Generalized Sugeno Integrals

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Abstract. Sugeno integrals are aggregation functions defined on a qualitative scale where only minimum, maximum and order-reversing maps are allowed. Recently, variants of Sugeno integrals based on Gödel implication and its contraposition were defined and axiomatized in the setting of bounded chain with an involutive negation. This paper proposes a more general approach. We consider totally ordered scales, multivalued conjunction operations not necessarily commutative, and implication operations induced from them by means of an involutive negation. In such a context, different Sugeno-like integrals are defined and axiomatized.

Keywords: Sugeno integral, conjunctions, implications, multifactorial evaluation

1 Introduction and prerequisites

In a recent paper [4], we introduced variants of Sugeno integrals based on Gödel implication and its contraposition using an involutive negation. It models qualitative aggregation methods that extend min and max, based on the idea of tolerance threshold beyond which a criterion is considered satisfied. These new aggregation operations have been axiomatized in [5] in the setting of a complete bounded chain with an involutive negation. In the present paper, we try to cast this approach in a more general totally ordered algebraic setting, using multivalued conjunction operations that are not necessarily commutative, and implication operations induced from them by means of an involutive negation.

We adopt the terminology and notations usual in multi-criteria decision making, where some alternatives are evaluated according to a common set $\mathcal{C} = \{1, \ldots, n\} = [n]$ of criteria. A common evaluation scale L is assumed to provide ratings according to the criteria: each alternative is thus identified with a function $f \in L^{\mathcal{C}}$ which maps every criterion i of $\mathcal{C}$ to the local rating f_i of the alternative with regard to this criterion. We assume that L is a totally ordered set with 1 and 0 as top and bottom, respectively (L may be the real unit interval $[0, 1]$ for instance). For any $a \in L$, we denote by $\mathbf{a}_{\mathcal{C}}$ the constant alternative equals to a on $\mathcal{C}$. In addition, we assume that L is equipped with a unary order reversing involutive operation $t \mapsto 1 - t$, that we call *negation*.

We denote by $\wedge$ and $\vee$ the minimum and maximum operation on L. These two aggregation schemes can be slightly generalised by means of importance levels or priorities $\pi_i \in L$, on the criteria $i \in [n]$. Suppose π_i is increasing with the importance of i. A fully important criterion has importance weight $\pi_i = 1$. In the following, we assume $\pi_i > 0$ for every $i \in [n]$, *i.e.*, there is no useless criterion. In this section, we also assume $\pi_i = 1$, for some criterion i (the most important one). It is a kind a normalization assumption that ensures that the whole scale L is useful, and that is typical of possibility theory. These importance levels can interact with each local evaluation f_i in different manners. Usually, a weight π_i acts as a saturation threshold that blocks the global score under or above a certain value dependent on the importance level of criterion i. Such weights truncate the evaluation scale from above or from below. The rating f_i is taken into acount in the form of either $(1-\pi_i)\vee f_i \in [1-\pi_i, 1]$, or $\pi_i \wedge f_i \in [0, \pi_i]$. A fully important criterion can alone bring the whole global score to 1 or to 0. The weighted minimum and maximum operations then take the following forms:

$$MIN_\pi(f) = \bigwedge_{i=1}^{n} \big((1-\pi_i) \vee f_i\big); \quad MAX_\pi(f) = \bigvee_{i=1}^{n} (\pi_i \wedge f_i). \tag{1}$$

It is well-known that if the evaluation scale L is reduced to $\{0, 1\}$ (Boolean criteria) then letting $A_f = \{i : f_i = 1\}$ be the set of criteria satisfied by alternative f, the function $\Pi \colon A_f \mapsto MAX_\pi(f) = \bigvee\{\pi_i : i \in A_f\}$ is a possibility measure on $\mathcal{C}$ [12] (*i.e*, a set function Π that satisfies $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for every $A, B \subseteq \mathcal{C}$), and $N \colon f \mapsto MIN_\pi(f) = \bigwedge\{1 - \pi_i : i \notin A_f\}$ is a necessity measure [3] (*i.e.*, a set function N that satisfies $N(A \cap B) = N(A) \wedge N(B)$ for every $A, B \subseteq \mathcal{C}$). Note that the well-known duality property $\Pi(A) = 1 - N(\overline{A})$, where $\overline{A}$ denotes the set complement of A in $\mathcal{C}$, immediately generalizes to the scale L in the following way:

$$MAX_\pi(f) = 1 - MIN_\pi(1 - f). \tag{2}$$

There are two possible lines of action to extend the definition of the aggregation operations in (1):

- Replacing possibility and necessity measures by more general monotonic set functions that attach weights to groups of criteria.
- Extending the rating modification schemes using more general conjunctions and implications.

Sugeno integral The first extension leads to modeling relative weights of the sets of criteria via a *capacity*, which is an order-preserving map $\gamma \colon 2^{\mathcal{C}} \to L$ that satisfies $\gamma(\varnothing) = 0$ and $\gamma(\mathcal{C}) = 1$. The *conjugate capacity* γ^c of γ is defined by $\gamma^c(A) = 1 - \gamma(\overline{A})$ for every $A \subseteq \mathcal{C}$. The Sugeno integral [11], of an alternative f can be defined by means of several expressions, among which the two following normal forms [9]:

$$\int_\gamma f = \bigvee_{A \subseteq \mathcal{C}} \big(\gamma(A) \wedge \bigwedge_{i \in A} f_i\big) = \bigwedge_{A \subseteq \mathcal{C}} (1 - \gamma^c(A)) \vee \bigvee_{i \in A} f_i. \tag{3}$$

These expressions, which generalise the conjunctive and disjunctive normal forms in logic, can be simplified as follows:

$$\int_\gamma f = \bigvee_{a\in L} \gamma(\{i : f_i \geq a\}) \wedge a = \bigwedge_{a\in L} \gamma(\{i : f_i > a\})) \vee a. \tag{4}$$

Moreover, for the necessity measure N associated with a possibility distribution π, we have $\int_N(f) = MIN_\pi(f)$; and for the possibility measure Π associated with π, we have $\int_\Pi(f) = MAX_\pi(f)$.

There is a duality relation between Sugeno integrals with respect to conjugate capacities, extending (2):

$$\int_\gamma f = 1 - \int_{\gamma^c}(1-f). \tag{5}$$

Two alternatives $f, g \in L^{\mathcal{C}}$ are said to be *comonotone* if for every $i, j \in [n]$, if $f(i) < f(j)$ then $g(i) \leq g(j)$ and if $g(i) < g(j)$ then $f(i) \leq f(j)$. By means of this notion, Sugeno integral can be characterized as follows:

Theorem 1 ([1]). *Let $I : L^{\mathcal{C}} \to L$. There is a capacity γ such that $I(f) = \int_\gamma f$ for every $f \in L^{\mathcal{C}}$ if and only if the following properties are satisfied*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotone $f, g \in L^{\mathcal{C}}$.*
2. *$I(a \wedge f) = a \wedge I(f)$, for every $a \in L$ and $f \in L^{\mathcal{C}}$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$.*

Equivalently, conditions (1-3) can be replaced by conditions (1'-3') below.

1'. *$I(f \wedge g) = I(f) \wedge I(g)$, for any comonotone $f, g \in L^{\mathcal{C}}$.*
2'. *$I(b \vee f) = b \vee I(f)$, for every $a \in L$ and $f \in L^{\mathcal{C}}$.*
3'. *$I(\mathbf{0}_{\mathcal{C}}) = 0$.*

The existence of these two equivalent characterisations is due to the possibility of writing Sugeno integral in conjunctive and disjunctive forms (3) equivalently.

Generalized rating modification The second extension yields weighted min and max operations of the form

$$MIN_\pi^{\to}(f) = \bigwedge_{i=1}^{n} \pi_i \to f_i; \quad MAX_\pi^{\otimes}(f) = \bigvee_{i=1}^{n} \pi_i \otimes f_i, \tag{6}$$

where $\to$ is an implication connective, and $\otimes$ a conjunction, understood as multi-valued connectives that coincide with Boolean implication and conjunction when restricted to $\{0, 1\}$. In order to preserve the duality property (2), these operations must be related by a property that we call *semi-duality*, defined by the equation $a \to b = 1 - (a \otimes (1 - b))$, or equivalently $a \otimes b = 1 - (a \to (1 - b))$.

One may then consider both generalizations together and define, given a pair of semi-dual implication $\to$ and conjunction $\otimes$, the integrals $\int^{\otimes}$ and $\int^{\to}$ by

$$\int_{\gamma}^{\otimes} f = \bigvee_{A \subseteq C} (\gamma(A) \otimes \bigwedge_{i \in A} f_i); \quad \int_{\gamma}^{\to} f = \bigwedge_{A \subseteq C} (\gamma^c(A) \to \bigvee_{i \in A} f_i), \tag{7}$$

for every capacity γ and every $f \in L^{\mathcal{C}}$. In what follows, we refer to expressions of the form $\int_{\gamma}^{\to}$ as *co-integrals*. The assumption of semi-duality ensures that the duality equation (5) holds between integrals and co-integrals.

Sugeno integral is a particular instance of (7), since the minimum $\wedge$ and the Kleene-Dienes implication $\to_K$ defined as $a \to_K b := (1-a) \vee b$ exchange by semi-duality. So, Equation (3) actually states that

$$\int_{\gamma} f \;=\; \int_{\gamma}^{\wedge} f \;=\; \int_{\gamma}^{\to_K} f$$

for every capacity γ and every $f \in L^{\mathcal{C}}$. It means that integrals and co-integrals defined by means of the operation $\wedge$ and $\to_K$, respectively, coincide. As we shall see in the sequel, this is not generally the case.

2 Variants of Sugeno integrals: an example.

Let us recall previous results [4] in the qualitative setting of a *complete bounded totally ordered set* $L = (L, \wedge, \to_G, 0, 1)$ where $\to_G$ is the Gödel implication defined by residuation of $\wedge$:

$$a \to_G b := \sup\{x : a \wedge x \leq b\} = \begin{cases} 1 \text{ if } a \leq b \\ b \text{ otherwise.} \end{cases} \tag{8}$$

As previously, L is equipped with an involutive operation $1 - \cdot$. The following (non-commutative) conjunction, introduced in [2] is defined by semi-duality:

$$a \otimes_G b := 1 - (a \to_G (1-b)) = \begin{cases} b \text{ if } a > 1-b, \\ 0 \text{ otherwise.} \end{cases} \tag{9}$$

The qualitative integral $\int_{\gamma}^{\otimes_G}$ and co-integral $\int_{\gamma}^{\to_G}$ have simplified expressions that extend those of Sugeno integrals, assuming $f_1 \leq \cdots \leq f_n$:

$$\int_{\gamma}^{\otimes} f = \bigvee_{i=1}^{n} \gamma(\{i, \cdots, n\}) \otimes f_{(i)} = \bigvee_{a \in L} \gamma(\{f \geq a\}) \otimes a \tag{10}$$

$$\int_{\gamma}^{\to} (f) \;=\; \bigwedge_{i=1}^{n} \gamma^c(\{1, \cdots, i\}) \to f_{(i)} \;=\; \bigwedge_{a \in L} \gamma^c(\{f \leq a\}) \to a. \tag{11}$$

Note also that if N is a necessity measure and Π is a possibility measure, then $\int_N^{\to_G} = MIN_{\pi}^{\to_G}$ and $\int_{\Pi}^{\otimes_G} = MAX_{\pi}^{\otimes_G}$. However we cannot exchange N and Π in those results.

As $\otimes_G$ is not commutative, there is an alternative definition for those aggregation operations, replacing $\otimes_G$ by the operation $\otimes_{GC}$ defined by $a \otimes_{GC} b := b \otimes_G a$, and the operation $\rightarrow_G$ by the implication $\rightarrow_{GC}$ associated with $\otimes_{GC}$ by semi-duality (*i.e.*, the operation $\rightarrow_{GC}$ is the contrapositive version of $\rightarrow_G$):

$$a \rightarrow_{GC} b := 1 - (a \otimes_{GC} (1-b)) = (1-b) \rightarrow_G (1-a) = \begin{cases} 1 \text{ if } a \leq b, \\ 1-a \text{ otherwise.} \end{cases}$$

Properties (11)-(10) hold for $\int_\gamma^{\rightarrow_{GC}} f$ and $\int_\gamma^{\otimes_{GC}} f$ as well as for their reductions to a form of weighted min and max for necessity and possibility measures.

Noticeably, the integral and co-integral based on Gödel implications and their associated semi-dual conjunctions do not coincide. We have proved [4] that

$$\int_\gamma^{\otimes_G} f \geq \int_\gamma^{\rightarrow_G} f \text{ and } \int_\gamma^{\otimes_{GC}} f \geq \int_\gamma^{\rightarrow_{GC}} f, \tag{12}$$

but the inequalities may be strict. For instance, $\int_\gamma^{\rightarrow_G}(f) = 1$ if for all $A \subseteq \mathcal{C}$, there is some $i \in A$ such that $f_i \geq \gamma_c(A)$, and $\int_\gamma^{\otimes_G}(f) = 1$ if there is some subset $A \subseteq \mathcal{C}$ such that $\gamma(A) > 0$ and $f_i = 1$ for every $i \in A$.

Some characterization theorems for these variants of Sugeno integrals have been obtained [5]:

Theorem 2. *Let $I\colon L^{\mathcal{C}} \to L$ be a mapping. There is a capacity γ such that $I(f) = \int_\gamma^{\otimes_G} f$ for every $f \in L^{\mathcal{C}}$ if and only if*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotone $f, g \in L^{\mathcal{C}}$.*
2. *There is a capacity $\lambda\colon 2^{\mathcal{C}} \to L$ such that $I(\mathbf{1}_A \otimes_G a) = \lambda(A) \otimes_G a$ for every $a \in L$ and every $A \subseteq \mathcal{C}$.*

In that case, we have $\gamma = \lambda$.

Theorem 3. *Let $I\colon L^{\mathcal{C}} \to L$ be a mapping. There is a capacity γ such that $I(f) = \int_\gamma^{\rightarrow_G} f$ for every $f \in L^{\mathcal{C}}$ if and only if*

1. *$I(f \wedge g) = I(f) \wedge I(g)$, for any comonotone $f, g \in L^{\mathcal{C}}$.*
2. *There is a capacity $\rho\colon 2^{\mathcal{C}} \to L$ such that $I(\mathbf{1}_A \rightarrow_G a) = \rho(A) \rightarrow_G a$ for every $a \in L$.*

If these conditions are satisfied then $\gamma = \rho^c$.

Similar theorems hold [5] for $\int_\gamma^{\otimes_{GC}}(f)$ and $\int_\gamma^{\rightarrow_{GC}}(f)$. The above results suggest that it is possible to find a more general algebraic structure to define generalized Sugeno integrals, while keeping the same properties.

Note that the three implications and conjunctions in the above setting are related in the following way. We consider the three following transformations that can be applied to any operation $\star$ on a bounded totally ordered set with involutive negation $L = (L, \vee, \wedge, 1 - \cdot, 0, 1)$:

- Residuation: $aRes(\star)b := \bigvee\{a : a \star b \leq c\}$ if this supremum exists,
- Semi-duality: $a\mathcal{S}(\star)b := 1 - a \star (1 - b)$,
- Contraposition: $a\mathcal{C}(\star)b := (1 - b) \star (1 - a)$.
- Argument exchange: $a\mathcal{A}(\star)b := b \star a$

Note that semi-duality and contraposition are involutive transformations. Moreover the diagram in Fig. 1 commutes [2].

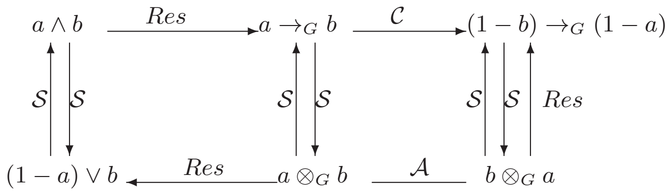

Fig. 1. Connectives induced by the minimum on a finite chain

In the sequel, we focus on generalized Sugeno integrals on a finite total order equipped with a conjunction that is not necessarily commutative, and the co-integral obtained by semi-duality. For simplicity, the word "q-integral" is used here in the sense of generalized Sugeno integrals on a qualitative scale.

3 Sugeno-like q-integrals based on left-conjunctions

We consider a bounded complete totally ordered value scale $(L, 0, 1, \leq)$, equipped with an operation $\otimes$ called *left-conjunction*, which has the following properties:

- the top element 1 is a left-identity: $1 \otimes x = x$,
- the bottom element 0 is a left-anihilator $0 \otimes x = 0$,
- the maps $x \mapsto a \otimes x$, $x \mapsto x \otimes a$ are order-preserving for every $a \in L$.

It follows that $a \otimes 0 = 0$ for every $a \in L$ (0 is an anihilator on both sides), and so, a left-conjunction coincides with a Boolean conjunction on $\{0, 1\}$; but we assume neither associativity nor commutativity. The following operations are examples of left-conjunctions.

- T-norms on $[0, 1]$, in particular $\wedge$, the product t-norm, the Łukasiewicz t-norm and the nilpotent minimum $\overline{\wedge}$ defined by $a\overline{\wedge}b = 0$ if $a + b \leq 1$ and $a\overline{\wedge}b = a \wedge b$ otherwise.
- Weak t-norms [8], i.e., left conjunctions such that $a \otimes 1 \leq a$.
- The non-commutative Gödel conjunction $\otimes_G$ previously introduced, and the non-commutative conjunction $\otimes_{rTC}$ defined by $a \otimes_{rTC} b = 0$ if $a = 0$, and $a \otimes_{rTC} b = b$ if $a \neq 0$ (see [5]).
- Pseudo-multiplications used by Klement et al. [10] in the definition of universal integrals. A pseudo-multiplication has genuine identity 1 and anihilator 0 (on both sides).

Definition 1. *Let $\otimes$ be a left-conjunction on L and $\gamma\colon 2^{\mathcal{C}} \to L$ be a capacity. The q-integral $\int_\gamma^\otimes$ is the mapping $\int_\gamma^\otimes\colon L^{\mathcal{C}} \to L$ defined by*

$$\int_\gamma^\otimes f = \bigvee_{A \subseteq C} \big(\gamma(A) \otimes \bigwedge_{i \in A} f_i\big), \text{ for all } f \in L^{\mathcal{C}}.$$

We show that q-integrals can be characterized similarly as in Theorem 2. In the following when we consider $f \in L^{\mathcal{C}}$, $(\cdot)$ denotes a permutation on the set of criteria such that $f_{(1)} \le \cdots \le f_{(n)}$ and we let $A_{(i)} = \{(i), \cdots, (n)\}$ with the convention $A_{(n+1)} = \varnothing$.

Lemma 1. *If $f \in L^{\mathcal{C}}$ then $f = \bigvee_{i=1}^n \mathbf{1}_{A_{(i)}} \otimes f_{(i)}$.*

Proof. For any $i, k \in [n]$, $f_{(i)} \le f_k$ if $k \in A_{(i)}$. It follows that $\mathbf{1}_{A_{(i)}}(k) \otimes f_{(i)} = 0$ if $f_k < f_{(i)}$ and $f_{(i)}$ otherwise; hence $\bigvee_{i=1}^n \mathbf{1}_{A_{(i)}}(k) \otimes f_{(i)} = \bigvee\{f_{(i)} \mid f_{(i)} \le f_k\} = f_k$.

Proposition 1. $\int_\gamma^\otimes f = \bigvee_{i=1}^n \gamma(A_{(i)}) \otimes f_{(i)}$.

Proof. $\int_\gamma^\otimes f = \bigvee_{A \subseteq C} \gamma(A) \otimes \bigwedge_{i \in A} f_i$. Let us denote $\bigwedge_{i \in A} f_i$ by f_{i_A}. It follows that $A \subseteq A_{(i_A)}$ which entails $\gamma(A) \le \gamma(A_{(i_A)})$ and $\gamma(A) \otimes f_{i_A} \le \gamma(A_{(i_A)}) \otimes f_{i_A}$.

Lemma 2. *For every capacity γ, the map $\int_\gamma^\otimes\colon L^{\mathcal{C}} \to L$ is order-preserving.*

Proof. Directly from the assumption that the map $x \mapsto a \otimes x$ is order-preserving.

Lemma 3. $\int_\gamma^\otimes f = \bigvee_{a \in L} \gamma(\{f \ge a\}) \otimes a$.

Proof. We use Proposition 1. Let $a \in L \setminus \{f_1, \ldots, f_n\}$.
If $a > f_{(n)}$ then $\gamma(\{f \ge a\}) \otimes a = 0 \otimes a = 0$.
If $a < f_{(1)}$ then $\gamma(\{f \ge a\}) \otimes a = \gamma(\{f \ge f_{(1)}\}) \otimes a \le \gamma(\{f \ge f_{(1)}\}) \otimes f_{(1)}$.
If $f_{(i-1)} < a < f_{(i)}$ then $\gamma(\{f \ge a\}) \otimes a = \gamma(\{f \ge f_{(i)}\}) \otimes a \le \gamma(\{f \ge f_{(i)}\}) \otimes f_{(i)}$.

Lemma 4. *For any comonotone $f, g \in L^{\mathcal{C}}$, we have $\int_\gamma^\otimes (f \vee g) = \int_\gamma^\otimes f \vee \int_\gamma^\otimes g$.*

Proof. The inequality $\int_\gamma^\otimes (f \vee g) \ge \int_\gamma^\otimes f \vee \int_\gamma^\otimes g$ follows from Lemma 2. Let us prove the other inequality. Let $a \in L$. For any two comonotone functions $f, g \in L^{\mathcal{C}}$ we have either $\{f \ge a\} \subseteq \{g \ge a\}$ or $\{g \ge a\} \subseteq \{f \ge a\}$.
If $\{f \ge a\} \subseteq \{g \ge a\}$ then $\{f \vee g \ge a\} = \{f \ge a\} \cup \{g \ge a\} = \{g \ge a\}$ and $\gamma(\{f \vee g \ge a\}) \otimes a = \gamma(\{g \ge a\}) \otimes a \le \big(\gamma(\{g \ge a\}) \otimes a\big) \vee \big(\gamma(\{f \ge a\}) \otimes a\big)$.
By symmetry, the inequality is also true when $\{g \ge a\} \subseteq \{f \ge a\}$; hence $\int_\gamma^\otimes (f \vee g) \le \bigvee_{a \in L} \Big(\big(\gamma(\{g \ge a\}) \otimes a\big) \vee \big(\gamma(\{f \ge a\}) \otimes a\big)\Big) = \int_\gamma^\otimes f \vee \int_\gamma^\otimes g$.

Lemma 5. *For every $f \in L^{\mathcal{C}}$ and every $\ell \in \{1, \ldots, n-1\}$, the maps $\mathbf{1}_{A_{(\ell)}} \otimes f_{(\ell)}$ and $\bigvee_{i=\ell+1}^n \mathbf{1}_{A_{(i)}} \otimes f_{(i)}$ are comonotone.*

Proof. We represent both maps as vectors of components ordered according to $(1), \ldots, (n)$, so that $A_{(\ell)} = \{(\ell), \ldots, (n)\}$. In consequence, $\mathbf{1}_{A_{(\ell)}} \otimes f_{(\ell)}(i) = \bigvee_{i=\ell+1}^n \mathbf{1}_{A_{(i)}} \otimes f_{(i)}(i) = 0$ if $i \le \ell$ while $\mathbf{1}_{A_{(\ell)}} \otimes f_{(\ell)}(i) = f_{(\ell)}$ and $\bigvee_{i=\ell+1}^n \mathbf{1}_{A_{(i)}} \otimes f_{(i)}(i) = f_{(i)}$ if $i > \ell$. Hence it is easy to check that the two maps are comonotone.

Lemma 6. *For any capacity* γ, *any* $B \subseteq \mathcal{C}$ *and any* $a \in L$ *we have* $\int_\gamma^\otimes (\mathbf{1}_B \otimes a) = \gamma(B) \otimes a$. *In particular* $\int_\gamma^\otimes \mathbf{1}_\mathcal{C} = 1$.

Proof. $\int_\gamma^\otimes f = \bigvee_{i=1}^n \gamma(A_{(i)}) \otimes f_{(i)}$, where $f = \mathbf{1}_B \otimes a$. Note that $\mathbf{1}_B(i) \otimes a = a$ if $i \in B$ and $\mathbf{1}_B(i) \otimes a = 0$ otherwise. So, there is j such that $B = A_{(j)} = \{(j), \ldots, (n)\}$. So we get $\int_\gamma^\otimes (\mathbf{1}_B \otimes a) = \bigvee_{i \geq j}^n \gamma(A_{(i)}) \otimes a$, and the maximum is attained for $i = j$. Further, $\int_\gamma^\otimes \mathbf{1}_\mathcal{C} = \int_\gamma^\otimes \mathbf{1}_\mathcal{C} \otimes 1 = \gamma(\mathcal{C}) \otimes 1 = 1$.

We can now prove our first characterization result.

Theorem 4. *Let* $I \colon L^\mathcal{C} \to L$ *be a mapping and* $\otimes$ *a left-conjunction. There is a capacity* γ *such that* $I(f) = \int_\gamma^\otimes f$ *for every* $f \in L^\mathcal{C}$ *if and only if*

1. $I(f \vee g) = I(f) \vee I(g)$, *for any comonotone* $f, g \in L^\mathcal{C}$.
2. *There is a capacity* $\lambda \colon 2^\mathcal{C} \to L$ *such that* $I(\mathbf{1}_A \otimes a) = \lambda(A) \otimes a$ *for every* $a \in L$ *and every* $A \subseteq \mathcal{C}$.

In that case, we have $\gamma = \lambda$.

Proof. Necessity is obtained by previous Lemmas. For sufficiency, assume that I is a mapping that satisfies conditions 1 and 2 and let $f \in L^\mathcal{C}$. We have $I(f) = I(\bigvee_{i=1}^n \mathbf{1}_{A_{(i)}} \otimes f_{(i)}) = \bigvee_{i=1}^n I(\mathbf{1}_{A_{(i)}} \otimes f_{(i)}) = \bigvee_{i=1}^n \lambda(A_{(i)}) \otimes f_{(i)} = \int_\lambda^\otimes f$.

Note that we have used all properties of left-conjunctions in our proof of the previous result. Moreover, contrary to universal integrals, $\int_\gamma^\otimes \mathbf{1}_A = \gamma(A) \otimes 1 \neq \gamma(A)$, generally, since 1 is not an identity on the right. To get the property $\int_\gamma^\otimes \mathbf{1}_A = \gamma(A)$, it is enough to assume the left-conjunction $\otimes$ is commutative. The set function $\hat{\gamma}(A) = \gamma(A) \otimes 1$ generally differs from γ. The following counter-example shows that we may have $\hat{\gamma}(A) \neq \gamma(A)$ and $\int_\gamma^\otimes \neq \int_{\hat{\gamma}}^\otimes$.

Example 1. $\mathcal{C} = \{1, 2\}$, $L = [0, 1]$ and $\otimes_G$ is the Gödel conjunction. We have $a \otimes_G 1 = 0$ if $a = 0$ and 1 otherwise so $a \otimes_G 1 \neq a$ if $a < 1$. Let us consider γ such that $\gamma(\{1\}) = 0$ and $\gamma(\{2\}) = 0.1$. If f is defined by $f_1 = 0$ and $f_2 = 0.8$ we obtain $\int_\gamma^{\otimes_G} f = (1 \otimes_G 0) \vee (0.1 \otimes 0.8) = 0.8$ and $\int_{\hat{\gamma}}^{\otimes_G} f = 1$ since $\hat{\gamma}(\{2\}) = 1$.

In the case when the functional I is maxitive, we prove the following:

Theorem 5. *Assume that* $\otimes$ *is a left conjunction that is right-cancellative, that is, for every* $a, b, c \in L$, *if* $a \otimes c = b \otimes c$ *then* $a = b$. *Let* $I \colon L^\mathcal{C} \to L$ *be a mapping. There is a possibility measure* Π *such that* $I(f) = \int_\Pi^\otimes f$ *for every* $f \in L^\mathcal{C}$ *if and only if* I *satisfies the following properties:*

1. $I(f \vee g) = I(f) \vee I(g)$, *for any* $f, g \in L^\mathcal{C}$.
2. *There is a capacity* $\lambda \colon 2^\mathcal{C} \to L$ *such that* $I(\mathbf{1}_A \otimes a) = \lambda(A) \otimes a$ *for every* $a \in L$ *and every* $A \subseteq \mathcal{C}$.

In that case, we have $\Pi = \lambda$.

Proof. A q-integral with respect to a possibility measure satisfies the requested properties. Let us prove the converse. According to the previous theorem, there exists a capacity λ such that $I(\mathbf{1}_A \otimes a) = \lambda(A) \otimes a$ for all a. For all $a \neq 0$, $(\mathbf{1}_A \otimes a) \vee (\mathbf{1}_B \otimes a) = (\mathbf{1}_{A \cup B} \otimes a)$ by the order-preservingness property; so $(\lambda(A) \otimes a) \vee (\lambda(B) \otimes a) = \lambda(A \cup B) \otimes a$. Again by order-preservingness, $(\lambda(A) \otimes a) \vee (\lambda(B) \otimes a) = (\lambda(A) \vee \lambda(B)) \otimes a$ hence the cancellativeness property allows us to conclude $\lambda(A \cup B) = \lambda(A) \vee \lambda(B)$.

The above result also holds for commutative conjunctions (so, for triangular norms), and also pseudo-multiplications, since in that case $I(\mathbf{1}_A) = I(\mathbf{1}_A \otimes \mathbf{1}_{\mathcal{C}}) = \Pi(A) \otimes 1 = \Pi(A)$. But it does not hold for the Gödel conjunction, nor the other non-commutative conjunction mentioned above.

4 Integrals defined with a right-conjunction

We consider a binary operation $\otimes_C$ defined by $a \otimes_C b := b \otimes a$ where $\otimes$ is a left-conjunction. Clearly, $a \otimes_C 1 = a$, $a \otimes_C 0 = 0$, $0 \otimes_C a = 0$, and the maps $x \mapsto a \otimes_C x$ and $x \mapsto x \otimes_C a$ are order-preserving. We call $\otimes_C$ a *right-conjunction*. The associated q-integral is

$$\int_\gamma^{\otimes_C} f = \bigvee_{A \subseteq C} (\bigwedge_{i \in A} f_i \otimes \gamma(A)).$$

It generally differs from $\int_\gamma^{\otimes} f$ [4]. Using the results presented in Section 3 it is easy to prove that if $f \in L^{\mathcal{C}}$, then $f = \bigvee_{i=1}^n f_{(i)} \otimes_C \mathbf{1}_{A_{(i)}}$ where for every $\ell \in \{1, \ldots, n-1\}$, the maps $f_{(\ell)} \otimes_C \mathbf{1}_{A_{(\ell)}}$ and $\bigvee_{i=\ell+1}^n f_{(i)} \otimes_C \mathbf{1}_{A_{(i)}}$ are comonotone.

Since $x \mapsto x \otimes_C a$ is increasing, $\int_\gamma^{\otimes_c} f = \bigvee_{i=1}^n \gamma(A_{(i)}) \otimes_C f_{(i)}$, and since $x \mapsto a \otimes_C x$ is increasing, the map $\int_\gamma^{\otimes_C} : L^{\mathcal{C}} \to L$ is order-preserving for every capacity γ. Moreover we have $\int_\gamma^{\otimes_C} f = \bigvee_{a \in L} (a \otimes \gamma(\{f \geq a\}))$ and for any comonotone $f, g \in L^{\mathcal{C}}$ we have $\int_\gamma^{\otimes_C} (f \vee g) = \int_\gamma^{\otimes_C} (f) \vee \int_\gamma^{\otimes_C} (g)$.

For every capacity γ, every $A \subseteq C$ and every $a \in L$, it holds

$$\int_\gamma^{\otimes_C} (a \otimes_C \mathbf{1}_A) = \bigvee_{B \subseteq C} \gamma(B) \otimes_C \bigwedge_{i \in B} (a \otimes_C \mathbf{1}_A(i)) = \bigvee_{B \subseteq C} \bigwedge_{i \in B} (\mathbf{1}_A(i) \otimes a) \otimes \gamma(B).$$

We have $\bigwedge_{i \in B} (\mathbf{1}_A(i) \otimes a) = a$ if $B \subseteq A$ and $\bigwedge_{i \in B} (\mathbf{1}_A(i) \otimes a) = 0$ otherwise, so

$$\int_\gamma^{\otimes_C} (a \otimes_C \mathbf{1}_A) = \bigvee_{B \subseteq A} a \otimes \gamma(B) = a \otimes \gamma(A).$$

In particular, as $1 \otimes a = a$, we have $\int_\gamma^{\otimes_C} \mathbf{1}_A = \gamma(A)$, and so $\int_\gamma^{\otimes_C} (a \otimes_C \mathbf{1}_A) = a \otimes \int_\gamma^{\otimes_C} \mathbf{1}_A$. We are ready to prove the following characterization result.

Theorem 6. *Let $I\colon L^{\mathcal{C}} \to L$ be a mapping and $\otimes_C$ a right-conjunction. There is a capacity γ such that $I(f) = \int_\gamma^{\otimes_C} f$ for every $f \in L^{\mathcal{C}}$ if and only if*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotone $f, g \in L^{\mathcal{C}}$.*
2. *For every $A \subseteq C$ and every $a \in L$ we have $I(\mathbf{1}_A \otimes_C a) = I(\mathbf{1}_A) \otimes_C a$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$.*

In that case γ is defined by $\gamma(A) = I(\mathbf{1}_A)$ for every $A \subseteq C$.

Proof. The proof that $I(f) = \bigvee_{i=1}^n I(\mathbf{1}_{A_{(i)}}) \otimes_C f_{(i)}$ is similar to that of Theorem 4. Then we must prove that the set function $\lambda\colon A \mapsto I(\mathbf{1}_A)$ is a capacity. We do have that $\lambda(\mathcal{C}) = 1$, and $\lambda(\varnothing) = I(0) = I(0 \otimes_C \mathbf{1}_{\mathcal{C}}) = 0 \otimes \lambda(\mathcal{C}) = 0$. Finally, for every $A \subseteq B$, as $1 \otimes_C \mathbf{1}_A \leq 1 \otimes_C \mathbf{1}_B$,we get by conditions 1 and 2 that $\lambda(A) = \lambda(A) \otimes 1 = I(1 \otimes_C \mathbf{1}_A) \leq I(1 \otimes_C \mathbf{1}_B) = 1 \otimes \lambda(B) = \lambda(B)$.

Note that if the maxitivity condition 1 is extended to any pair of mappings f, g, then $I(\mathbf{1}_B) = \Pi(B)$ and $I(f) = MAX_\pi^{\otimes_C}(f) = \bigvee_{i=1}^n f_i \otimes \pi_i$. The contraposed Gödel conjunction $\otimes_{GC}$ and the right conjunction associated with the conjunction $\otimes_{rTC}$ introduced above are examples of right-conjunctions.

5 Q-cointegrals defined from left-conjunctions

As L is equipped with an involutive negation $t \mapsto 1 - t$, we can define an implication $\to$ from $\otimes$ by semi-duality: $a \to b := 1-(a\otimes(1-b))$. This implication satisfies the following very usual properties:

- $a \to 1 = 1$, $0 \to b = 1$ and $1 \to b = b$,
- $\to$ is decreasing according to its first argument,
- $\to$ is increasing according to the second one.

Under property $1 \to b = b$, implication $\to$ is called a *border implication*. The implication $\to$ and the conjunction $\otimes$ exchange via semi-duality.

The following implication operations satisfy the properties presented above.

- S-implications obtained from triangular norms by semi-duality, among them, the Kleene-Dienes implication, the Łukasiewicz implication $a \to_{\text{Ł}} b = \max(1-a+b, 0)$, the nilpotent implication induced by the nilpotent minimum $a \to b = 1$ if $a \geq b$ and $(1-a) \vee b$ otherwise, Reichenbach implication (induced by product).
- Implications obtained from triangular norms by residuation Res, among which the Gödel and Łukasiewicz implications, the nilpotent implication, Goguen implication (induced by product). Operations of the form $\otimes = \mathcal{S}(Res(\star))$, where $\star$ is a left-continuous t-norm, are (generally non-commutative) left-conjunctions instrumental in the construction of Section 4.
- $a \to_{XC} b = 1$ if $a = 0$, and b if $a \neq 0$ [5].

Definition 2. *Let* $\rightarrow$ *be a border implication as defined above on* L *and* $\gamma\colon 2^{\mathcal{C}} \rightarrow L$ *be a capacity. The q-cointegral* $\int_\gamma^{\rightarrow} \colon L^{\mathcal{C}} \rightarrow L$ *is the mapping*

$$\int_\gamma^{\rightarrow} f = \bigwedge_{A \subseteq C} \big(\gamma^c(A) \rightarrow \bigvee_{i \in A} f_i\big), \text{ for all } f \in L^{\mathcal{C}}.$$

Using semi-duality, q-cointegrals can be expressed in terms of q-integrals.

Proposition 2. $\int_\gamma^{\rightarrow}(f) = 1 - \int_{\gamma^c}^{\otimes}(1-f)$.

As in [5], using semi-duality, we derive the following results from Section 4.

For any $f \in L^{\mathcal{C}}$, we have $f = \bigwedge_{i=1}^n \mathbf{1}_{\overline{A_{(i+1)}}} \rightarrow f_{(i)}$ where for every $\ell \in \{1,\ldots,n-1\}$, the maps $\mathbf{1}_{\overline{A_{(\ell+1)}}} \rightarrow f_{(\ell)}$ and $\bigwedge_{i=\ell+1}^n \mathbf{1}_{\overline{A_{(i+1)}}} \rightarrow f_{(i)}$ are comonotone. Also, $\int_\gamma^{\rightarrow}(f) = \bigwedge_{i=1}^n \gamma^c(\overline{A_{(i+1)}}) \rightarrow f_{(i)} = \bigwedge_{a \in L} \gamma^c(\{f \leq a\}) \rightarrow a$.

Moreover we have the following characterisation result.

Theorem 7. *Let* $I\colon L^{\mathcal{C}} \rightarrow L$ *be a mapping. There is a capacity* γ *such that* $I(f) = \int_\gamma^{\rightarrow} f$ *for every* $f \in L^{\mathcal{C}}$ *if and only if the following properties are satisfied.*

1. $I(f \wedge g) = I(f) \wedge I(g)$, *for any comonotone* $f, g \in L^{\mathcal{C}}$.
2. *There is a capacity* $\rho\colon 2^{\mathcal{C}} \rightarrow L$ *such that* $I(\mathbf{1}_A \rightarrow a) = \rho^c(A) \rightarrow a, \forall a \in L$.

In that case $\rho = \gamma$.

We can run a similar study for q-cointegrals induced by a right-conjunction. They use implications that differ from the above ones by the property $a \rightarrow 0 = 1 - a$ that replaces $1 \rightarrow b = b$, i.e. these implications reconstruct the involutive negation.

Remark 1. The equality (3) between the q-cointegral and the q-integral, in the case of conjunction $\wedge$ and its semi-dual $(1-a) \vee b$ does not extend to conjunction-based q-integrals. For example only the inequality (12): $\int_\gamma^{\otimes} f \geq \int_\gamma^{\rightarrow} f$ holds for $(\otimes, \rightarrow) \in \{(\otimes_G, \rightarrow_G), (\otimes_{GC}, \rightarrow_{GC})\}$ [4]. This inequality cannot even be generalised to other conjunctions. For example, for the nilpotent minimum and the nilpotent maximum ($a \underline{\vee} b = a \vee b$, if $a \leq 1-b$ and 1 otherwise) the relations $a \overline{\wedge} b \leq a \wedge b$ and $a \vee b \leq a \underline{\vee} b$ imply the opposite inequality $\int_\gamma^{\overline{\wedge}} f \leq \int_\gamma^{\Rightarrow} f$, where $\Rightarrow$ is $\mathcal{S}(\overline{\wedge})$, but $\int_\gamma^{\overline{\wedge}} f \neq \int_\gamma^{\Rightarrow} f$.

Example 2. Consider $\mathcal{C} = \{1,2\}$, $L = [0,1]$, a capacity γ defined by $\gamma(\{1\}) = \gamma(\{2\}) = 0$ and f such that $f_1 = 0.5$ and $f_2 = 0.5$; then $\int_\gamma^{\overline{\wedge}} f = 0.5 < \int_\gamma^{\Rightarrow} f = 1$.

Remark 2. The diagram on Fig. 1 holds for transforms of many more operations $\star$ than $\wedge$. Fodor [8] has shown that the existence of $Res(\star)$ is a necessary and sufficient condition for the square on the left-hand side to commute. The whole diagram commutes if and only if moreover $\star$ is commutative. $Res(\star)$ exists for left and right conjunctions. So from any q-integral based on a left-conjunction $\otimes$, one can generate two q-cointegrals (based on $Res(\otimes)$ and $\mathcal{S}(\otimes)$), and another q-integral based on $Res \circ \mathcal{S}(\otimes) = \mathcal{S} \circ Res(\otimes)$. We can do likewise for the associated right-conjunction $\mathcal{A}(\otimes)$.

Remark 3. The q-cointegral-like expression defined on a complete residuated lattice in [6] is based on an anticapacity ν i.e., a set function such that $\nu(\emptyset) = 1$, $\nu(C) = 0$ and $A \subseteq B$ implies $\nu(A) \geq \nu(B)$. For all $f \in L^{\mathcal{C}}$, it takes the form: $\oint_{\nu}^{\to} f = \bigwedge_{A \subseteq C} \bigvee_{i \in A} (f_i \to \nu(A))$, for all $f \in L^{\mathcal{C}}$. It is what we call a desintegral in [4] as it is decreasing with f_i.

6 Conclusion

In this paper, we have proposed a very general setting for generalized forms of Sugeno integrals where the inside operation is either a not-necessarily commutative multivalued conjunction or a multivalued implication. The properties in the algebraic setting were chosen to be minimal in order to preserve representation theorems by means of comonotonic minitive or maxitive functionals: integrals are maxitive, while cointegrals are minitive and differ from each other, in contrast with the case of standard Sugeno integrals. One remaining open problem is to find necessary and sufficient conditions for a conjunction $\otimes$ to ensure the equality between integrals and their semi-dual cointegrals.

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