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Low-order finite element method for the well-posed bidimensional Stokes problem

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We work on a low-order finite element approximation of the vorticity, velocity and pressure formulation of the bidimensional Stokes problem. In a previous paper, we have introduced the adequate space in which to look for the vorticity in order to have a well-posed problem. In this paper, we deal with the numerical approximation of this space, prove optimal convergence of the scheme and show numerical experiments in good accordance with the theory. We remark that despite one supplementary unknown (the vorticity), results of the scheme are much better than the ones obtained with the $\mathbb{P}^1$ plus bubble$-\mathbb{P}^1$ element in the velocity–pressure formulation.

Keywords: Stokes problem; vorticity–velocity–pressure formulation; mixed formulation; finite elements method; harmonic functions; integral method.

1. Introduction

1.1 *Motivation*

Let Ω be an open bounded domain of $\mathbb{R}^2$ with a regular boundary $\partial\Omega \equiv \Gamma$. Modelling of the equilibrium of an incompressible and viscous fluid leads to the Navier–Stokes problem (Landau & Lifchitz, 1971). If we neglect convection terms (when the viscosity is sufficiently important or the velocity of the fluid sufficiently small), we obtain the stationary Stokes problem which is (in primitive variables, i.e., velocity u and pressure p)

$$\begin{cases} -\nu\Delta u + \nabla p = f \quad \text{in } \Omega, \\ \operatorname{div} u = 0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where ν is the kinematic viscosity and f is a field of given external forces. For the sake of simplicity, we shall take $\nu = 1$ in all the following. We will, in the sequel, consider this problem in a vorticity–velocity–pressure setting which is well adapted to nonstandard boundary conditions (Girault, 1988 or Bramble & Lee, 1994) and was already studied in the framework of least-square finite element methods (see, e.g., Bochev & Gunzburger, 1994). For obtaining such a scheme, we introduce the vorticity ω which is the curl of the velocity, as a new unknown. Hence, the equations of the Stokes problem become

$$\begin{cases} \omega - \operatorname{curl} u = 0 \quad \text{in } \Omega, \\ \operatorname{curl} \omega + \nabla p = f \quad \text{in } \Omega, \\ \operatorname{div} u = 0 \quad \text{in } \Omega. \end{cases} \tag{1.2}$$

Moreover, we suppose that the velocity is zero on the boundary, which will be written here

$$u \cdot n = 0 \text{ on } \Gamma \quad \text{and} \quad u \cdot t = 0 \text{ on } \Gamma, \tag{1.3}$$

where $u \cdot n$ and $u \cdot t$ stand, respectively, for the normal and the tangential components of the velocity, n being the unit outer normal vector to the boundary Γ and t the tangent vector, chosen such that (n, t) is direct.

REMARK 1.1 The original scheme allows one to decompose the boundary Γ of the domain Ω with the help of *two* independent partitions

$$\begin{cases} \Gamma = \bar{\Gamma}_m \cup \bar{\Gamma}_p \quad \text{with } \Gamma_m \cap \Gamma_p = \emptyset; \\ \Gamma = \bar{\Gamma}_\theta \cup \bar{\Gamma}_t \quad \text{with } \Gamma_\theta \cap \Gamma_t = \emptyset. \end{cases} \tag{1.4}$$

Thus, the general boundary conditions for the Stokes problem read

$$\begin{cases} u \cdot n = 0 \quad \text{on } \Gamma_m, \\ p = \Pi_0 \quad \text{on } \Gamma_p, \\ \omega = 0 \quad \text{on } \Gamma_\theta, \\ u \cdot t = \sigma_0 \quad \text{on } \Gamma_t. \end{cases} \tag{1.5}$$

In all the following, we restrict ourselves to the most favourable case (see Dubois *et al.*, 2003b), which is $\Gamma_p = \emptyset$ so $\Gamma_m = \Gamma$. It means that the normal velocity is zero on the whole boundary

$$u \cdot n = 0 \quad \text{on } \Gamma. \tag{1.6}$$

Moreover, we will suppose, for convenience only, that $\Gamma_\theta = \emptyset$, so $\Gamma_t = \Gamma$, and that $\sigma_0 = 0$, which means

$$u \cdot t = 0 \quad \text{on } \Gamma. \tag{1.7}$$

This scheme, introduced by Dubois (1992), extends to arbitrary triangular meshes the very reliable method to solve the complete Navier–Stokes equations on quadrilateral and regular meshes, the HAWAY one (Harlow and Welch MAC scheme: Harlow & Welch, 1965; Arakawa C-grid: Arakawa, 1966; Yee staggered grids for Maxwell equations: Yee, 1966). In particular, it is now a basic method in the Computer Graphics community to simulate realistic movements of fluids; see, e.g., Génevaux *et al.* (2003). The idea of this formulation is to use exactly the same degrees of freedom as in the HAWAY method (see Fig. 1). Note that we will have to deal with only 7 degrees of freedom and that the pressure is discontinuous, which leads to a low-order approximation. In comparison, the lowest $\mathbb{P}^1 - \mathbb{P}^0$ approximation of the Stokes problem in primitive variables asks also for 7 degrees of freedom. But as it is well

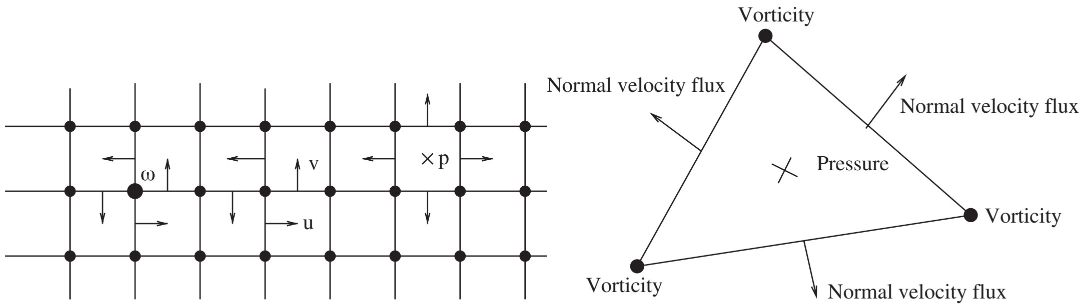

FIG. 1. Left panel: HAWAY discretization on a cartesian mesh. Right panel: Degrees of freedom on a triangular mesh.

known for a long time, this scheme needs stabilization because the inf–sup condition is not verified for this element (see, for example, Girault & Raviart, 1986). Nevertheless, there is still work done on this lowest approximation; see for an example of recent stabilization (Wang *et al.*, 2012). Compared with it, additional work for stability of our scheme is of the same order, but in addition, our scheme leads to an exactly divergence-free velocity.

We have intensively studied this three-fields mixed formulation in vorticity–velocity–pressure (see Dubois, 2002; Dubois *et al.*, 2003a,b; Salaün & Salmon, 2007) looking for the vorticity in $H^1(\Omega)$, the velocity in $H(\mathrm{div}, \Omega)$ and the pressure in $L^2(\Omega)$, and exploring finite element discretization. As it allows a wide range of boundary conditions, this formulation is becoming more and more interesting, see, for example, Amara *et al.* (2004) for a stabilized version looking for vorticity in $L^2(\Omega)$, the velocity in $H(\mathrm{rot}, \mathrm{div}, \Omega)$ and the pressure in $L^2(\Omega)$. We can also cite works of Bernardi and co-authors for a spectral discretization of the formulation in the case of homogeneous Dirichlet boundary conditions on the velocity; see, for example, Bernardi & Chorfi (2006), Amoura *et al.* (2007b) with extensions to multiply connected domains and extensions to Navier–Stokes equations in Azaïez *et al.* (2006), Amoura *et al.* (2007a).

We have proved in Dubois *et al.* (2003b) that a good way of obtaining a well-posed and stable problem is to look for the vorticity in a less regular functions space, denoted by M, of square integrable functions whose curl is in the dual space of $H_0(\mathrm{div}, \Omega)$ (see Equation (2.2)). We have also shown in this paper that in two dimensions and in the particular case of Dirichlet boundary conditions (where the velocity u is null on the whole boundary), this space reduces to the one previously introduced for the stream function–vorticity formulation by Bernardi *et al.* (1992) of square integrable functions whose Laplacian is in $H^{-1}(\Omega) = (H_0^1(\Omega))'$. From our point of view, the well-posedness of the problem should lead to a good numerical scheme. So, we work with the well-posed vorticity–velocity–pressure variational formulation. We propose, in the sequel, to study a natural discretization of the 'good' space for the vorticity, which leads to a numerical scheme using harmonic functions to compute the vorticity along the boundary.

Then, the scope of this work is the following. In Section 2, we recall the variational formulation and the needed properties of the space of vorticity. Section 3 is devoted to the numerical discretization of the scheme. In Section 4, we prove the stability and the optimal convergence of the scheme. The last section contains numerical experiments that confirm the theory and shows that, despite its very low order, results of our scheme are much better that the one obtained with the famous $\mathbb{P}^1$ plus bubble$-\mathbb{P}^1$ element in the velocity–pressure formulation (which, furthermore, asks for 11 degrees of freedom for each triangle).

1.2 *Functional spaces and notation*

Let Ω be a given open bounded and simply connected domain of $\mathbb{R}^2$ with a boundary Γ whose regularity will be precised later. We refer the reader to Adams (1975) for more details on the Sobolev spaces. We shall consider the following spaces: $\mathscr{D}(\Omega)$ is the space of all infinitely differentiable functions from Ω to $\mathbb{R}$ with compact support and $L^2(\Omega)$ is the space of all classes of square integrable functions. The subspace of $L^2(\Omega)$ containing square integrable functions whose mean value is zero over Ω is denoted by $L^2_0(\Omega)$. For any integer $m \geqslant 0$ and any real p such that $1 \leqslant p \leqslant \infty$, $W^{m,p}(\Omega)$ is the space of all functions $v \in L^p(\Omega)$ whose partial derivatives in the distribution sense, up to the total order m, belong to $L^p(\Omega)$. We define as usual $H^1(\Omega) = W^{1,2}(\Omega)$ and $H^2(\Omega) = W^{2,2}(\Omega)$. We denote by $\|\cdot\|_{m,p,\Omega}$ (respectively, $|\cdot|_{m,p,\Omega}$) norms (respectively, semi-norms) in Sobolev spaces $W^{m,p}(\Omega)$. We make the usual modification for $p = \infty$ and we agree to drop index 2 when $p = 2$. Space $H^1_0(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ with respect to the norm $\|\cdot\|_{1,\Omega}$. In the following, $(\cdot,\cdot)$ denotes the standard inner product in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{-1,1}$ the duality product between $H^1_0(\Omega)$ and its topological dual space $H^{-1}(\Omega)$. Finally, γ shall denote the trace operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$, or from $H^2(\Omega)$ onto $H^{3/2}(\Gamma)$ (see Lions & Magenes, 1968). We shall also need the dual spaces of $\mathscr{D}(\Omega)$ denoted by $\mathscr{D}'(\Omega)$ and of $H^{1/2}(\Gamma)$ (respectively, $H^{3/2}(\Gamma)$) denoted by $H^{-1/2}(\Gamma)$ (respectively, $H^{-3/2}(\Gamma)$) (see again Lions & Magenes, 1968).

- For any vector field v in $\mathbb{R}^2$, the divergence of v is defined by

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

Then, $H(\operatorname{div},\Omega)$ is the space of vector fields that belong to $(L^2(\Omega))^2$ with divergence (in the distribution sense) in $L^2(\Omega)$. We have classically

$$H(\operatorname{div},\Omega) = \{v \in (L^2(\Omega))^2 / \operatorname{div} v \in L^2(\Omega)\}, \tag{1.8}$$

which is a Hilbert space for the norm

$$\| v \|_{\operatorname{div},\Omega} = \left(\sum_{j=1}^{2} \| v_j \|^2_{0,\Omega} + \| \operatorname{div} v \|^2_{0,\Omega} \right)^{1/2}. \tag{1.9}$$

We recall that functions of $H(\operatorname{div},\Omega)$ have a normal trace in $H^{1/2}(\Gamma)$ that we will shortly denote by $v \cdot n$.

- Finally, let us recall that if v is a vector field in a bidimensional domain, then curl v is the scalar field defined by

$$\operatorname{curl} v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}. \tag{1.10}$$

In the following, we shall also use the curl of a scalar field, say φ, which is the bidimensional field defined by

$$\operatorname{curl} \varphi = \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right)^t. \tag{1.11}$$

2. Continuous variational formulation

2.1 *The problem to be solved*

• Suppose that we are looking for the vorticity in a space M, for the velocity in a space X and for the pressure in a space Y (these spaces will be detailed below). To obtain the variational formulation, we multiply the first equation of (1.2) by a test function $\varphi \in M$ and we formally integrate by parts

$$(\omega, \varphi) - (\operatorname{curl} u, \varphi) = (\omega, \varphi) - \langle \operatorname{curl} \varphi, u \rangle - \langle u \cdot t, \gamma\varphi \rangle_\Gamma .$$

In this expression, $\langle ., . \rangle_\Gamma$ stands for a boundary integral while $\langle ., . \rangle$ will appear further as a duality bracket. Then, with boundary condition (1.7), we obtain, for any φ,

$$(\omega, \varphi) - \langle \operatorname{curl} \varphi, u \rangle = 0 \quad \forall \varphi \in M .$$

The second equation of (1.2) is multiplied by a field $v \in X$. As we have

$$(\nabla p, v) = -(p, \operatorname{div} v) + \langle p, v \cdot n \rangle_\Gamma ,$$

with the boundary condition (1.6), we obtain

$$\langle \operatorname{curl} \omega, v \rangle - (p, \operatorname{div} v) = (f, v) \quad \forall v \in X .$$

Finally, the third equation of (1.2) is multiplied by q in Y and becomes

$$(\operatorname{div} u, q) = 0 \quad \forall q \in Y .$$

Then, the vorticity–velocity–pressure formulation is the following:

$$\begin{cases} \text{Find } (\omega,\ u,\ p) \text{ in } M \times X \times Y \text{ such that} \\ \quad (\omega, \varphi) - \langle \operatorname{curl} \varphi, u \rangle = 0 \quad \forall \varphi \in M , \\ \quad \langle \operatorname{curl} \omega, v \rangle - (p, \operatorname{div} v) = (f, v) \quad \forall v \in X , \\ \quad (\operatorname{div} u, q) = 0 \quad \forall q \in Y . \end{cases} \tag{2.1}$$

• Let us now introduce the spaces M, X and Y.

○ For the velocity, we define the space X by

$$X = H_0(\operatorname{div}, \Omega) = \{ v \in H(\operatorname{div}, \Omega) / v \cdot n = 0 \text{ on } \Gamma \}. \tag{2.2}$$

○ For the pressure, the space is

$$Y = L^2_0(\Omega) = \left\{ \varphi \in L^2(\Omega) \Big/ \int_\Omega \varphi \, \mathrm{d}x = 0 \right\} . \tag{2.3}$$

○ Here, we shall define the new space M where we search for the vorticity as announced above. Looking at the variational formulation (2.1), our problem is to give sense to the term $\langle \operatorname{curl} \varphi, u \rangle$ when u belongs to $H_0(\operatorname{div}, \Omega)$. Previously (see Dubois *et al.*, 2003a), we took φ in $H^1(\Omega)$ and $\langle \operatorname{curl} \varphi, u \rangle$ was simply the L^2 scalar product. However, it appears to be too restrictive: It is sufficient to take $\operatorname{curl} \varphi$ in the dual space of $H_0(\operatorname{div}, \Omega)$. It is the choice which is followed in Dubois *et al.* (2003b) where in the considered case $\Gamma_t \equiv \Gamma$, the complete equivalence between the vorticity–velocity–pressure and the classical stream function–vorticity formulation has been proved. Starting from this last formulation and following Ruas (1991) and Bernardi *et al.* (1992), we introduce the space for the vorticity as

$$M = \{\varphi \in L^2(\Omega) / \Delta\varphi \in H^{-1}(\Omega)\}, \tag{2.4}$$

where $H^{-1}(\Omega)$ is the topological dual space of $H_0^1(\Omega)$ with the associated norm

$$H^{-1}(\Omega) \ni \theta \longmapsto \| \theta \|_{-1,\Omega} = \sup_{v \in H_0^1(\Omega)} \frac{\langle \theta, v \rangle_{-1,1}}{\| \nabla v \|_{0,\Omega}}. \tag{2.5}$$

Consequently, the norm on M is defined by the relation

$$\| \varphi \|_M = (\| \varphi \|_{0,\Omega}^2 + \| \Delta\varphi \|_{-1,\Omega}^2)^{1/2}, \tag{2.6}$$

and M is a Hilbert space for this norm.

• The first question that arises is why choosing ω in M gives sense to the duality bracket $\langle \operatorname{curl} \varphi, v \rangle$ for $v \in X$. To answer this question, we need to study some properties of the space M.

2.2 *Properties of the vorticity space*

This section gives the main properties of the space M we shall use in the sequel. The extensive proofs of the three following propositions can be found in Dubois *et al.* (2003b).

PROPOSITION 2.1 Equivalence of norms.
○ The Sobolev space $H^1(\Omega)$ is contained in M with continuous embedding. We have $\|\varphi\|_M \leqslant \|\varphi\|_{1,\Omega}$ for any function φ in $H^1(\Omega)$.
○ Moreover, if $\varphi \in M \cap H_0^1(\Omega)$, its M-norm is equal to its H_0^1-norm.

PROPOSITION 2.2 Trace in M.
Let Ω be a simply connected, open bounded domain in $\mathbb{R}^2$, with a Lipschitz boundary Γ. Then, there exists a trace operator, still denoted by γ, which is a continuous application from M in $(H^{1/2}(\Gamma))' = H^{-1/2}(\Gamma)$.

• Now, let us explain the meaning of $\langle \operatorname{curl} \varphi, v \rangle$ when φ belongs to M and v to $X = H_0(\operatorname{div}, \Omega)$. However, the curl of an element of M is generally not in $L^2(\Omega)$. Thus, we have to check that it belongs to the dual space X'.

PROPOSITION 2.3 Curl of an element of M belongs to X'.
Let Ω be a simply connected, open bounded domain in $\mathbb{R}^2$, with a Lipschitz boundary Γ. Then the curl of any element φ of M belongs to the dual space $X' = (H_0(\operatorname{div}, \Omega))'$.

Proof.
○ Let φ be an element of M. First, we introduce function ψ, which is the unique solution in $H_0^1(\Omega)$ of the homogeneous Dirichlet problem

$$\begin{cases} \Delta\psi = \Delta\varphi & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma. \end{cases}$$

As usual, for such problems there exists $C > 0$ such that

$$\| \psi \|_{1,\Omega} \leqslant C \| \Delta\varphi \|_{-1,\Omega} . \tag{2.7}$$

Second, as we have $\Delta(\varphi - \psi) = 0$, and since we are in two dimensions, this is equivalent to $\operatorname{curl}(\varphi - \psi) = 0$. Hence, let $w = \operatorname{curl}(\varphi - \psi)$; it is in $(H^{-1}(\Omega))^2$ and its curl is zero. As Ω is simply connected, there exists a unique χ in $L^2_0(\Omega)$ such that $w = \nabla\chi$, and the inf–sup condition for the divergence (see Girault & Raviart, 1986) implies that

$$\| \chi \|_{0,\Omega} \leqslant C \| w \|_{-1,\Omega} \leqslant C(\| \operatorname{curl} \psi \|_{-1,\Omega} + \| \operatorname{curl} \varphi \|_{-1,\Omega}). \tag{2.8}$$

Now, let us observe that, for any η in $L^2(\Omega)$,

$$\| \operatorname{curl} \eta \|_{-1,\Omega} = \sup_{v \in H^1_0(\Omega)} \frac{\langle \operatorname{curl} \eta, v \rangle_{-1,1}}{\| \nabla v \|_{0,\Omega}} = \sup_{v \in H^1_0(\Omega)} \frac{(\eta, \operatorname{curl} v)}{\| \nabla v \|_{0,\Omega}} \leqslant \| \eta \|_{0,\Omega} .$$

Hence, we deduce from (2.7) and (2.8)

$$\| \chi \|_{0,\Omega} \leqslant C(\| \psi \|_{0,\Omega} + \| \varphi \|_{0,\Omega}) \leqslant C \| \varphi \|_M . \tag{2.9}$$

∘ Hence, we have obtained the following decomposition $\operatorname{curl} \varphi = \operatorname{curl} \psi + \nabla\chi$, with ψ in $H^1_0(\Omega)$ and χ in $L^2_0(\Omega)$. Now, let us consider a function v in $(\mathscr{D}(\Omega))^2$, which is contained in X. Then, let us calculate, in the distribution sense, $\langle \operatorname{curl} \varphi, v \rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}$ for any φ in M and v in $(\mathscr{D}(\Omega))^2$. Using the previous decomposition, we have

$$\langle \operatorname{curl} \varphi, v \rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = (\operatorname{curl} \psi, v) + \langle \nabla\chi, v \rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = (\operatorname{curl} \psi, v) - (\chi, \operatorname{div} v).$$

Finally, using (2.7) and (2.9), we obtain (C denotes various constants)

$$\begin{aligned} |\langle \operatorname{curl} \varphi, v \rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}| &\leqslant \| \operatorname{curl} \psi \|_{0,\Omega} \| v \|_{0,\Omega} + \| \chi \|_{0,\Omega} \| \operatorname{div} v \|_{0,\Omega} \\ &\leqslant C(\| \Delta\varphi \|_{-1,\Omega} \| v \|_{0,\Omega} + \| \varphi \|_M \| \operatorname{div} v \|_{0,\Omega}) \\ &\leqslant C \| \varphi \|_M \| v \|_{\operatorname{div},\Omega} . \end{aligned}$$

This inequality proves that $\operatorname{curl} \varphi$ defines a linear functional on $(\mathscr{D}(\Omega))^2$, which is continuous for the $H(\operatorname{div}, \Omega)$-topology. As X is the closure of $(\mathscr{D}(\Omega))^2$ with respect to the $H(\operatorname{div}, \Omega)$-norm, $\operatorname{curl} \varphi$ effectively belongs to X' for any φ of $H^1(\Omega)$. Let us remark that this density argument forces us to restrict ourselves to the case Γ_m equal to Γ.

∘ As, in the above inequality, the continuity constant depends on the M-norm, we deduce that any function φ of M has a weak curl which belongs to X', by density of $H^1(\Omega)$ in M. Finally, the previous inequality shows that, for any φ of M,

$$\| \operatorname{curl} \varphi \|_{X'} = \sup_{v \in X} \frac{\langle \operatorname{curl} \varphi, v \rangle}{\| v \|_{\operatorname{div},\Omega}} \leqslant C \| \varphi \|_M . \tag{2.10}$$

□

• Let us introduce now the space of harmonic functions of $L^2(\Omega)$

$$\mathscr{H}(\Omega) = \{\varphi \in L^2(\Omega), \Delta\varphi = 0 \in \mathscr{D}'(\Omega)\}.$$

We have the following decomposition of M (see Abboud *et al.*, 2004).

PROPOSITION 2.4 Decomposition of M.
∘ Any φ of M can be split as $\varphi = \varphi^0 + \varphi^\Delta$, with $\varphi^0 \in H_0^1(\Omega)$ and $\varphi^\Delta \in \mathscr{H}(\Omega)$ (i.e., harmonic). We have

$$M = H_0^1(\Omega) \oplus \mathscr{H}(\Omega).$$

∘ If Ω is convex and if φ belongs to $H^2(\Omega) \cap M = H^2(\Omega)$, then both φ^0 and φ^Δ belong also to $H^2(\Omega)$ and there exists a constant $C > 0$ such that

$$\begin{cases} \|\varphi^0\|_{2,\Omega}, & \leqslant C\|\varphi\|_{2,\Omega}, \\ \|\varphi^\Delta\|_{2,\Omega}, & \leqslant C\|\varphi\|_{2,\Omega}. \end{cases}$$

Finally, the following orthogonality property occurs.

PROPOSITION 2.5 Orthogonality property.
Let Ω be a simply connected, open bounded domain in $\mathbb{R}^2$, with a Lipschitz boundary. Then, for any harmonic function φ^Δ of $\mathscr{H}(\Omega)$ and for any velocity field v of $H_0(\text{div}, \Omega)$ which is divergence-free, we have

$$\langle \text{curl}\,\varphi^\Delta, v\rangle = 0. \tag{2.11}$$

Proof. This result is an immediate consequence of the following inequality:

$$|\langle \text{curl}\,\varphi, v\rangle| \leqslant C(\|\,\Delta\varphi\,\|_{-1,\Omega}\|\,v\,\|_{0,\Omega} + \|\,\varphi\,\|_M\|\,\text{div}\,v\,\|_{0,\Omega}),$$

which was obtained during the proof of Proposition 2.3. □

• In this paper, we will not deal with the hypotheses which make the continuous problem (2.1) well-posed in the most general case: Some results were established in Dubois *et al.* (2003b). We will only focus on numerical aspects of the above bidimensional case when Ω is convex with a Lipschitz boundary.

3. Numerical discretization

Let $\mathscr{T}$ be a triangulation of the domain Ω. As we want to use a finite element method, we shall assume that Ω is convex and polygonal (therefore with a Lipschitz boundary) in such a way that it is entirely covered by the mesh $\mathscr{T}$ and that the needed regularity assumptions in the previous section are verified. Moreover, we will assume that the mesh $\mathscr{T}$ belongs to a regular family of triangulations in the sense of Ciarlet (1987). Finally, $h_{\mathscr{T}}$ will be the maximum of the diameters of the triangles of $\mathscr{T}$.

Let us now introduce finite-dimensional spaces, say $M_{\mathscr{T}}$, $X_{\mathscr{T}}$ and $Y_{\mathscr{T}}$, which are, respectively, contained in M, X and Y.

3.1 *Numerical discretization of the pressure and velocity spaces*

The velocity is given by its fluxes through edges of the triangles, by the use of the Raviart–Thomas finite element of degree 1, say $RT^0_{\mathscr{T}}$, see Raviart & Thomas (1977). Then the discrete space for velocity reads as

$$X_{\mathscr{T}} = \{v \in RT^0_{\mathscr{T}} / v \cdot n = 0 \text{ on } \varGamma\}, \tag{3.1}$$

and if $\varPi^{\text{div}}_{\mathscr{T}}$ stands for the interpolation operator of any vector field v in $(H^1(\varOmega))^2$, the interpolation error is given by the following theorem:

THEOREM 3.1 Interpolation error for velocity, see Thomas (1980).

Let us assume that the mesh $\mathscr{T}$ belongs to a regular family of triangulations. Then, there exists a strictly positive constant C, independent of $h_{\mathscr{T}}$, such that, for any v in $(H^1(\varOmega))^2$, we have

$$\| v - \varPi^{\text{div}}_{\mathscr{T}} v \|_{0,\varOmega} \leqslant C h_{\mathscr{T}} \| v \|_{1,\varOmega} .$$

• For pressure interpolation, the space $P^0_{\mathscr{T}}$ of piecewise constants is chosen. Then, we set

$$Y_{\mathscr{T}} = \left\{ q \in P^0_{\mathscr{T}} \Big/ \int_{\varOmega} q \, \mathrm{d}x = 0 \right\}. \tag{3.2}$$

If we introduce the L^2 projection operator on $Y_{\mathscr{T}}$, denoted by $\varPi^0_{\mathscr{T}}$, we recall the following result (see, e.g., Girault & Raviart, 1986):

THEOREM 3.2 Interpolation error for pressure.

Let us assume that the mesh $\mathscr{T}$ belongs to a regular family of triangulations. There exists a strictly positive constant C, independent of $h_{\mathscr{T}}$, such that, for any $q \in H^1(\varOmega)$, we have

$$\| q - \varPi^0_{\mathscr{T}} q \|_{0,\varOmega} \leqslant C h_{\mathscr{T}} |q|_{1,\varOmega}.$$

• To conclude this section, let us recall the following basic property (see Brezzi & Fortin, 1991):

PROPOSITION 3.3 For any v in $(H^1(\varOmega))^2$ and for any q in $Y_{\mathscr{T}}$, we have

$$\int_{\varOmega} q \, \mathrm{div}(\varPi^{\text{div}}_{\mathscr{T}} v - v) \, \mathrm{d}x = 0.$$

3.2 *Numerical discretization of the vorticity space*

Starting from the decomposition of M given in Proposition 2.4, we begin by the discretization of $\mathscr{H}(\varOmega)$.

We recall that $\varOmega$ being polygonal allows us to entirely cover it with a mesh $\mathscr{T}$. We introduce the trace of mesh $\mathscr{T}$ on the boundary $\varGamma$. This is a set $\mathscr{A}(\mathscr{T}, \varGamma)$ of edges of triangles of the mesh which are contained in $\varGamma$. If $N_a(\mathscr{T}, \varGamma)$ is the number of these edges, we label them $\varGamma_i$, $1 \leqslant i \leqslant N_a(\mathscr{T}, \varGamma)$. As $\varGamma$ is closed, $N_a(\mathscr{T}, \varGamma)$ is also equal to the number of vertices of the mesh $\mathscr{T}$ on the boundary $\varGamma$. Then, we define the vector space $\mathscr{C}_{\mathscr{T}}$ generated by the characteristic functions of the edges $\varGamma_i \in \mathscr{A}(\mathscr{T}, \varGamma)$ of $\varGamma$

$$\mathscr{C}_{\mathscr{T}} = \mathrm{Span}\{q_i = \mathbb{1}_{\varGamma_i} / \varGamma_i \in \mathscr{A}(\mathscr{T}, \varGamma), 1 \leqslant i \leqslant N_a(\mathscr{T}, \varGamma)\}, \tag{3.3}$$

where $\mathbb{1}_{\varGamma_i}$ is the function from $\varGamma$ to $\mathbb{R}$ defined by

$$\mathbb{1}_{\varGamma_i}(x) = \begin{cases} 1 & \text{if } x \in \varGamma_i, \\ 0 & \text{if } x \notin \varGamma_i. \end{cases}$$

The dimension of $\mathscr{C}_{\mathscr{T}}$ is clearly equal to $N_a(\mathscr{T},\Gamma)$. Then, we denote by $\mathscr{S}$ the 'single layer' operator applied to functions of $\mathscr{C}_{\mathscr{T}}$, defined by

$$\mathscr{S}:\mathscr{C}_{\mathscr{T}} \ni q_i \longmapsto \varphi_i \in \mathscr{H}_{\mathscr{T},I}, \quad 1 \leqslant i \leqslant N_a(\mathscr{T},\Gamma),$$

where

$$\varphi_i(x) = \mathscr{S} q_i(x) = \int_\Gamma G(x,y)\, q_i(y)\, \mathrm{d}\gamma_y \quad \forall x \in \bar{\Omega},$$

and $G(x,y) = (1/2\pi)\log|x-y|$ is the Green kernel. Moreover, $\mathscr{H}_{\mathscr{T},I}$ is the discrete space spanned by functions $\varphi_i = \mathscr{S} q_i$ for any $q_i \in \mathscr{C}_{\mathscr{T}}$, $1 \leqslant i \leqslant N_a(\mathscr{T},\Gamma)$. This space is finite-dimensional and, clearly, its dimension is equal to the dimension of $\mathscr{C}_{\mathscr{T}}$. By construction, functions of $\mathscr{H}_{\mathscr{T},I}$ are harmonic. We shall denote by $S = \gamma\mathscr{S}$ the operator $\mathscr{S}$ on the boundary. Finally, for any x on the boundary Γ and for any i, $1 \leqslant i \leqslant N_a(\mathscr{T},\Gamma)$, we introduce

$$\gamma\varphi_i(x) = Sq_i(x) = \int_{\Gamma_i} G(x,y)\, \mathrm{d}\gamma_y.$$

It is important to remark that our discretization will be conforming as $\mathscr{H}_{\mathscr{T},I}$ is contained in $\mathscr{H}(\Omega) \subset M$, and that the boundary traces of functions in $\mathscr{H}_{\mathscr{T},I}$ are in $H^{-1/2}(\Gamma)$ (see Proposition 2.2).

• Now, we introduce the space $H^1_{\mathscr{T}}$ of continuous functions defined on $\bar{\Omega}$, polynomial of degree $\leqslant 1$ in each triangle of $\mathscr{T}$ and $H^1_{0,\mathscr{T}} = H^1_{\mathscr{T}} \cap H^1_0(\Omega)$. Then, for the discretization of M, we set

$$M_{\mathscr{T}} = H^1_{0,\mathscr{T}} \oplus \mathscr{H}_{\mathscr{T},I}. \tag{3.4}$$

More precisely, let us introduce the projection operator on $\mathscr{C}_{\mathscr{T}}$ by the following definition.

DEFINITION 3.4 The L^2-projection on the space of piecewise constants $\mathscr{C}_{\mathscr{T}}$ is

$$p_{\mathscr{C}}: L^2(\Gamma) \longrightarrow \mathscr{C}_{\mathscr{T}},$$
$$\rho \longmapsto p_{\mathscr{C}}\rho \text{ such that } \int_\Gamma (p_{\mathscr{C}}\rho - \rho) q\, \mathrm{d}\gamma = 0 \quad \forall q \in \mathscr{C}_{\mathscr{T}}.$$

Then, if $\Pi^1_{\mathscr{T}}: H^2(\Omega) \longrightarrow H^1_{\mathscr{T}}$ is the classical Lagrange interpolation operator associated with mesh $\mathscr{T}$, using the previous results and the above-mentioned decomposition of M (see Proposition 2.4), we set the following definition.

DEFINITION 3.5 Interpolation operators for the vorticity.
∘ The interpolation operator for a harmonic function φ^Δ is defined by

$$\phi_{\mathscr{T}}: \mathscr{H}(\Omega) \cap H^2(\Omega) \longrightarrow \mathscr{H}_{\mathscr{T},I},$$

where

$$\phi_{\mathscr{T}}\varphi^\Delta(x) = \mathscr{S} p_{\mathscr{C}}(\gamma\varphi^\Delta)(x) = \int_\Gamma G(x,y) p_{\mathscr{C}}(\gamma\varphi^\Delta)(y)\, \mathrm{d}\gamma_y \quad \forall x \in \bar{\Omega}.$$

∘ The interpolation operator for a vorticity field φ is associated with the decomposition $\varphi = \varphi^0 + \varphi^\Delta$, by

$$\mathscr{P}_{\mathscr{T}} : M \cap H^2(\Omega) \longrightarrow M_{\mathscr{T}} = H^1_{0,\mathscr{T}} \oplus \mathscr{H}_{\mathscr{T},1},$$

where

$$\mathscr{P}_{\mathscr{T}}\varphi = \Pi^1_{\mathscr{T}}\varphi^0 + \phi_{\mathscr{T}}\varphi^\Delta.$$

REMARK 3.6 Assuming that the function φ^Δ is in $\mathscr{H}(\Omega) \cap H^2(\Omega)$ allows us to define the projection of its trace $p_{\mathscr{C}}(\gamma\varphi^\Delta)$.

To conclude, let us observe that $M_{\mathscr{T}}$ has exactly the same dimension as $H^1_{\mathscr{T}}$. But, on the boundary, the classical piecewise linear continuous functions are replaced by harmonic functions.

3.3 *Interpolation error for the vorticity*

Let us begin with the following important result, whose proof can be found, e.g., in Nédélec (1977) or Dautray & Lions (1985).

THEOREM 3.7 The operator S is an isomorphism from the Sobolev space $H^s(\Gamma)$ onto $H^{s+1}(\Gamma)$ for any real number s.

Then, let us mention an important property of the 'single layer' operator $\mathscr{S}$. If we define the following subspace of $H^{-3/2}(\Gamma)$,

$${}_\circ H^{-3/2}(\Gamma) = \{\mu \in H^{-3/2}(\Gamma) / \langle \mu, 1\rangle_{-3/2,3/2} = 0\},$$

we have the following result (see Abboud *et al.*, 2004).

PROPOSITION 3.8 For any $q \in {}_\circ H^{-3/2}(\Gamma)$, if the harmonic function $\mathscr{S}q$ of $L^2(\Omega)$ is defined by

$$\mathscr{S}q(x) = \int_\Gamma G(x,y)q(y)\,\mathrm{d}\gamma_y \quad \forall x \in \bar{\Omega},$$

there exists a strictly positive constant C such that, for any $q \in {}_\circ H^{-3/2}(\Gamma)$,

$$\|\mathscr{S}q\|_{0,\Omega} \leqslant C\|q\|_{-3/2,\Gamma}. \tag{3.5}$$

• Now, let us come back to the L^2-projection operator $p_{\mathscr{C}}$ on the space of piecewise constants $\mathscr{C}_{\mathscr{T}}$ (see (3.3)). If $h_{\mathscr{T}}$ is the maximum diameter of triangles in $\mathscr{T}$, the standard interpolation error (Ciarlet, 1987) gives, for any ρ in $H^1(\Gamma)$,

$$\|\rho - p_{\mathscr{C}}\rho\|_{0,\Gamma} \leqslant Ch_{\mathscr{T}}|\rho|_{1,\Gamma}. \tag{3.6}$$

Then, using the classical result of interpolation between Sobolev spaces (Lions & Magenes, 1968), we obtain the following inequality for any $\rho \in H^{1/2}(\Gamma)$:

$$\|\rho - p_{\mathscr{C}}\rho\|_{0,\Gamma} \leqslant Ch^{1/2}_{\mathscr{T}}\|\rho\|_{1/2,\Gamma}. \tag{3.7}$$

Owing to this result, we obtain the two following propositions, already proved in Abboud *et al.* (2004), but written again for completeness of the study:

PROPOSITION 3.9
∘ For any $\rho \in H^{1/2}(\Gamma)$, we have

$$\|\rho - p_{\mathscr{C}}\rho\|_{-3/2,\Gamma} \leqslant C h_{\mathscr{T}}^{3/2} \|\rho\|_{1/2,\Gamma}. \tag{3.8}$$

∘ For any $\rho \in H^{1}(\Gamma)$, we have

$$\|\rho - p_{\mathscr{C}}\rho\|_{-3/2,\Gamma} \leqslant C h_{\mathscr{T}}^{2} \|\rho\|_{1,\Gamma}. \tag{3.9}$$

Proof.
∘ By definition of the norm in $H^{-3/2}(\Gamma)$, we have

$$\|\rho - p_{\mathscr{C}}\rho\|_{-3/2,\Gamma} = \sup_{\eta \in H^{3/2}(\Gamma)} \frac{\langle \rho - p_{\mathscr{C}}\rho, \eta\rangle_{-3/2,3/2}}{\|\eta\|_{3/2,\Gamma}}.$$

In the two cases, as ρ and η belong also to $L^2(\Gamma)$, the duality product can be rewritten as

$$\langle \rho - p_{\mathscr{C}}\rho, \eta\rangle_{-3/2,3/2} = \int_{\Gamma} (\rho - p_{\mathscr{C}}\rho)\eta \, \mathrm{d}\gamma.$$

By definition, $\int_{\Gamma} (\rho - p_{\mathscr{C}}\rho)\chi \, \mathrm{d}\gamma = 0$ for any $\chi \in \mathscr{C}_{\mathscr{T}}$, thus

$$\langle \rho - p_{\mathscr{C}}\rho, \eta\rangle_{-3/2,3/2} = \int_{\Gamma} (\rho - p_{\mathscr{C}}\rho)(\eta - \chi) \, \mathrm{d}\gamma \leqslant \|\rho - p_{\mathscr{C}}\rho\|_{0,\Gamma} \|\eta - \chi\|_{0,\Gamma}.$$

This result being true for any $\chi \in \mathscr{C}_{\mathscr{T}}$, we can choose $\chi = p_{\mathscr{C}}\eta$. Then, using (3.6), we obtain

$$\langle \rho - p_{\mathscr{C}}\rho, \eta\rangle_{-3/2,3/2} \leqslant \|\rho - p_{\mathscr{C}}\rho\|_{0,\Gamma} C h_{\mathscr{T}} \|\eta\|_{1,\Gamma} \leqslant C h_{\mathscr{T}} \|\rho - p_{\mathscr{C}}\rho\|_{0,\Gamma} \|\eta\|_{3/2,\Gamma},$$

which leads to

$$\|\rho - p_{\mathscr{C}}\rho\|_{-3/2,\Gamma} \leqslant C h_{\mathscr{T}} \|\rho - p_{\mathscr{C}}\rho\|_{0,\Gamma}. \tag{3.10}$$

∘ Hence, if ρ belongs to $H^{1/2}(\Gamma)$, inequality (3.8) is a direct consequence of (3.10) and (3.7). And when ρ belongs to $H^1(\Gamma)$, (3.9) results from (3.10) and (3.6). □

We can now state the main result of this section.

THEOREM 3.10 Error estimates.
We suppose that the mesh $\mathscr{T}$ belongs to a regular family of triangulations. Let φ be a given element of M decomposed into φ^0 and φ^Δ. We assume $\varphi \in H^2(\Omega)$ and $\varphi^0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there exists

some strictly positive constants, say C, only depending on the mesh family, such that

$$\|\varphi^0 - \Pi^1_{\mathscr{T}}\varphi^0\|_M \leqslant C h_{\mathscr{T}} |\varphi^0|_{2,\Omega}, \tag{3.11}$$

$$\|\varphi^\Delta - \phi_{\mathscr{T}}\varphi^\Delta\|_M \leqslant C h_{\mathscr{T}}^{3/2} \|\varphi\|_{2,\Omega}. \tag{3.12}$$

Moreover, if φ belongs to $H^{5/2}(\Omega)$, we have

$$\|\varphi^\Delta - \phi_{\mathscr{T}}\varphi^\Delta\|_M \leqslant C h_{\mathscr{T}}^2 \|\varphi\|_{5/2,\Omega}. \tag{3.13}$$

Proof.
○ As $H^2(\Omega)$ is a subset of $\mathscr{C}^0(\Omega)$ when Ω is two-dimensional, we can use the classical interpolation operator and we have the following interpolation error estimate (Ciarlet, 1987):

$$\|\varphi^0 - \Pi^1_{\mathscr{T}}\varphi^0\|_{1,\Omega} \leqslant C h_{\mathscr{T}} |\varphi^0|_{2,\Omega}.$$

But $\|\varphi^0 - \Pi^1_{\mathscr{T}}\varphi^0\|_M = \|\varphi^0 - \Pi_{\mathscr{T}}\varphi^0\|_{1,\Omega}$ because $\varphi^0 - \Pi^1_{\mathscr{T}}\varphi^0$ belongs to $H^1_0(\Omega)$ (see Proposition 2.1), finally, relation (3.11) is established.
○ We now interpolate the harmonic part φ^Δ of φ. By definition, φ^Δ verifies

$$\begin{cases} \Delta\varphi^\Delta = 0 & \text{in } \Omega, \\ \gamma\varphi^\Delta = \gamma\varphi & \text{on } \Gamma. \end{cases}$$

As φ is assumed to be in $H^2(\Omega)$, its trace $\gamma\varphi$ belongs to $H^{3/2}(\Gamma)$. As S is an isomorphism from $H^s(\Gamma)$ onto $H^{s+1}(\Gamma)$ (see Theorem 3.7), there exists a unique q in $H^{1/2}(\Gamma)$ such that $Sq = \gamma\mathscr{S}q = \gamma\varphi$ on Γ. And because of uniqueness, $\mathscr{S}q = \varphi^\Delta$ on Ω. Let us set now $q_{\mathscr{T}} = p_{\mathscr{C}}q$ in $\mathscr{C}_{\mathscr{T}}$. We recall that $\phi_{\mathscr{T}}\varphi^\Delta$ is given by

$$\phi_{\mathscr{T}}\varphi^\Delta(x) = \mathscr{S}q_{\mathscr{T}}(x) = \int_\Gamma G(x,y)q_{\mathscr{T}}(y)\,\mathrm{d}\gamma_y \quad \forall x \in \bar{\Omega}.$$

As these functions are harmonic, we obtain

$$\|\varphi^\Delta - \phi_{\mathscr{T}}\varphi^\Delta\|_M = \|\varphi^\Delta - \phi_{\mathscr{T}}\varphi^\Delta\|_{0,\Omega} = \|\mathscr{S}q - \mathscr{S}q_{\mathscr{T}}\|_{0,\Omega}.$$

As constants belong to $\mathscr{C}_{\mathscr{T}}$, $\int_\Gamma (q - q_{\mathscr{T}})\,\mathrm{d}\gamma = 0$, which means $q - q_{\mathscr{T}}$ belongs to ${}_\circ H^{-3/2}(\Gamma)$. Hence, Proposition 3.8 gives

$$\|\mathscr{S}q - \mathscr{S}q_{\mathscr{T}}\|_{0,\Omega} \leqslant C\|q - q_{\mathscr{T}}\|_{-3/2,\Gamma}.$$

Then, inequality (3.8) leads to

$$\|q - q_{\mathscr{T}}\|_{-3/2,\Gamma} = \|q - p_{\mathscr{C}}q\|_{-3/2,\Gamma} \leqslant C h_{\mathscr{T}}^{3/2}\|q\|_{1/2,\Gamma},$$

and, finally,

$$\|\varphi^\Delta - \phi_{\mathscr{T}}\varphi^\Delta\|_M \leqslant C h_{\mathscr{T}}^{3/2}\|q\|_{1/2,\Gamma} = C h_{\mathscr{T}}^{3/2}\|S^{-1}(\varphi)\|_{1/2,\Gamma} \leqslant C h_{\mathscr{T}}^{3/2}\|\gamma\varphi\|_{3/2,\Gamma}$$

because S is an isomorphism. Then, the continuity of the trace operator leads to the announced result.
○ Finally, noting that if φ is assumed to be in $H^{5/2}(\Omega)$, its trace $\gamma\varphi$ belongs to $H^2(\Gamma)$. Therefore, there exists a unique q in $H^1(\Gamma)$ such that $Sq = \gamma\mathscr{S}q = \gamma\varphi$ on Γ. The same arguments as above lead to the

inequality

$$\|\mathscr{S}q - \mathscr{S}q_{\mathscr{T}}\|_{0,\Omega} \leqslant C\|q - q_{\mathscr{T}}\|_{-3/2,\Gamma}.$$

Then, from formula (3.9), we have

$$\|q - q_{\mathscr{T}}\|_{-3/2,\Gamma} = \|q - p_{\mathscr{C}}q\|_{-3/2,\Gamma} \leqslant Ch_{\mathscr{T}}^2\|q\|_{1,\Gamma}.$$

Finally, (3.13) is obtained from the previous inequality by using exactly the same arguments (S is an isomorphism and trace continuity) as for (3.12). □

4. Stability and convergence results

For the sake of simplicity, in what follows, we will denote $(\operatorname{curl}\varphi, v)$ for the duality product between X' and X.

4.1 *Discrete inf–sup conditions*

As we work with a three-fields formulation, the analysis of this mixed problem leads to two inf–sup conditions (see Ladyzhenskaya & Ural'tseva, 1968; Babuška, 1971; Brezzi, 1974): A first classical one between pressure and velocity and a second one between vorticity and velocity. First, we give the discrete inf–sup condition between velocity and pressure, whose proof can be found in Raviart & Thomas (1977).

Proposition 4.1 Inf–sup condition on velocity and pressure.

Let us assume that Ω is polygonal and bounded, and that the mesh $\mathscr{T}$ belongs to a regular family of triangulations. Then, there exists a strictly positive constant a, independent of $h_{\mathscr{T}}$, such that

$$\inf_{q_{\mathscr{T}}\in Y_{\mathscr{T}}} \sup_{v_{\mathscr{T}}\in X_{\mathscr{T}}} \frac{(q_{\mathscr{T}}, \operatorname{div} v_{\mathscr{T}})}{\| v_{\mathscr{T}} \|_{\operatorname{div},\Omega}\| q_{\mathscr{T}} \|_{0,\Omega}} \geqslant a. \tag{4.1}$$

• Let us now express the link between vorticity and velocity. In a first step, we have to define the discrete kernel of the divergence operator. We set

$$V_{\mathscr{T}} = \{v \in X_{\mathscr{T}} / (\operatorname{div} v, q) = 0, \text{ for all } q \in Y_{\mathscr{T}}\}.$$

Then, this space is characterized by (see Dubois *et al.*, 2003a)

$$V_{\mathscr{T}} = \{v \in X_{\mathscr{T}} / \operatorname{div} v = 0 \text{ in } \Omega\}. \tag{4.2}$$

Moreover, the following link occurs between velocity and vorticity (see Dubois *et al.*, 2003a).

Lemma 4.2 Let us assume that Ω is simply connected. For any vector field v of $RT^0_{\mathscr{T}}$, divergence-free, such that $v \cdot n = 0$ on Γ, there exists a scalar field φ in $H^1_{\mathscr{T}}$ such that $\gamma\varphi = 0$ on Γ and $v = \operatorname{curl}\varphi$ in Ω. Conversely, for any scalar field φ in $H^1_{\mathscr{T}}$ such that $\gamma\varphi = 0$ on Γ, $v = \operatorname{curl}\varphi$ is a divergence-free vector field of $RT^0_{\mathscr{T}}$ such that $v \cdot n = 0$ on Γ.

This lemma leads naturally to the following result.

PROPOSITION 4.3 Inf–sup condition on vorticity and velocity. Let us assume that Ω is a polygonal convex domain. Then, there exists a strictly positive constant b, independent of $h_{\mathscr{T}}$, such that

$$\inf_{v_{\mathscr{T}} \in V_{\mathscr{T}}} \sup_{\varphi_{\mathscr{T}} \in M_{\mathscr{T}}} \frac{(v_{\mathscr{T}}, \operatorname{curl} \varphi_{\mathscr{T}})}{\| v_{\mathscr{T}} \|_{\mathrm{div},\Omega} \| \varphi_{\mathscr{T}} \|_M} \geqslant b. \tag{4.3}$$

Proof.
∘ The convexity of the domain allows us to write $(v_{\mathscr{T}}, \operatorname{curl} \varphi_{\mathscr{T}})$ for $v_{\mathscr{T}} \in V_{\mathscr{T}}$ and $\varphi_{\mathscr{T}} \in M_{\mathscr{T}}$.
∘ Let $v_{\mathscr{T}}$ be an arbitrary element of $V_{\mathscr{T}}$. Then, due to Lemma 4.2, we know that there exists a scalar field φ_0 in $H^1_{\mathscr{T}}$ such that $\gamma\varphi_0 = 0$ on Γ and $v_{\mathscr{T}} = \operatorname{curl} \varphi_0$ on Ω. Then φ_0 belongs to $H^1_{0,\mathscr{T}}$ and then to $M_{\mathscr{T}}$, and $\|\varphi_0\|_M = \|\varphi_0\|_{1,\Omega}$; see Proposition 2.1. Thus,

$$\sup_{\varphi_{\mathscr{T}} \in M_{\mathscr{T}}} \frac{(v_{\mathscr{T}}, \operatorname{curl} \varphi_{\mathscr{T}})}{\| \varphi_{\mathscr{T}} \|_M} \geqslant \frac{(v_{\mathscr{T}}, \operatorname{curl} \varphi_0)}{\| \varphi_0 \|_M} = \frac{\| v_{\mathscr{T}} \|^2_{0,\Omega}}{\| \varphi_0 \|_{1,\Omega}}.$$

Let us observe that, as $v_{\mathscr{T}}$ is divergence-free, we have $\| v_{\mathscr{T}} \|^2_{0,\Omega} = \| v_{\mathscr{T}} \|^2_{\mathrm{div},\Omega}$. Moreover, using the Poincaré inequality, there exists a strictly positive constant C, independent of $h_{\mathscr{T}}$, such that

$$\| \varphi_0 \|_{1,\Omega} \leqslant C \| \nabla\varphi_0 \|_{0,\Omega} = C \| \operatorname{curl} \varphi_0 \|_{0,\Omega} = C \| v_{\mathscr{T}} \|_{0,\Omega} .$$

These results lead to the expected inequality with $b = 1/C$. □

REMARK 4.4 Both inf–sup conditions are very classical because of the boundary condition considered: u is null on the whole boundary Γ. We can prove, for example, an inf–sup condition on velocity and pressure with the velocity null only on a part of the boundary or an inf–sup condition on vorticity and velocity with Dirichlet boundary conditions for the vorticity and the velocity on the same part of the boundary (see Dubois *et al.*, 2003a).

4.2 *Well-posedness of the discrete problem*

In order to use the previous results, let us recall that we must assume that Ω is a convex domain with a Lipschitz boundary. In the frame of a finite element discretization, it leads naturally to the following hypotheses on Ω, stated in the next result.

PROPOSITION 4.5 We assume that Ω is a polygonal convex domain and that the mesh $\mathscr{T}$ belongs to a regular family of triangulations.

Then, the discrete problem which consists in finding $(\omega_{\mathscr{T}}, u_{\mathscr{T}}, p_{\mathscr{T}})$ in $M_{\mathscr{T}} \times X_{\mathscr{T}} \times Y_{\mathscr{T}}$ such that

$$\begin{cases} (\omega_{\mathscr{T}}, \varphi_{\mathscr{T}}) - (\operatorname{curl} \varphi_{\mathscr{T}}, u_{\mathscr{T}}) = 0 \quad \forall \varphi_{\mathscr{T}} \in M_{\mathscr{T}}, \\ (\operatorname{curl} \omega_{\mathscr{T}}, v_{\mathscr{T}}) - (p_{\mathscr{T}}, \operatorname{div} v_{\mathscr{T}}) = (f, v_{\mathscr{T}}) \quad \forall v_{\mathscr{T}} \in X_{\mathscr{T}}, \\ (\operatorname{div} u_{\mathscr{T}}, q_{\mathscr{T}}) = 0 \quad \forall q_{\mathscr{T}} \in Y_{\mathscr{T}} \end{cases} \tag{4.4}$$

has a unique solution.

Proof. First, let us observe that the hypotheses are such that the two inf–sup conditions (4.1) and (4.3) are true. Secondly, as we consider a finite-dimensional square linear system, the only point to prove is that the solution associated with f equal to zero, is zero. For this, in the above system, we choose

$\varphi_{\mathscr{T}} = \omega_{\mathscr{T}}$, $v_{\mathscr{T}} = u_{\mathscr{T}}$ and $q_{\mathscr{T}} = p_{\mathscr{T}}$, and we add the three equations. We obtain

$$(\omega_{\mathscr{T}}, \omega_{\mathscr{T}}) = 0,$$

which implies $\omega_{\mathscr{T}} = 0$. Then, the second equation becomes

$$(p_{\mathscr{T}}, \operatorname{div} v_{\mathscr{T}}) = 0 \quad \forall v_{\mathscr{T}} \in X_{\mathscr{T}}.$$

Then, inf–sup condition (4.1) leads to $p_{\mathscr{T}} = 0$. Finally, the third equation shows that $u_{\mathscr{T}}$ belongs to $V_{\mathscr{T}}$, and the first one becomes

$$(\operatorname{curl} \varphi_{\mathscr{T}}, u_{\mathscr{T}}) = 0 \quad \forall \varphi_{\mathscr{T}} \in M_{\mathscr{T}},$$

as $\omega_{\mathscr{T}} = 0$. Finally, $u_{\mathscr{T}}$ is zero owing to inf–sup condition (4.3). □

4.3 *Stability of the discrete problem*

We can now study the stability of the discrete problem. Let (ω, u, p) be the solution in $M \times X \times Y$ of the continuous problem

$$\begin{cases} (\omega, \varphi) - (\operatorname{curl} \varphi, u) = 0 \quad \forall \varphi \in M, \\ (\operatorname{curl} \omega, v) - (p, \operatorname{div} v) = (f, v) \quad \forall v \in X, \\ (\operatorname{div} u, q) = 0 \quad \forall q \in Y, \end{cases}$$

and $(\omega_{\mathscr{T}}, u_{\mathscr{T}}, p_{\mathscr{T}})$ in $M_{\mathscr{T}} \times X_{\mathscr{T}} \times Y_{\mathscr{T}}$ be the solution of the discrete problem (4.4). As discrete spaces $M_{\mathscr{T}}, X_{\mathscr{T}}$ and $Y_{\mathscr{T}}$ are, respectively, contained in the continuous ones M, X and Y, we can take $\varphi = \varphi_{\mathscr{T}}$, $v = v_{\mathscr{T}}$ and $q = q_{\mathscr{T}}$ in the continuous problem. Then, subtracting each corresponding equation in the two systems, and setting

- $f = \omega - \mathscr{P}_{\mathscr{T}}\omega$, which belongs to $L^2(\Omega)$,
- $g = -u + \Pi^{\operatorname{div}}_{\mathscr{T}} u$, which belongs to X and is divergence-free (Proposition 3.3),
- $k = \operatorname{curl}(\omega - \mathscr{P}_{\mathscr{T}}\omega)$, which is in the dual space X',
- $l = -p + \Pi^0_{\mathscr{T}} p$, which is in $L^2(\Omega)$,

the following auxiliary problem appears:

$$\begin{cases} \text{Find } (\theta_{\mathscr{T}}, w_{\mathscr{T}}, r_{\mathscr{T}}) \text{ in } M_{\mathscr{T}} \times X_{\mathscr{T}} \times Y_{\mathscr{T}} \text{ such that} \\ \quad (\theta_{\mathscr{T}}, \varphi_{\mathscr{T}}) - (\operatorname{curl} \varphi_{\mathscr{T}}, w_{\mathscr{T}}) = (f, \varphi_{\mathscr{T}}) + (\operatorname{curl} \varphi_{\mathscr{T}}, g) \quad \forall \varphi_{\mathscr{T}} \in M_{\mathscr{T}}, \\ \quad (\operatorname{curl} \theta_{\mathscr{T}}, v_{\mathscr{T}}) - (r_{\mathscr{T}}, \operatorname{div} v_{\mathscr{T}}) = (k, v_{\mathscr{T}}) + (l, \operatorname{div} v_{\mathscr{T}}) \quad \forall v_{\mathscr{T}} \in X_{\mathscr{T}}, \\ \quad (\operatorname{div} w_{\mathscr{T}}, q_{\mathscr{T}}) = 0 \quad \forall q_{\mathscr{T}} \in Y_{\mathscr{T}}. \end{cases} \tag{4.5}$$

• Now, we can prove a stability result, which is the key point that fails when looking for the vorticity in $H^1(\Omega)$.

PROPOSITION 4.6 Stability of the discrete variational formulation.

Let us assume that Ω is a convex polygonal domain and that the mesh $\mathscr{T}$ belongs to a regular family of triangulations. Then, the problem (4.5) is well-posed and there exists a strictly positive constant C, independent of the mesh, such that

$$\| \theta_{\mathscr{T}} \|_M + \| w_{\mathscr{T}} \|_{\mathrm{div},\Omega} + \| r_{\mathscr{T}} \|_{0,\Omega} \leqslant C(\| f \|_{0,\Omega} + \| g \|_{0,\Omega} + \| k \|_{X'} + \| l \|_{0,\Omega}).$$

Proof.
○ We observe that the hypotheses are such that the two inf–sup conditions (4.1) and (4.3) are true. Then, exactly as in Proposition 4.5, the problem (4.5) is well-posed. Moreover, we remark that the third equation of (4.5) shows that $w_{\mathscr{T}}$ is divergence-free (see Proposition 4.2). Then, we have

$$\| w_{\mathscr{T}} \|_X = \| w_{\mathscr{T}} \|_{\mathrm{div},\Omega} = \| w_{\mathscr{T}} \|_{0,\Omega} .$$

Moreover, we recall that the M-norm is defined by

$$\| \varphi \|_M = (\| \varphi \|^2_{0,\Omega} + \| \Delta\varphi \|^2_{-1,\Omega})^{1/2}.$$

Then, using the decomposition of M (see Proposition 2.4), any φ of M can be split as $\varphi = \varphi^0 + \varphi^{\Delta}$, with $\varphi^0 \in H^1_0(\Omega)$ and φ^{Δ} harmonic. Hence,

$$\| \varphi \|_M = (\| \varphi \|^2_{0,\Omega} + \| \Delta\varphi^0 \|^2_{-1,\Omega})^{1/2},$$

with $\| \Delta\varphi^0 \|_{-1,\Omega} = \| \nabla\varphi^0 \|_{0,\Omega}$ (see Proposition 2.1), and, as a bidimensional problem is considered, we finally obtain

$$\| \varphi \|_M = (\| \varphi \|^2_{0,\Omega} + \| \mathrm{curl}\, \varphi^0 \|^2_{0,\Omega})^{1/2}. \tag{4.6}$$

The proof of the inequality is given in six steps, in which C will denote various constants, independent of the mesh.
○ **First step**. We take $\varphi_{\mathscr{T}} = \theta_{\mathscr{T}}$, $v_{\mathscr{T}} = w_{\mathscr{T}}$ and $q_{\mathscr{T}} = r_{\mathscr{T}}$ in (4.5). As $w_{\mathscr{T}}$ is divergence-free, after adding the three equations, we obtain

$$\begin{aligned} \| \theta_{\mathscr{T}} \|^2_{0,\Omega} &= (f, \theta_{\mathscr{T}}) + (\mathrm{curl}\, \theta_{\mathscr{T}}, g) + (k, w_{\mathscr{T}}) \\ &\leqslant \| f \|_{0,\Omega} \| \theta_{\mathscr{T}} \|_{0,\Omega} + |(\mathrm{curl}\, \theta_{\mathscr{T}}, g)| + \| k \|_{X'} \| w_{\mathscr{T}} \|_{0,\Omega} . \end{aligned}$$

Then, using the classical inequality $\alpha\beta \leqslant \frac{1}{2}(\alpha^2 + \beta^2)$, we deduce

$$\| \theta_{\mathscr{T}} \|^2_{0,\Omega} \leqslant \| f \|^2_{0,\Omega} + 2|(\mathrm{curl}\, \theta_{\mathscr{T}}, g)| + 2 \| k \|_{X'} \| w_{\mathscr{T}} \|_{0,\Omega} . \tag{4.7}$$

○ **Second step**. We apply the inf–sup condition (4.3) to $w_{\mathscr{T}}$, which is divergence-free, in the first equation of (4.5). We deduce

$$b \| w_{\mathscr{T}} \|_{\mathrm{div},\Omega} \leqslant \sup_{\varphi \in M_{\mathscr{T}}} \frac{(\mathrm{curl}\, \varphi, w_{\mathscr{T}})}{\| \varphi \|_M} \leqslant \sup_{\varphi \in M_{\mathscr{T}}} \frac{(\theta_{\mathscr{T}}, \varphi) - (f, \varphi) - (\mathrm{curl}\, \varphi, g)}{\| \varphi \|_M}.$$

Using $|(\mathrm{curl}\, \varphi, g)| \leqslant \| \mathrm{curl}\, \varphi \|_{X'} \| g \|_X$, the fact that g is divergence-free and (2.10), we obtain

$$b \| w_{\mathscr{T}} \|_{\mathrm{div},\Omega} \leqslant \| \theta_{\mathscr{T}} \|_{0,\Omega} + \| f \|_{0,\Omega} + C \| g \|_{0,\Omega} . \tag{4.8}$$

○ **Third step**. Let us recall that the discrete vorticity field $\theta_{\mathscr{T}}$ can also be split in $\theta_{\mathscr{T}} = \theta^0_{\mathscr{T}} + \theta^{\Delta}_{\mathscr{T}}$, with $\theta^0_{\mathscr{T}}$ in $H^1_{0,\mathscr{T}}$ and $\theta^{\Delta}_{\mathscr{T}}$ harmonic (see (3.4)). Moreover, Proposition 4.2 shows that $v_{\mathscr{T}} \equiv \mathrm{curl}\, \theta^0_{\mathscr{T}}$ belongs to $RT^0_{\mathscr{T}}$, is divergence-free and such that $v_{\mathscr{T}} \cdot n = 0$ on Γ. Then, $v_{\mathscr{T}}$ belongs to $X_{\mathscr{T}}$ and, introducing this velocity field in the second equation of (4.5), we obtain

$$(\operatorname{curl}\theta_{\mathscr{T}}, v_{\mathscr{T}}) = (k, v_{\mathscr{T}}),$$

as it is divergence-free. Moreover, we have

$$(\operatorname{curl}\theta_{\mathscr{T}}, v_{\mathscr{T}}) = (\operatorname{curl}\theta^0_{\mathscr{T}}, v_{\mathscr{T}}) + (\operatorname{curl}\theta^\Delta_{\mathscr{T}}, v_{\mathscr{T}}) = (\operatorname{curl}\theta^0_{\mathscr{T}}, v_{\mathscr{T}})$$

as harmonic functions and divergence-free ones are orthogonal (see (2.11)). Finally, replacing $v_{\mathscr{T}}$ by its value, we obtain $\| \operatorname{curl}\theta^0_{\mathscr{T}} \|^2_{0,\Omega} = (k, \operatorname{curl}\theta^0_{\mathscr{T}})$, which obviously leads to

$$\| \operatorname{curl}\theta^0_{\mathscr{T}} \|_{0,\Omega} \leqslant \| k \|_{X'} . \tag{4.9}$$

○ **Fourth step**. Let us go back to the term $|(\operatorname{curl}\theta_{\mathscr{T}}, g)|$. Using again the splitting of $\theta_{\mathscr{T}}$ and (2.11) as g is also divergence-free, we obtain

$$(\operatorname{curl}\theta_{\mathscr{T}}, g) = (\operatorname{curl}\theta^0_{\mathscr{T}}, g) + (\operatorname{curl}\theta^\Delta_{\mathscr{T}}, g) = (\operatorname{curl}\theta^0_{\mathscr{T}}, g).$$

Hence, we deduce that

$$|(\operatorname{curl}\theta_{\mathscr{T}}, g)| \leqslant \| \operatorname{curl}\theta^0_{\mathscr{T}} \|_{0,\Omega} \| g \|_{0,\Omega} . \tag{4.10}$$

○ **Fifth step**. Inequalities (4.7), (4.9) and (4.10) lead to

$$\| \theta_{\mathscr{T}} \|^2_{0,\Omega} \leqslant \| f \|^2_{0,\Omega} + 2 \| g \|_{0,\Omega} \| k \|_{X'} + 2 \| k \|_{X'} \| w_{\mathscr{T}} \|_{0,\Omega},$$

or else, using again $\alpha\beta \leqslant \frac{1}{2}(\alpha^2 + \beta^2)$, we obtain

$$\| \theta_{\mathscr{T}} \|^2_{0,\Omega} \leqslant \| f \|^2_{0,\Omega} + \| g \|^2_{0,\Omega} + \| k \|^2_{X'} + 2 \| k \|_{X'} \| w_{\mathscr{T}} \|_{0,\Omega} .$$

Finally, introducing (4.8) in the above inequality, we have

$$\begin{aligned}\| \theta_{\mathscr{T}} \|^2_{0,\Omega} &\leqslant \| f \|^2_{0,\Omega} + \| g \|^2_{0,\Omega} + \| k \|^2_{X'} + \frac{2}{b} \| k \|_{X'} (\| \theta_{\mathscr{T}} \|_{0,\Omega} + \| f \|_{0,\Omega} + C \| g \|_{0,\Omega}) \\ &\leqslant C(\| f \|^2_{0,\Omega} + \| g \|^2_{0,\Omega} + \| k \|^2_{X'}) + \frac{2}{b} \| k \|_{X'} \| \theta_{\mathscr{T}} \|_{0,\Omega},\end{aligned}$$

where C is a constant equal to $1 + (2/b)\max(C, 1)$. Now, we use the classical inequality $2\alpha\beta \leqslant \alpha^2/\varepsilon + \varepsilon\beta^2$, true for any strictly positive real number ε, to obtain

$$\| \theta_{\mathscr{T}} \|^2_{0,\Omega} \leqslant C(\| f \|^2_{0,\Omega} + \| g \|^2_{0,\Omega} + \| k \|^2_{X'}) + \frac{1}{b\varepsilon} \| k \|^2_{X'} + \frac{\varepsilon}{b} \| \theta_{\mathscr{T}} \|^2_{0,\Omega} .$$

Taking $\varepsilon \leqslant b/2$, we finally obtain

$$\| \theta_{\mathscr{T}} \|^2_{0,\Omega} \leqslant C(\| f \|^2_{0,\Omega} + \| g \|^2_{0,\Omega} + \| k \|^2_{X'}). \tag{4.11}$$

The inequalities (4.6), (4.9) and (4.11) lead to

$$\| \theta_{\mathscr{T}} \|^2_{M} \leqslant C(\| f \|^2_{0,\Omega} + \| g \|^2_{0,\Omega} + \| k \|^2_{X'}),$$

and then

$$\| \theta_{\mathscr{T}} \|_M \leqslant C(\| f \|_{0,\Omega} + \| g \|_{0,\Omega} + \| k \|_{X'}). \tag{4.12}$$

Finally, introducing (4.12) in (4.8) gives

$$\| w_{\mathscr{T}} \|_{\mathrm{div},\Omega} \leqslant C(\| f \|_{0,\Omega} + \| g \|_{0,\Omega} + \| k \|_{X'}). \tag{4.13}$$

○ **Sixth step**. We use the inf–sup condition (4.1) in the second equation of (4.5) and obtain

$$a \| r_{\mathscr{T}} \|_{0,\Omega} \leqslant \sup_{v \in X_{\mathscr{T}}} \frac{(\mathrm{div}\, v, r_{\mathscr{T}})}{\| v \|_{\mathrm{div},\Omega}} \leqslant \sup_{v \in X_{\mathscr{T}}} \frac{(\mathrm{curl}\, \theta_{\mathscr{T}}, v) - (l, \mathrm{div}\, v) - (k, v)}{\| v \|_{\mathrm{div},\Omega}}.$$

As $X_{\mathscr{T}}$ is a subspace of X, it is obvious that this inequality leads to

$$a \| r_{\mathscr{T}} \|_{0,\Omega} \leqslant \sup_{v \in X} \frac{(\mathrm{curl}\, \theta_{\mathscr{T}}, v) - (l, \mathrm{div}\, v) - (k, v)}{\| v \|_{\mathrm{div},\Omega}}.$$

Then, the norm in X being the norm in $H(\mathrm{div}, \Omega)$, we finally have

$$a \| r_{\mathscr{T}} \|_{0,\Omega} \leqslant \| \mathrm{curl}\, \theta_{\mathscr{T}} \|_{X'} + \| l \|_{0,\Omega} + \| k \|_{X'}\,. \tag{4.14}$$

Let us recall that $\| \mathrm{curl}\, \theta_{\mathscr{T}} \|_{X'} \leqslant C \| \theta_{\mathscr{T}} \|_M$ because $\theta_{\mathscr{T}}$ belongs to M (see (2.10)). Then, the final inequality, given in the proposition, is a direct consequence of (4.12–4.14). □

4.4 *Convergence of the discrete problem*

We can now state the convergence result associated with our numerical scheme.

THEOREM 4.7 Convergence of the discrete variational formulation.

Let us assume that Ω is a polygonal convex domain, that the mesh $\mathscr{T}$ belongs to a regular family of triangulations and that $h_{\mathscr{T}}$ is small enough. Let (ω, u, p) be the solution in $M \times X \times Y$ of the continuous problem (2.1) and $(\omega_{\mathscr{T}}, u_{\mathscr{T}}, p_{\mathscr{T}})$ in $M_{\mathscr{T}} \times X_{\mathscr{T}} \times Y_{\mathscr{T}}$ be the solution of the discrete problem (4.4). We suppose that the solution is such that $u \in (H^1(\Omega))^2$, with $\mathrm{div}\, u \in H^1(\Omega)$, $p \in H^1(\Omega)$ and $\omega \in H^2(\Omega)$. Then, there exists a strictly positive constant C, independent of the mesh, such that

$$\begin{aligned} &\| \omega - \omega_{\mathscr{T}} \|_M + \| u - u_{\mathscr{T}} \|_{\mathrm{div},\Omega} + \| p - p_{\mathscr{T}} \|_{0,\Omega} \\ &\quad \leqslant C h_{\mathscr{T}} (\| \omega \|_{2,\Omega} + \| u \|_{1,\Omega} + \| \mathrm{div}\, u \|_{1,\Omega} + \| p \|_{1,\Omega}). \end{aligned}$$

Proof. First, let us recall the basic inequalities

$$\begin{aligned} \| \omega - \omega_{\mathscr{T}} \|_M &\leqslant \| \omega - \mathscr{P}_{\mathscr{T}} \omega \|_M + \| \mathscr{P}_{\mathscr{T}} \omega - \omega_{\mathscr{T}} \|_M, \\ \| u - u_{\mathscr{T}} \|_{\mathrm{div},\Omega} &\leqslant \| u - \overset{\mathrm{div}}{\Pi}_{\mathscr{T}} u \|_{\mathrm{div},\Omega} + \| \overset{\mathrm{div}}{\Pi}_{\mathscr{T}} u - u_{\mathscr{T}} \|_{\mathrm{div},\Omega}, \\ \| p - p_{\mathscr{T}} \|_{0,\Omega} &\leqslant \| p - \Pi^0_{\mathscr{T}} p \|_{0,\Omega} + \| \Pi^0_{\mathscr{T}} p - p_{\mathscr{T}} \|_{0,\Omega}\,. \end{aligned} \tag{4.15}$$

In these relations, the first terms are well known: They are the classical interpolation errors. And the second terms are precisely the solutions of the auxiliary problem (4.5), where we have

$$\theta_{\mathscr{T}} = \omega_{\mathscr{T}} - \mathscr{P}_{\mathscr{T}} \omega, \quad w_{\mathscr{T}} = u_{\mathscr{T}} - \overset{\mathrm{div}}{\Pi}_{\mathscr{T}} u, \quad r_{\mathscr{T}} = p_{\mathscr{T}} - \Pi^0_{\mathscr{T}} p.$$

Then, Proposition 4.6 ensures that there exists a strictly positive constant C, independent of the mesh, such that

$$\begin{aligned}
&\| \omega_{\mathscr{T}} - \mathscr{P}_{\mathscr{T}}\omega \|_M + \| u_{\mathscr{T}} - \overset{\text{div}}{\Pi}_{\mathscr{T}} u \|_{\text{div},\Omega} + \| p_{\mathscr{T}} - \Pi^0_{\mathscr{T}} p \|_{0,\Omega} \\
&\quad \leqslant C(\| f \|_{0,\Omega} + \| g \|_{0,\Omega} + \| k \|_{X'} + \| l \|_{0,\Omega}),
\end{aligned}$$

where we have set $f = \omega - \mathscr{P}_{\mathscr{T}}\omega$, $g = -u + \overset{\text{div}}{\Pi}_{\mathscr{T}} u$, $k = \text{curl}\,(\omega - \Pi^1_{\mathscr{T}}\omega)$ and $l = -p + \Pi^0_{\mathscr{T}} p$. Then, the above inequality and (4.15) lead to

$$\begin{aligned}
&\| \omega - \omega_{\mathscr{T}} \|_M + \| u - u_{\mathscr{T}} \|_{\text{div},\Omega} + \| p - p_{\mathscr{T}} \|_{0,\Omega} \\
&\quad \leqslant C(\| \omega - \mathscr{P}_{\mathscr{T}}\omega \|_M + \| u - \overset{\text{div}}{\Pi}_{\mathscr{T}} u \|_{\text{div},\Omega} + \| p - \Pi^0_{\mathscr{T}} p \|_{0,\Omega}) \\
&\quad \leqslant C(\| \omega^0 - \Pi^1_{\mathscr{T}}\omega^0 \|_M + \| \omega^\Delta - \phi_{\mathscr{T}}\omega^\Delta \|_M \\
&\qquad + \| u - \overset{\text{div}}{\Pi}_{\mathscr{T}} u \|_{\text{div},\Omega} + \| p - \Pi^0_{\mathscr{T}} p \|_{0,\Omega}),
\end{aligned}$$

where C is another constant independent of the mesh size. Finally, using the interpolation errors recalled in Theorems 3.1, 3.2, 3.10 and Proposition 2.4, we obtain the announced result, as far as $h_{\mathscr{T}}$ is small enough to overestimate $h^{3/2}_{\mathscr{T}}$ by $h_{\mathscr{T}}$. □

• Let us make one comment on this result. If ω belongs to $H^{5/2}(\Omega)$, we have seen in Proposition 3.10 that

$$\|\omega^\Delta - \omega^\Delta_{\mathscr{T}}\|_{0,\Omega} \leqslant Ch^2_{\mathscr{T}}\|\omega\|_{5/2,\Omega}.$$

As Ω is also assumed to be convex and always under the assumptions of the previous theorem, using the classical Aubin–Nitsche argument, which says that the regularity on the adjoint problem is obtained (Aubin, 1967; Nitsche, 1968), we can expect that

$$\| \omega - \omega_{\mathscr{T}} \|_{0,\Omega} \leqslant Ch^2_{\mathscr{T}} \| \omega \|_{5/2,\Omega}\,. \tag{4.16}$$

5. Numerical experiments

The first numerical experiments have been performed on a unit square with an analytical solution (test of Bercovier & Engelman, 1979). The velocity is zero on the whole boundary Γ and there is no boundary condition on the pressure and the vorticity. The external force field is given by

$$f_1(x, y) = g(x, y) + (y - \tfrac{1}{2}), \quad f_2(x, y) = -g(y, x) + (x - \tfrac{1}{2}),$$

with

$$g(x, y) = 256(x^2(x-1)^2(12y-6) + y(y-1)(2y-1)(12x^2 - 12x + 2))$$

for which we obtain

$$\begin{aligned}
\omega(x, y) &= 256(y^2(y-1)^2(6x^2 - 6x + 1) + x^2(x-1)^2(6y^2 - 6y + 1)),\\
u_1(x, y) &= -256x^2(x-1)^2 y(y-1)(2y-1),\\
u_2(x, y) &= 256y^2(y-1)^2 x(x-1)(2x-1),\\
p(x, y) &= (x - \tfrac{1}{2})(y - \tfrac{1}{2}).
\end{aligned}$$

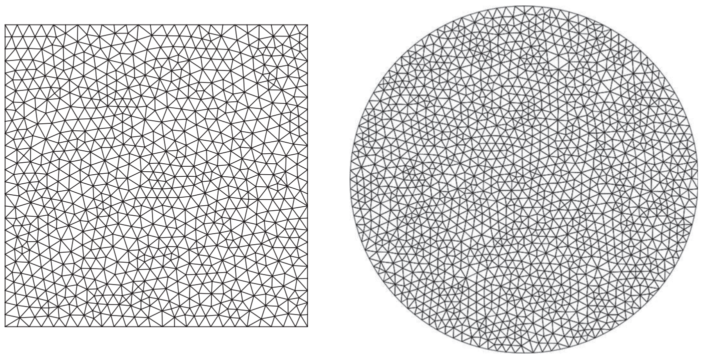

FIG. 2. Two unstructured meshes obtained by EMC2.

• For the second numerical experiments, we have considered circular domains. They have been performed on a circle of radius 2 with an analytical solution (test suggested by Ruas, 1997). The boundary conditions are exactly the same as in the previous case. The external force field is given by

$$f_1(x,y) = -32y, \quad f_2(x,y) = 32x,$$

which gives

$$\omega(x,y) = 32 - 16x^2 - 16y^2,$$
$$u_1(x,y) = -4y(4 - x^2 - y^2),$$
$$u_2(x,y) = 4x(4 - x^2 - y^2),$$

and the pressure p is constant (equal to 1) on the whole domain.

REMARK 5.1

- For error estimates on the circle, let us note that we should add a boundary approximation error. However, the boundary is approximated by continuous polynomials of degree 1. Nevertheless, the forthcoming results show that this error does not pollute the numerical scheme.
- All the integrals for assembling the mass matrix in the first equation of (4.4) were computed with the help of a Gauss formula using 13 quadrature points, and we obtain results in accordance with the theory. Although errors due to numerical integration were not studied here, they seem to be dominated by other errors and again do not pollute results.

• In these two cases, we have worked with unstructured meshes obtained with EMC2, mesh generator of Modulef (Bernadou *et al.*, 1988); see Fig. 2.

For the first test, the analytical vorticity attains its extremum on the middle of each edge of the square and its value is then +16.00. And, for the second one, the extremum (−32) is attained on the whole boundary. We recall that the method introduced in Dubois *et al.* (2003a) uses piecewise linear functions to approximate the vorticity on the whole domain. On structured meshes with regular functions, we have optimal convergence for the three fields in L^2-norm: $\mathcal{O}(h^2)$ for the vorticity, $\mathcal{O}(h)$ for velocity and pressure, where h stands for the mesh size parameter (we think that this is due to superconvergence

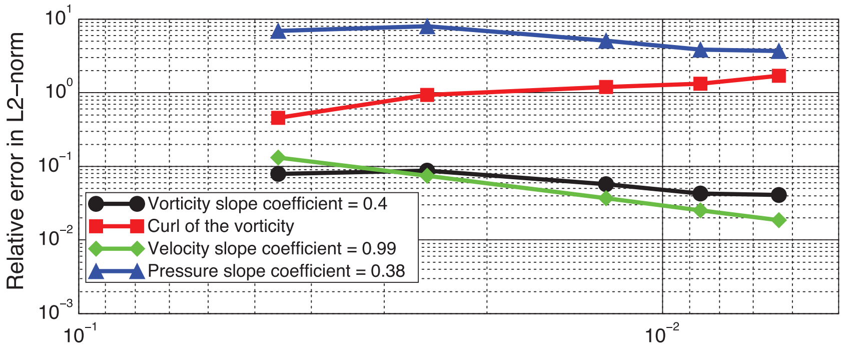

FIG. 3. Convergence curves without harmonic functions—Bercovier–Engelman's test.

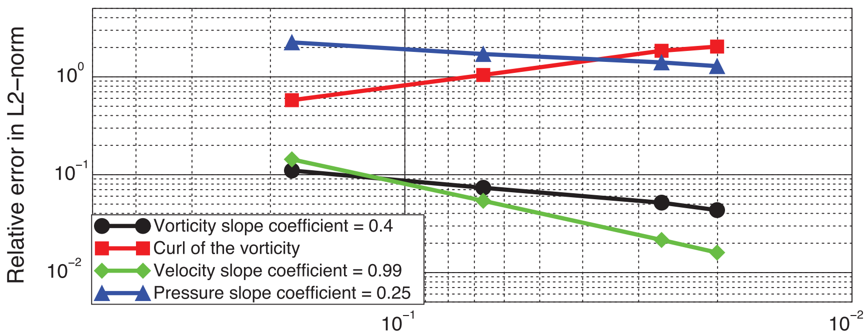

FIG. 4. Convergence curves without harmonic functions—Test proposed by Ruas.

properties on regular meshes; see Girault & Raviart, 1986). But on unstructured meshes, results were really not satisfactory: Vorticity and pressure fields are not well approximated. In particular, on tests introduced above, for which an analytical solution is known, we observe that values of vorticity and pressure are far from the expected ones along the boundary, even if the mesh is refined. Moreover, the order of convergence for all these fields, except the velocity, is more or less $\mathcal{O}(\sqrt{h})$, as numerically illustrated in Figs 3 and 4. The theoretical study of convergence shows that the problem is a stability one: The curl of the vorticity, which appears in the formulation, is not bounded except if we suppose that the velocity and the vorticity are given on the same part of the boundary ($\Gamma_\theta = \Gamma_m$).

REMARK 5.2 However, in the very particular case of $\Gamma_\theta = \Gamma_m$, an optimal rate of convergence is achieved, even on unstructured meshes (see Dubois *et al.*, 2003a). Nevertheless, this condition is clearly too restrictive.

The new numerical scheme replaces piecewise linear functions on the boundary by harmonic functions obtained by a 'single layer' potential. The number of harmonic functions is equal to the number of vertices on the boundary. Figure 5 gives the values of the vorticity along the boundary obtained in Dubois *et al.* (2003a) without harmonic functions and by the new method on the *same* mesh. In fact, in

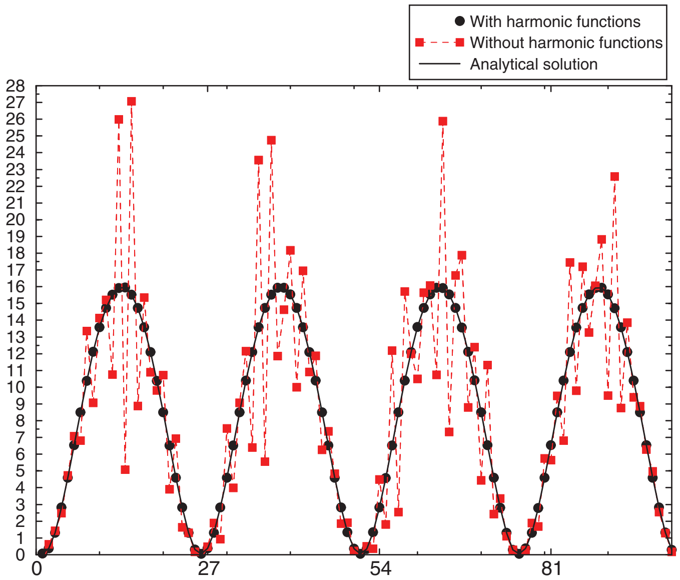

FIG. 5. Comparison: vorticity along the boundary—Bercovier–Engelman's test.

Dubois *et al.* (2003a) extrema of the vorticity exploded on the boundary when the mesh is not regular (see Fig. 5). Moreover, for both tests, the exact solution is very regular and the domain is convex. Then, it is not surprising to obtain, as expected by theorem 4.7 and (4.16), a convergence of order 2 for the L^2-norm of the vorticity (see Figs 6 and 7).

With regard to the pressure, in Dubois *et al.* (2003a), the error remains at a too important level: More than 200% in relative error for the quadratic norm (see Figs 3 and 4). For instance, the pressure varies between -7.67 and 6.44 instead of -0.25 and 0.25 in the Bercovier–Engelman case, and between -17.56 to 12.83 instead of the constant value in the Ruas test. With the new scheme, results are as expected for the pressure. We also observe that the rate of convergence for the pressure goes from approximatively $\mathscr{O}(\sqrt{h_{\mathscr{T}}})$ (Dubois *et al.*, 2003a) to $\mathscr{O}(h_{\mathscr{T}})$ with the new scheme (see Figs 6 and 7), as expected by Theorem 4.7.

Let us conclude with a comparison between the vorticity–velocity–pressure formulation and the classical formulation in velocity–pressure using $\mathbb{P}^1$ plus bubble$-\mathbb{P}^1$ element. This element is the well-known lowest one verifying the inf–sup condition, and it asks for 11 degrees of freedom (4 for each component of the velocity and 3 for the pressure which is continuous). Figure 8, obtained by the free software FreeFem++[1], shows that, for Bercovier–Engelman test, result on the pressure is far from the expected one, even if theoretical convergence results prove that error will converge to zero. In Fig. 9, we can see that the result of the vorticity–velocity–pressure scheme, asking for only 7 degrees of freedom (3 for the vorticity, 3 for the velocity and 1 for the pressure, which is discontinuous), is much better and very close to the analytical solution, presented in Fig. 8.

[1] http://www.freefem.org.

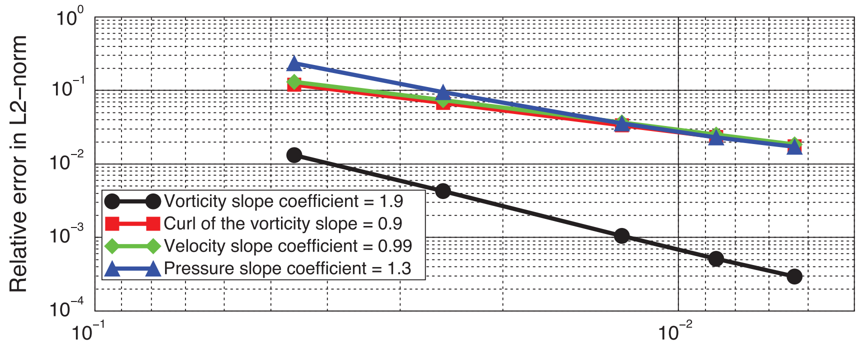

FIG. 6. Convergence curves with harmonic functions—Bercovier–Engelman's test.

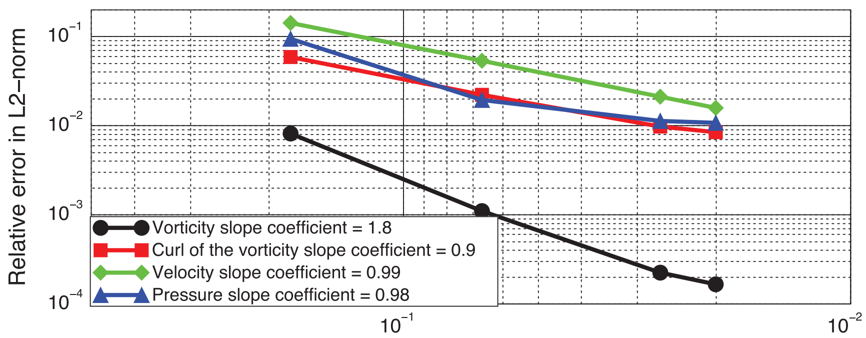

FIG. 7. Convergence curves with harmonic functions—Test proposed by Ruas.

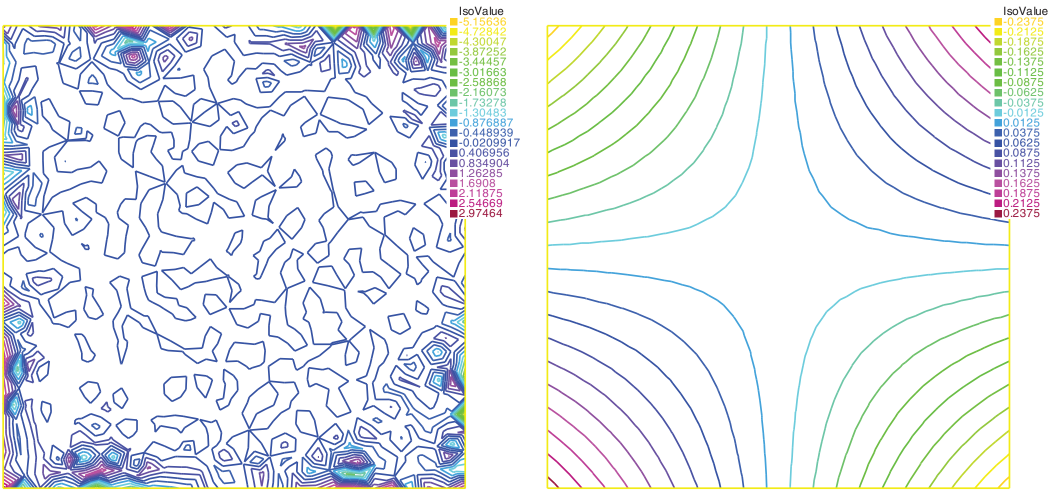

FIG. 8. Left panel: Computed pressure with the $\mathbb{P}^1$+bubble$-\mathbb{P}^1$ element, FreeFem++ result on the mesh shown in Fig. 2, extrema: -5.15 to 2.97. Right panel: Analytical pressure interpolated on the same mesh, extrema: -0.25 to 0.25.

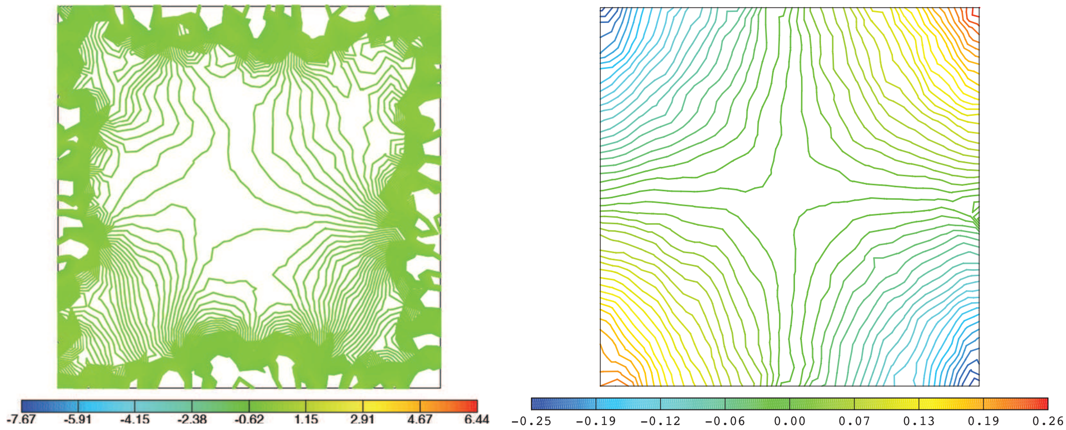

FIG. 9. Computed pressure without (left panel) and with (right panel) harmonic functions in the vorticity–velocity–pressure scheme, extrema: −7.67 to 6.44 (left) and −0.25 to 0.26 (right).

6. Conclusion

We have introduced in Dubois *et al.* (2003a) a vorticity–velocity–pressure variational formulation of the bidimensional Stokes problem. For this formulation, we have defined a natural numerical scheme which can be viewed as an extension of the popular MAC scheme on triangular meshes. We have numerically studied this scheme and observed that it is not stable in the general case of boundary conditions. If it gives correct results on structured meshes, improvable ones are obtained on unstructured meshes.

In this paper, we have studied the well-posed bidimensional Stokes problem in the vorticity–velocity–pressure form we have introduced in Dubois *et al.* (2003b). We have shown theoretically and numerically that approximating numerically the space of real harmonic functions with the help of an integral representation is sufficient to obtain, on the one hand, a better numerical solution and, on the other hand, better estimations on the convergence than those obtained previously. Actually, we obtain convergence with an optimal rate in the Dirichlet boundary conditions case on the quadratic norm of the vorticity. We stress on the facts that first, the scheme is a very low order one and, second, the velocity is exactly divergence-free. Finally, the only additional cost (computation of the mass matrix of harmonic functions, which is on the order of the square of the number of boundary nodes) needs to be done only once. Then, the extension of our scheme to the nonstationary Stokes problem does not incur any significant additional numerical cost.

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