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Observer-based controllers for fractional differential systems

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Abstract

The goal of this paper is to propose observer-based controllers, either in state-space form or in polynomial representation, for fractional differential systems. As for linear differential systems of integer order, polynomial representation will allow us to take advantage of the Youla parametrization in order to asymptotically reject some perturbations. This will be illustrated on a worked-out example.

Introduction

The fractional differential systems we consider have previously been defined in [1, 2, 3] where the *stability* question was solved both from an algebraic and an analytic point of view. Moreover *controllability* and *observability* notions for such systems, given in state-space form or in polynomial representation, were defined in [4, 3].

Making use of these controllability and observability properties, we are able to build state-feedback controllers, asymptotic observers and more generally observer-based controllers to solve regulation or tracking problems.

The paper is organized as follows: the first part presents the theory of observer-based controllers for fractional differential systems in state-space form, which was announced in [5]. The second part is devoted to observer-based controllers for fractional differential systems in polynomial representation, and takes advantage of the Youla-Jabr-Bongiorno parametrization [6, 7, 8] in order to asymptotically reject some perturbations. The polynomial representation can be compared to the module theoretic framework using operational calculus, developped in [9]. In the last part, we illustrate our results on the case study of a mechanical system made of an oscillator with both classical fluid damping of order one and frictious damping of order one-half.

1 Observer-based controllers for fractional differential systems in state-space form

In this section, we recall useful criteria for structural properties of fractional differential systems such as stability, controllability and observability. We then apply them to the design of *static state feedback controllers*, the design of *asymptotic observers* and the synthesis of *observer-based controllers.*

1.1 Some structural results

We recall some structural results on the system:

$$\left\{ \begin{array}{lcl} d^\alpha x & = & Ax + Bu \\ y & = & Cx \end{array} \right. , \quad x(0) = x_0 \tag{1}$$

where $0 < \alpha < 1$, $u \in \mathbb{R}^m$ is the control, $x \in \mathbb{R}^n$ is the state, and $y \in \mathbb{R}^p$ is the observation. Moreover $d^\alpha x$ is the *smooth* derivative of order α of x (see [1, 10]), which proves to be at least continuous if x has a locally integrable first derivative:

$$d^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(t-\tau)}{\tau^\alpha} \, d\tau$$

where Γ denotes the usual Euler Gamma function.

Let us recall the following important properties, a detailed proof of which can be found in [2, 4] for example:

- System (1) is **asymptotically stable** iff $|\arg(\operatorname{spec} A)| > \alpha\pi/2$,

  (in this case the asymptotics of the state displays a so-called ultra-slow behaviour: it goes like $t^{-\alpha}$ as t tends to infinity, which is much slower than any exponential decay, and very typical for fractional differential systems).

- System (1) is **controllable** iff $\mathcal{C} \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has rank n,

  (in this case, an explicit formula can be given for the control; a fractional definition of the controllability Gramian is therefore being used).

- System (1) is **observable** iff $\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has rank n,

  (in this case, an explicit formula can be given for the initial state recovery; a fractional definition of the observability Gramian is therefore being used).

1.2 Static state feedback controller

The aim is to change the dynamics of the system (1), or equivalently to place the spectrum of $A - BK$; in other words we want to stabilize system (1) with a control law $u = -K\,x + r$. We have the following result:

Theorem 1. *System (1) is stabilizable by static state feedback iff the uncontrollable modes of A are asymptotically stable.*

The proof can be established as in the usual integer order case, since it involves *algebraic* properties of the pair (A, B) only.

Remark 1.1. In particular if $(A,\ B)$ is controllable, the spectrum of $A - BK$ can be assigned anywhere in the complex region of asymptotic stability $|\arg(\sigma)| > \alpha\pi/2$.

1.3 Asymptotic observer

The aim is to build a fractional differential system, the state $\hat{x}$ of which asymptotically converges to x, with the following dynamics:

$$\begin{cases} d^\alpha \hat{x} = A\hat{x} + Bu - L(\hat{y} - y) \\ \hat{y} = C\hat{x} \end{cases}$$

Theorem 2. *System (1) is detectable by an asymptotic observer iff the unobservable modes of A are asymptotically stable.*

The proof can be established as in the usual integer order case, since it involves *algebraic* properties of the pair (C, A) only.

Remark 1.2. In particular if $(C,\ A)$ is observable, the spectrum of $A - LC$ (i.e. the matrix of dynamics of the state error $\hat{x} - x$) can be assigned anywhere in the complex region of asymptotic stability $|\arg(\sigma)| > \alpha\pi/2$.

1.4 Observer-based controller

Now replacing x by its estimate $\hat{x}$ in the control law, i.e. taking $u = -K\,\hat{x} + r$, leads to an observer-based controller for which we can formulate the following **separation principle**:

Theorem 3. *An observer-based controller can be synthesized for system (1) iff the non asymptotically stable modes of A are both controllable and observable.*

The proof is straightforward when expressing the global system in the appropriate basis, namely:

$$\begin{aligned} d^\alpha \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix} &= \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix} \\ &\quad + \begin{bmatrix} B \\ 0 \end{bmatrix} r \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix} \end{aligned}$$

Hence the global dynamics is determined by

$$\operatorname{spec}(A - BK) \cup \operatorname{spec}(A - LC)$$

which is nothing but the classical separation principle.

2 Observer-based controllers for fractional differential systems in polynomial representation

In the *multivariable* case, when the poles of $A-BK$ and $A-LC$ are placed from an observer-based controller synthesis (see section 1.4), some degrees of freedom are left : $n_d = n(m+p-2)$.

As for classical linear differential systems of integer order these degrees of freedom will be used to place some zeros in the closed-loop transfer functions from perturbations to outputs, called *regulation zeros*, in order to asymptotically reject some perturbations.

The Youla-Jabr-Bongiorno parametrization of the controller turns out to be very appropriate for this task and is easily obtained from a polynomial representation of the fractional differential system.

2.1 Notion of zeros

A transfer function of the input/output relation of a scalar system (1) is

$$\mathcal{H}(s) = C(s^\alpha I - A)^{-1}B = \frac{n(\sigma)}{d(\sigma)}$$

with

$$\sigma = s^\alpha$$

The zeros of the system are the complex roots of $n(\sigma) = 0$.

For example, when $\alpha = 1/2$, a zero located at $\sigma = 0$ enables to asymptotically reject $Y_1(t) = 1_{t\geq 0}$ the Heaviside unit step; a double zero located at $\sigma = 0$ also enables to asymptotically reject $Y_{3/2}(t) \propto \sqrt{t_+}$; a triple zero at $\sigma = 0$ also enables to asymptotically reject $Y_2(t) = t_+$ the unit slope, and so forth.

To prove this, we use the input $U(s) = s^{-\beta}$, compute the corresponding output $Y(s)$ and use the final value theorem for a stable system $\lim_{t\to+\infty} y(t) = \lim_{s\to 0^+} sY(s)$ with $sY(s) = \sigma^{2(1-\beta)}n(\sigma)/d(\sigma)$.

2.2 Some structural properties

A fractional differential input/output relation can be written in a polynomial representation of the following form:

$$\left\{\begin{array}{rcl} P(\sigma)\,\xi &=& Q(\sigma)\,u \\ y &=& R(\sigma)\,\xi \end{array}\right. \qquad (2)$$

where $\xi \in \mathbb{R}^{\bar{n}}$ is the partial state (see for example [11, 8]). Let us recall that:

- System (2) is **BIBO** (bounded input–bounded output) iff $\forall\sigma, |\arg(\sigma)| \leq \alpha\pi/2$, $\det P(\sigma) \neq 0$ when the triplet (P, Q, R) is minimal; otherwise, the criterium applies to the minimum degree polynomial $d(\sigma)$ of the denominator of the irreducible form of the transfer matrix $R(\sigma)P^{-1}(\sigma)Q(\sigma)$.
- System (2) is **controllable** iff P and Q are left coprime;

  in this case, system (2) is equivalent to a canonical polynomial controller form:

$$\left\{\begin{array}{rcl} P_c(\sigma)\,\xi_c &=& u \\ y &=& R_c(\sigma)\,\xi_c \end{array}\right. \qquad (3)$$

- System (2) is **observable** iff P and R are right coprime;

  in this case, system (2) is equivalent to a canonical polynomial observer form:

$$\left\{\begin{array}{rcl} P_o(\sigma)\,\xi_o &=& Q_o(\sigma)\,u \\ y &=& \xi_o \end{array}\right. \qquad (4)$$

Therefore if (2) is supposed to be *minimal*, then the following Bezout identity holds:

$$P_o(\sigma)E_o(\sigma) + Q_o(\sigma)F_o(\sigma) = I \qquad (5)$$

and the following identity is true:

$$P_o(\sigma)R_c(\sigma) = Q_o(\sigma)P_c(\sigma) \qquad (6)$$

2.3 Youla parametrization of the controller

This Youla parametrization, which will be presented now, has been used in [7] to stabilize a double inverted pendulum with asymptotic rejection of some perturbations.

Since we want to asymptotically reject some perturbations, let us consider the perturbed polynomial representation:

$$P_o(\sigma)\,y = Q_o(\sigma)\,u + Q_w(\sigma)\,w \qquad (7)$$

where $w \in \mathbb{R}^d$ is a *disturbance vector* which directly acts on the dynamics.

We also consider an *error measurement* vector $v \in \mathbb{R}^p$ which directly acts on the output y; after some tedious computations (see [8, section 8.7.]) the general perturbed polynomial representation takes the following controller form:

$$\begin{aligned} P_c(\sigma)\,\xi_c &= u + F_o(\sigma)Q_w(\sigma)\,w && (8)\\ y &= R_c(\sigma)\,\xi_c + E_o(\sigma)Q_w(\sigma)\,w + v && (9)\end{aligned}$$

where E_o and F_o are given by (5). This representation is well adapted to the computation of the closed-loop transfer functions, the feedback law being expressed directly from the partial state ξ_c.

More precisely we consider the most general linear, rational and causal control law of the form:

$$\bar{P}(\sigma)\,u + \bar{Q}(\sigma)\,y = r \tag{10}$$

where r is some *reference signal.* For the control law to be causal, $\bar{P}$ and $\bar{Q}$ are such that:

$$\bar{P}(\sigma)^{-1}\bar{Q}(\sigma) \quad \text{is strictly proper} \tag{11}$$

Let us denote

$$\Delta(\sigma) = \bar{P}P_c + \bar{Q}R_c \tag{12}$$

an arbitrary $m \times m$ polynomial matrix; Δ will be chosen such that the roots of its determinant are the closed-loop poles. We can take for example those of $A - BK$ and $A - LC$ obtained from a classical observer-based controller design (see section 1.4).

Moreover P_c and R_c being right coprime, there exists $\bar{P}^0$, $\bar{Q}^0$ such that

$$\bar{P}^0 P_c + \bar{Q}^0 R_c = \Delta \tag{13}$$

Then, for a prescribed Δ, the general solution of (12) is obtained from a particular solution given by (13), as follows:

$$\left\{\begin{array}{lllll} \bar{P} & = & \bar{P}^0 & + & \Lambda Q_o \\ \bar{Q} & = & \bar{Q}^0 & - & \Lambda P_o \end{array}\right. \tag{14}$$

making use of (6). In (14), Λ is a *free* polynomial matrix, which can suitably be chosen in order to place some *regulation zeros*, the poles remaining unchanged.

2.4 Transfer matrix of perturbations to outputs

We are now interested in the real output $\tilde{y} = y - v$. Replacing in (10) the controller form (8)-(9) and using (12), we obtain the different transfer functions:

$$\tilde{y} = T_{r\tilde{y}}\, r + T_{w\tilde{y}}\, w + T_{v\tilde{y}}\, v$$

with

$$\begin{aligned} T_{r\tilde{y}} &= R_c\Delta^{-1} && (15)\\ T_{w\tilde{y}} &= \left(R_c\Delta^{-1}(\bar{P}F_o - \bar{Q}E_o) + E_o\right)Q_w && (16)\\ T_{v\tilde{y}} &= -R_c\Delta^{-1}\bar{Q} && (17)\end{aligned}$$

3 A worked-out example

We now illustrate our results with the case study of a mechanical system composed of an oscillator with both classical fluid damping of order one and frictious damping of order one-half:

$$\ddot{z} + \alpha\dot{z} + \beta d^{1/2}z + \gamma z = u \tag{18}$$

The measured outputs y are position z and speed $\dot{z}$, which can be *physically* measured, contrarily to $d^{1/2}z$.

Remark 3.1. For technical reasons, namely regularity conditions (see [5]), it will be assumed that $d^{1/2}z(0) = 0$ and $d^{1/2}\dot{z}(0) = 0$.

3.1 State-space form of the system

In this system $\alpha = 1/2$, the state $x = \begin{bmatrix} z & d^{1/2}z & \dot{z} & d^{1/2}\dot{z}\end{bmatrix}'$ has dimension $n = 4$, the input u is a scalar $m = 1$ and the outputs $y = \begin{bmatrix} z & \dot{z}\end{bmatrix}'$ have dimension $p = 2$.

The matrices of the state-space form (1) are:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\gamma & -\beta & -\alpha & 0\end{bmatrix} \quad B = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

The open-loop system is assumed to be unstable, in the sense that there exists $\sigma \in \operatorname{spec}(A)$ such that $|\arg(\sigma)| \leq \pi/4$ and $\sigma^4+\alpha\sigma^2+\beta\sigma+\gamma = 0$. The triplet (A, B, C) is clearly minimal, thus permitting the construction of any observer-based controller with appropriate pole placement, as presented in section 1.4.

Let us point out that the number of degrees of freedom left to place some regulation zeros is $n_d = n(m+p-2) = 4$. We will present the polynomial representations of this system in section 3.2 and compute Youla parametrization of the controller in section 3.3 in order to place some regulation zeros in section 3.4.

3.2 Polynomial representations of the system

The canonical polynomial observer form (4) is easily obtained from (18) with:

$$P_o = \begin{bmatrix} \sigma^2 & -1 \\ \beta\sigma+\gamma & \sigma^2+\alpha \end{bmatrix} \quad \text{and} \quad Q_o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The canonical polynomial controller form (3) is given by:

$$\xi_c = z \quad P_c = \sigma^4+\alpha\sigma^2+\beta\sigma+\gamma \quad \text{and} \quad R_c = \begin{bmatrix} 1 \\ \sigma^2 \end{bmatrix}$$

We will denote $\beta_1 = 1$ and $\beta_2 = \sigma^2$ the components of R_c.

It can be easily checked that (6) holds, and that the solution of the Bezout identity (5) is given by:

$$E_o = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad F_o = \begin{bmatrix} \sigma^2+\alpha & 1 \end{bmatrix}$$

The error on the dynamics is supposed to act as a perturbation on the control u, therefore we choose $Q_w = Q_o$ in (7); there is also a measurement error v on the output y as in (9).

3.3 Youla parametrization of the controller

Let us parameterize the controller to point out the $n_d = 4$ degrees of freedom.

The closed-loop polynomial $\Delta(\sigma)$ being fixed, we have to find the expression (10) of the controller, more precisely we have to find polynomials $\bar{P}$ and $\bar{Q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$ such that:

$$\bar{P}(\sigma)P_c(\sigma) + q_1(\sigma)\beta_1(\sigma) + q_2(\sigma)\beta_2(\sigma) = \Delta(\sigma) \tag{19}$$

3.3.1 General solution: if $\bar{P}^0$, q_1^0, q_2^0 are a particular solution, we have:

$$(\bar{P}^0 - \bar{P})P_c = (q_1 - q_1^0)\beta_1 + (q_2 - q_2^0)\beta_2 \tag{20}$$

Moreover, β_1 and β_2 being coprime, there exists polynomials r_1 and r_2 allowing the following decomposition for P_c:

$$r_1\beta_1 + r_2\beta_2 = -P_c \tag{21}$$

A solution is for example:

$$r_1 = -\beta\sigma - \gamma \quad \text{and} \quad r_2 = -\sigma^2 - \alpha$$

Multiplying (21) by $\bar{P} - \bar{P}^0$, we obtain:

$$(\bar{P}^0 - \bar{P})P_c = (\bar{P} - \bar{P}^0)r_1\beta_1 + (\bar{P} - \bar{P}^0)r_2\beta_2 \tag{22}$$

Equations (20) and (22) are Bezout relations with respective solutions $q_1 - q_1^0$, $q_2 - q_2^0$ on the one hand and $(\bar{P} - \bar{P}^0)r_1$, $(\bar{P} - \bar{P}^0)r_2$ on the other hand; consequently there exists polynomials $l(\sigma)$ and $k(\sigma)$ such that:

$$\begin{aligned} \bar{P} &= \bar{P}^0 + l & (23) \\ q_1 &= q_1^0 + lr_1 + k\beta_2 & (24) \\ q_2 &= q_2^0 + lr_2 - k\beta_1 & (25) \end{aligned}$$

which is the general parametrization referred to in (14).

3.3.2 Particular solution: we now seek a particular solution of

$$\Delta = \bar{P}^0 P_c + q_1^0\beta_1 + q_2^0\beta_2 \tag{26}$$

We apply the division algorithm $\Delta = \bar{P}^0 P_c + r^0$ with $d^o r^0 < d^o P_c = 4$; then a unique solution of $r^0 = q_1^0\beta_1 + q_2^0\beta_2$ must be found, involving:

$$\begin{aligned} d^o q_1^0 &< d^o\beta_2 = 2 & (27) \\ d^o q_2^0 &< d^o\beta_1 = 0 & (28) \end{aligned}$$

which yields $d^o q_1^0 \leq 1$ and $q_2^0 = 0$.

Now let us perform a careful analysis on the degrees of the polynomials. Let $\bar{n} = d^o\Delta$, then from (23)-(25) we get:

$$\begin{aligned} d^o q_1 &= \max(1,\, d^o l + 1,\, d^o k + 2) \\ d^o q_2 &= \max(d^o l + 2,\, d^o k) \\ d^o \bar{P} &= \max(d^o \bar{P}^0,\, d^o l) = \max(\bar{n} - 4,\, d^o l) \end{aligned}$$

Considering strict causality condition (11), which writes here $d^o\bar{P} > d^o q_1$ and $d^o\bar{P} > d^o q_2$, some computations lead to:

$$\bar{n} - 4 > d^o l + 2 \qquad (29)$$
$$\bar{n} - 4 > d^o k + 2 \qquad (30)$$

Note that when $\bar{n} = 8$ as in our example, we find $d^o l \leq 1$ and $d^o k \leq 1$, which gives exactly 4 degrees of freedom, as expected.

3.4 Placement of regulation zeros

From (16)-(17) the transfer functions from the perturbations to the outputs are:

$$T_{w\tilde{y}} = \frac{1}{\Delta(\sigma)} \begin{bmatrix} \bar{P}(\sigma) \\ \sigma^2 \bar{P}(\sigma) \end{bmatrix}$$
$$T_{v\tilde{y}} = -\frac{1}{\Delta(\sigma)} \begin{bmatrix} q_1(\sigma) & q_2(\sigma) \\ \sigma^2 q_1(\sigma) & \sigma^2 q_2(\sigma) \end{bmatrix}$$

In order to asymptotically reject a step perturbation w on $y_1 = z$, we can choose l such that $\bar{P}(0) = 0$. On $y_2 = \dot{z}$ a slope will be naturally rejected (see section 2.1).

Moreover if we want to asymptotically reject a perturbation w of the form $\sqrt{t_+}$ on $y_1 = z$, we could take l such that $\bar{P}(0) = 0$ and $\bar{P}'(0) = 0$. In this case, a perturbation $w = t_+^{3/2}$ would be asymptotically rejected on $y_2 = \dot{z}$.

The same kind of analysis can be performed for the perturbation v_2 on z and $\dot{z}$ by using the two remaining degrees of freedom left by $k(\sigma)$.

From (24) we can see that no choice of k would allow to asymptotically reject a step perturbation v_1 on z, since $d^o\beta_2 = 2$.

Remark 3.2. Anyway it can be noticed from (19) that the choice $\bar{P}(0) = 0$, $q_1(0) = 0$ and $q_2(0) = 0$ is impossible, otherwise it would lead to the existence of an unstable closed-loop pole $\sigma = 0$.

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