lambda$ is balanced and $(a_1, \ldots, a_n) \cong^\lambda (b_1, \ldots, b_n)$. If $(a_1, \ldots, a_n) \not\cong^\lambda (b_1, \ldots, b_n)$, then there exists a not-iff formula $\phi(x_1, \ldots, x_n)$ in $\Phi(\lambda)$ such that $\phi(a_1, \ldots, a_n) \not\in Th(\lambda)$ and $\phi(b_1, \ldots, b_n) \not\in Th(\lambda)$. Without loss of generality, let us assume that $\phi(a_1, \ldots, a_n) \not\in Th(\lambda)$ and $\phi(b_1, \ldots, b_n) \not\in Th(\lambda)$. Since $\lambda$ is balanced, there exists a countable frame $\mathcal{F} \in CF(\lambda)$ and there exists a balanced valuation $V$ on $\mathcal{F}$ such that $(\mathcal{F}, V) \not\models \phi(a_1, \ldots, a_n)$. By Lemma 2, $(\mathcal{F}, V) \models \phi(\mathcal{F}, V, V, V, V) \models (b_1, \ldots, b_n)$. Since $\lambda$ is balanced, there exists a countable frame $\mathcal{F}' \in CF(\lambda)$ and there exists a balanced valuation $V'$ on $\mathcal{F}'$ such that $(\mathcal{F}', V') \models \phi(\mathcal{F}', V') \models (a_1, \ldots, a_n)$. Suppose $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$. Now, consider $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}^n$. If $(b_1^1 \cap \ldots \cap b_n^1)^V \not= 0$, then $(\mathcal{F}, V) \models b_1^1 \cap \ldots \cap b_n^1 \not\models 0$. Hence, $(\mathcal{F}', V) \models b_1^1 \cap \ldots \cap b_n^1 \not\models 0$. Similarly, the reader may easily verify that if $(a_1^1 \cap \ldots \cap a_n^1)^V \not= 0$, then $(\mathcal{F}, V) \models (a_1^1 \cap \ldots \cap a_n^1)^V \not\models 0$. To prove the equivalence, let $f$ be the union of all $f'_1, \ldots, f'_n$ when $(\epsilon_1, \ldots, \epsilon_n)$ describes $\{0, 1\}$.
6 Admissibility: decidability

Let $\lambda$ be an extension of $\lambda_0$. By Proposition 13 and Lemmas 5–8 and 10, wADM($\lambda$) would be decidable if $\lambda$ is balanced, $Th(\lambda)$ is decidable and a complete set of representatives for each class on $A_n$ modulo $\equiv^*_n$ could be computed. Let $k$ be a nonnegative integer. Given $(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \in A_n$, we define on \{0, 1\}$^k$ the equivalence relation $\sim_k^{(a_1, \ldots, a_n)}$ as follows:

- $(\epsilon_1, \ldots, \epsilon_k) \sim_k^{(a_1, \ldots, a_n)} (\epsilon'_1, \ldots, \epsilon'_k)$ if for all positive integers $i$, if $i \leq n$, then $z_1^i \cap \ldots \cap z_k^i \in \text{mon}(a_i(z_1, \ldots, z_k))$ if $z_1^i \cap \ldots \cap z_k^i \in \text{mon}(a_i(z_1, \ldots, z_k))$.

Obviously,

**Lemma 11.** $\sim_k^{(a_1, \ldots, a_n)}$ has at most $2^n$ equivalence classes on \{0, 1\}$^k$.

Hence, there exists a one-to-one function $f$ assigning to each equivalence class $[ (\epsilon_1, \ldots, \epsilon_k) |_{\sim_k^{(a_1, \ldots, a_n)}} ]$ an $n$-tuple $f([ (\epsilon_1, \ldots, \epsilon_k) |_{\sim_k^{(a_1, \ldots, a_n)}} ]) \in \{0, 1\}^n$. By means of the one-to-one function $f$, for all positive integers $i$, if $i \leq n$, then we define the term $b_i(x_1, \ldots, x_n)$ as follows:

- $b_i(x_1, \ldots, x_n) = \bigcup \{ (x_1^i \cap \ldots \cap x_n^i : z_1^i \cap \ldots \cap z_k^i \in \text{mon}(a_i(z_1, \ldots, z_k)) \}$ and $f([ (\epsilon_1, \ldots, \epsilon_k) |_{\sim_k^{(a_1, \ldots, a_n)}} ]) = (\epsilon_1^i, \ldots, \epsilon_n^i)$.

Given a nonempty set $W$, the reader may easily verify the following

**Lemma 12.** for all valuations $V$ on $W$, there exists a valuation $V'$ on $W$ such that for all positive integers $i$, if $i \leq n$, then $(a_i(z_1, \ldots, z_k))^V = (b_i(x_1, \ldots, x_n))^V$, for all valuations $V$ on $W$ such that for all positive integers $i$, if $i \leq n$, then $(b_i(x_1, \ldots, x_n))^V = (a_i(z_1, \ldots, z_k))^V$.

The key thing to note about the terms $b_i(x_1, \ldots, x_n), b_i(x_1, \ldots, x_n)$ is contained in the following

**Lemma 13.** $(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \equiv^*_n (b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$.

**Proof.** Suppose $(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \not\equiv^*_n (b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$. Hence, there exists a $C$-free weak formula $\phi(y_1, \ldots, y_n)$ in $\Phi_n$ such that $\phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \in Th(\lambda)$ and there exists a frame $F \in CF(\lambda)$ such that $\phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n)) \not\in Th(\lambda)$. This, thus, we have to consider the following two cases.

**Case.** $\phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \in Th(\lambda)$ and $\phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n)) \not\in Th(\lambda)$. Hence, there exists a frame $F \in CF(\lambda)$ and there exists a valuation $V$ on $F$ such that $(F, V) \not\models \phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$.

By Lemma 12, there exists a valuation $V'$ on $F$ such that for all positive integers $i$, if $i \leq n$, then $(b_i(x_1, \ldots, x_n))^V = (a_i(z_1, \ldots, z_k))^V$. Since $(F, V) \not\models \phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$, $(F, V') \not\models \phi(a_i(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k))$. Thus, $F \not\models \phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k))$. Since $F \in CF(\lambda), \ldots$
\( \phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \notin Th(\lambda) \): a contradiction.

Case. \( \phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \notin Th(\lambda) \) and \( b_n(x_1, \ldots, x_n) \in Th(\lambda) \). Similar to the case \( \phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \in Th(\lambda) \) and \( \phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n)) \notin Th(\lambda) \).

By means of the lemmas presented above, let us prove the following

**Proposition 14.** A complete set of representatives for each class on \( A_n \) modulo \( \approx^\lambda_n \) can be computed.

**Proof.** By Lemma 13, the set of all \( n \)-tuples of terms on \( \{x_1, \ldots, x_n\} \) constitutes a complete set of representatives for each class on \( A_n \) modulo \( \approx^\lambda_n \). Since there exists exactly \( 2^{2^n} \) pairwise non-equivalent terms of the form \( b(x_1, \ldots, x_n) \), a complete set of representatives for each class on \( A_n \) modulo \( \approx^\lambda_n \) can be computed.

As a result,

**Proposition 15.** If \( \lambda \) is balanced and there exists a finite set \( A \) of axioms such that \( \lambda = \lambda_0(A) \), then \( wADM(\lambda) \) is decidable.

**Proof.** Suppose \( \lambda \) is balanced and there exists a finite set \( A \) of axioms such that \( \lambda = \lambda_0(A) \). We define an algorithm taking as input a weak inference rule \( (I(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n)) \) and returning the value \( true \) iff \( (I(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n)) \) is admissible in \( \lambda \) as follows:

- compute a complete set \( \{(a_1^1, \ldots, a_n^1), \ldots, (a_1^N, \ldots, a_n^N)\} \) of representatives for each class on \( A_n \) modulo \( \approx^\lambda_n \);
- if there exists a positive integer \( k \) such that \( k \leq N, I(a_1^k, \ldots, a_n^k) \subseteq Th(\lambda) \) and \( \phi(a_1^k, \ldots, a_n^k) \notin Th(\lambda) \) then return \( false \) else return \( true \).

By Propositions 13 and 14 and Lemmas 5–8 and 10, this algorithm is sound and complete with respect to \( wADM(\lambda) \) and can be executed.

However, the exact complexity of \( wADM(\lambda) \) is not known.

## 7 Unifiability

Let \( \lambda \) be an extension of \( \lambda_0 \). A formula \( \phi \) is said to be unifiable in \( \lambda \) iff there exists a substitution \( s \) such that \( s(\phi) \in Th(\lambda) \). It happens that if \( \lambda \) is consistent, then unifiability is a special case of admissibility.

**Proposition 16.** Let \( \phi \) be a formula. If \( \lambda \) is consistent, then the following conditions are equivalent:

- \( \phi \) is unifiable in \( \lambda \),
- \( (\{\phi\}, \bot) \) is not admissible in \( \lambda \).
Lemma 14. For all weak formulas \( \phi(x_1, \ldots, x_n) \), the following conditions are equivalent:

- \( \phi \) is unifiable in \( \lambda \),
- there exists \((\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n\) such that \( \phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda) \).

Proof. (\(\Rightarrow\)) Suppose \( \phi(x_1, \ldots, x_n) \) is unifiable in \( \lambda \). Hence, there exists a substitution \( s \) such that \( s(\phi(x_1, \ldots, x_n)) \in Th(\lambda) \). Let \( t \) be a ground substitution. Since \( s(\phi(x_1, \ldots, x_n)) \in Th(\lambda) \), \( t(s(\phi(x_1, \ldots, x_n))) \in Th(\lambda) \). Let \((\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n\) be obtained from \( t(s(\phi(x_1, \ldots, x_n))) \) by applying ordinary reasoning in non-degenerate Boolean algebras. Since \( t(s(\phi(x_1, \ldots, x_n))) \in Th(\lambda) \), \( \phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda) \).

(\(\Leftarrow\)) Suppose there exists \((\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n\) such that \( \phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda) \). Let \( s \) be a substitution such that for all positive integers \( i \), if \( i \leq n \), then \( s(x_i) = \epsilon_i \). Since \( \phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda) \), \( s(\phi(x_1, \ldots, x_n)) \in Th(\lambda) \). Thus, \( \phi \) is unifiable in \( \lambda \).

Hence, it is easy to check that when \( Th(\lambda) \) is decidable, \( wUNI(\lambda) \) is decidable. Now, remark that for all weak formulas \( \phi(x_1, \ldots, x_n) \) and for all \((\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n\), \( \phi(\epsilon_1, \ldots, \epsilon_n) \) is equivalent modulo \( \equiv^\lambda \) to one of the following elementary formulas: \( \bot \), \( \top \), \( C(1,1) \), \( \bar{C}(1,1) \). Moreover, even when \( Th(\lambda) \) is undecidable, the elementary formula in \( \{\bot, \top, C(1,1), \bar{C}(1,1)\} \) that is equivalent modulo \( \equiv^\lambda \) to \( \phi(\epsilon_1, \ldots, \epsilon_n) \) in \( \lambda \) can be computed. As a result, in all cases, i.e. whatever is the decidability status of \( Th(\lambda) \),

**Proposition 17.** \( wUNI(\lambda) \) is decidable.

Remark that the elementary formula in \( \{\bot, \top, C(1,1), \bar{C}(1,1)\} \) that is equivalent modulo \( \equiv^\lambda \) to \( \phi(\epsilon_1, \ldots, \epsilon_n) \) in \( \lambda \) can be computed in linear time. As a result,

**Proposition 18.** \( wUNI(\lambda) \) is in \( NP \).

It happens that if \( \lambda \) is consistent, then the satisfiability problem in Boolean Logic is reducible to \( wUNI(\lambda) \).

**Proposition 19.** Let \( a(x_1, \ldots, x_n) \) be a term. If \( \lambda \) is consistent, then the following conditions are equivalent:

Proof. Suppose \( \lambda \) is consistent.

(\(\Rightarrow\)) Suppose \( \phi \) is unifiable in \( \lambda \). If \((\phi, \bot)\) is admissible in \( \lambda \), then for all substitutions \( s \), if \( s(\phi) \in Th(\lambda) \), then \( \bot \in Th(\lambda) \). Since \( \lambda \) is consistent, \( \bot \not\in Th(\lambda) \). Hence, \( \phi \) is not unifiable in \( \lambda \): a contradiction.

(\(\Leftarrow\)) Suppose \((\phi, \bot)\) is not admissible in \( \lambda \). Hence, there exists a substitution \( s \) such that \( s(\phi) \in Th(\lambda) \). Thus, \( \phi \) is unifiable in \( \lambda \).
Proposition 21. Let \( \phi(x_1, \ldots, x_n) \) be a weak formula. If \( \lambda \) is consistent and \( C(1, 1) \in Th(\lambda) \), then the following conditions are equivalent:

- \( \phi(x_1, \ldots, x_n) \) is not unifiable in \( \lambda \),
- \( \phi(x_1, \ldots, x_n) \to \bigvee \{ x_i \neq 0 \land x_i \neq 1 : 1 \leq i \leq n \} \in Th(\lambda) \).

Proof. Suppose \( \lambda \) is consistent and \( C(1, 1) \in Th(\lambda) \).

(\( \Rightarrow \)) Suppose \( \phi(x_1, \ldots, x_n) \) is not unifiable in \( \lambda \). If \( \phi(x_1, \ldots, x_n) \to \bigvee \{ x_i \neq 0 \land x_i \neq 1 : 1 \leq i \leq n \} \notin Th(\lambda) \), then there exists a frame \( \mathcal{F} \in CF(\lambda) \) and there exists a valuation \( V \) on \( \mathcal{F} \) such that \( (\mathcal{F}, V) \not\models \phi(x_1, \ldots, x_n) \to \bigvee \{ x_i \neq 0 \land x_i \neq 1 : 1 \leq i \leq n \} \). Hence, \( (\mathcal{F}, V) \models \phi(x_1, \ldots, x_n) \) and \( (\mathcal{F}, V) \not\models \bigvee \{ x_i \neq 0 \land x_i \neq 1 : 1 \leq i \leq n \} \). Suppose \( \mathcal{F} = (W, R) \). Thus, for all positive integers \( i \), if \( i \leq n \), then either \( V(x_i) = \emptyset \), or \( V(x_i) = W \). Let \( s \) be a substitution such that for all positive integers \( i \), if \( i \leq n \), then

- if \( V(x_i) = \emptyset \), then \( s(x_i) = 0 \),
- if \( V(x_i) = W \), then \( s(x_i) = 1 \).
Therefore, for all positive integers $i$, if $i \leq n$, then $V(x_i) = (s(x_i))^V$. Since $(\mathcal{F}, V) \models \text{Phil}(x_1, \ldots, x_n)$, $(\mathcal{F}, V) \models \phi(s(x_1), \ldots, s(x_n))$. Since for all positive integers $i$, if $i \leq n$, then $s(x_i)$ is either equal to $0$, or equal to $1$, $\phi(s(x_1), \ldots, s(x_n))$ is equivalent modulo $\equiv^\lambda$ to an elementary formula in $\{ \bot, \top, C(1,1), C(1,1) \}$. Since $\mathcal{F} \models Th(\lambda)$, $C(1,1) \in Th(\lambda)$ and $(\mathcal{F}, V) \models \phi(s(x_1), \ldots, s(x_n))$, $\phi(s(x_1), \ldots, s(x_n))$ is equivalent modulo $\equiv^\lambda$ to $\top$. Consequently, $\phi(x_1, \ldots, x_n)$ is unifiable in $\lambda$: a contradiction.

$\leftarrow$ Suppose $\phi(x_1, \ldots, x_n) \rightarrow \bigvee \{ x_i \neq 0 \land x_i \neq 1: 1 \leq i \leq n \} \in Th(\lambda)$. If $\phi(x_1, \ldots, x_n)$ is unifiable in $\lambda$, then by Lemma 14, there exists $(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$ such that $\phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda)$. Let $s$ be a substitution such that for all positive integers $i$, if $i \leq n$, then $s(x_i) = \epsilon_i$. Since $\phi(x_1, \ldots, x_n) \rightarrow \bigvee \{ x_i \neq 0 \land x_i \neq 1: 1 \leq i \leq n \} \in Th(\lambda)$, $s(\phi(x_1, \ldots, x_n)) \rightarrow \bigvee \{ s(x_i) \neq 0 \land s(x_i) \neq 1: 1 \leq i \leq n \} \in Th(\lambda)$. Since for all positive integers $i$, if $i \leq n$, then $s(x_i) = \epsilon_i$, $\phi(s(x_1), \ldots, s(x_n)) \rightarrow \bigvee \{ \epsilon_i \neq 0 \land \epsilon_i \neq 1: 1 \leq i \leq n \} \in Th(\lambda)$. Since $\phi(1, \ldots, n) \in Th(\lambda)$, $\bigvee \{ \epsilon_i \neq 0 \land \epsilon_i \neq 1: 1 \leq i \leq n \} \in Th(\lambda)$. Since $(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, $\land \{ \epsilon_i \equiv 0 \lor \epsilon_i \equiv 1: 1 \leq i \leq n \} \in Th(\lambda)$. Since $\bigvee \{ \epsilon_i \neq 0 \land \epsilon_i \neq 1: 1 \leq i \leq n \} \in Th(\lambda)$, $\lambda$ is not consistent: a contradiction.

8 Conclusion

Admissibility problems and unifiability problems are decidable in many modal logics [1–3, 13, 14], but modal logics for which they become undecidable are known [18]. Nevertheless, very little is known about these problems in some of the most important modal logics considered in Computer Science and Artificial Intelligence. For example, the decidability and the complexity of the unification problem for the following modal logics remains open: modal logic $K$, multi-modal variants of $K$, sub-Boolean modal logics.

In this paper, we have examined variants of contact logics. The central result in this paper is the proof that the weak admissibility problem and the weak unifiability problem are decidable in these variants.

Much remains to be done. For example, $\lambda$ being a consistent extension of $\lambda_0$, Propositions 16 and 20 imply that $wADM(\lambda)$ is coNP-hard, but the exact complexity of $wADM(\lambda)$ is not known. One may also consider the admissibility problem $ADM(\lambda)$ defined in Section 4 and the following unifiability problem: given a formula $\phi$, determine whether $\phi$ is unifiable in $\lambda$. Finally, there is also the related question of the unification type of $\lambda$. Our conjecture is that the unification type of most extensions of $\lambda_0$ considered in [5, 17] is finitary.

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