Recovering the initial state of dynamical systems using observers

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Introduction

In many areas of science, we need to recover the initial data of a physical system from partial observation over some finite time interval. In oceanography and meteorology, this problem is known as ***data assimilation***. It also arises in medical imaging, for instance in thermoacoustic tomography where the problem is to recover an initial data for a 3D wave type equation from surface measurement [8].

In the last decade, new algorithms based on time reversal (see Fink [4, 5]) appeared to answer this question. We can mention, for instance, the Back and Forth Nudging proposed by Auroux and Blum [1], the Time Reversal Focusing by Phung and Zhang [10], the algorithm proposed by Ito, Ramdani and Tucsnak [7] and finally, the one we will consider here, the forward–backward observers based algorithm proposed by Ramdani, Tucsnak and Weiss [11].

Under some assumptions, we prove that this algorithm allows to reconstruct the ***observable part*** of the initial state, *i.e.* the part which contributes to the measurement, and give necessary and sufficient conditions to get exponential decay.

The dynamical system

Let X and Y be two Hilbert spaces, $A : \mathcal{D}(A) \to X$ a skew-adjoint operator (possibly unbounded), and $C \in \mathcal{L}(X, Y)$. We consider the system

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \geq 0, \\ z(0) = z_0 \in X. \end{cases} \tag{1}$$

Such a system is often used to model vibrating system (acoustic or elastic waves) or quantum system (Schrödinger equation).

We observe system (1) *via* the operator C during a finite time interval $(0, \tau)$ with $\tau > 0$, leading to the measurement

$$y(t) = Cz(t), \quad \forall\, t \in (0, \tau). \tag{2}$$

The inverse problem

Considering systems (1)–(2), a natural question arises

Is it possible to recover z_0 of (1) from the measurement y given by (2) ?

If we denote $\Psi_\tau : X \to L^2_{\ell oc}([0, \infty), Y)$ the continuous linear operator which associates z_0 to the measurement y, *i.e.* $y = \Psi_\tau z_0$, it is clear that the problem is well-posed if Ψ_τ is left-invertible. In other words, it is well-posed if there exists a constant $k_\tau > 0$ such that

$$\|\Psi_\tau x\| \geq k_\tau \|x\|, \quad \forall\, x \in X. \tag{3}$$

If the above inequality (3) holds, we say that (1)–(2) (or (A, C)) is exactly observable.

The iterative algorithm

Under some assumptions, Ramdani, Tucsnak and Weiss [11] proposed the iterative use of the following back and forth ***observers*** (4)–(5) to reconstruct z_0 from y. In our case, these assumptions are equivalent to the exact observability assumption (3) (see [11, Proposition 3.7]).

Let $A^+ = A - \gamma C^*C$ and $A^- = -A - \gamma C^*C$ be the generator of the two exponentially stable C_0-semigroups (denoted $\mathbb{T}^+$ and $\mathbb{T}^-$ respectively) for all $\gamma > 0$ (see Liu [9] or [11, Proposition 3.7]), and let $z_0^+ \in X$ be the initial guess (usually $z_0^+ = 0$). Then the algorithm reads: for all $n \in \mathbb{N}^*$

$$\begin{cases} \dot{z}_n^+(t) = A^+ z_n^+(t) + \gamma C^* y(t), & \forall\, t \in (0, \tau), \\ z_1^+(0) = z_0^+, \\ z_n^+(0) = z_{n-1}^-(0), & \forall\, n \geq 2, \end{cases} \tag{4}$$

$$\begin{cases} \dot{z}_n^-(t) = -A^- z_n^-(t) - \gamma C^* y(t), & \forall\, t \in (0, \tau), \\ z_n^-(\tau) = z_n^+(\tau), & \forall\, n \geq 1. \end{cases} \tag{5}$$

Convergence of the algorithm

Denoting $e_n^+ = z_n^+ - z$ and $e_n^- = z_n^- - z$, one can easily show that e_n^+ and e_n^- are the respective solutions of

$$\begin{cases} \dot{e}_n^+(t) = A^+ e_n^+(t), & \forall\, t \in (0, \tau), \\ e_1^+(0) = z_0^+ - z_0, \\ e_n^+(0) = z_{n-1}^-(0) - z_0, & \forall\, n \geq 2, \end{cases}$$

$$\begin{cases} \dot{e}_n^-(t) = -A^- e_n^-(t), & \forall\, t \in (0, \tau), \\ e_n^-(\tau) = z_n^+(\tau) - z(\tau), & \forall\, n \geq 1. \end{cases}$$

Then for all $n \in \mathbb{N}^*$

$$\left\| e_n^-(0) \right\| = \left\| \left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right)^n e_1^+(0) \right\| \leq \left\| \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right\|^n \left\| e_1^+(0) \right\|,$$

which can be rewritten

$$\left\| z_n^-(0) - z_0 \right\| \leq \left\| \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right\|^n \left\| z_0^+ - z_0 \right\|, \quad \forall\, n \in \mathbb{N}^*.$$

By the exponential stability of $\mathbb{T}^+$ and $\mathbb{T}^-$, there exists a $\tau > 0$ such that

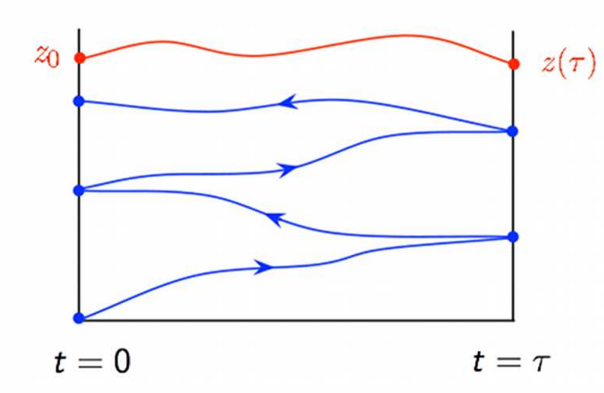
$$\left\| \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right\| < 1. \tag{6}$$

Ito, Ramdani and Tucsnak [7, Lemma 2.2] show that every time of observability (*i.e.* every time $\tau > 0$ such that (3) holds) leads to (6).

In other words, $z_n^-(0)$ converges exponentially to z_0: there exists a constant $\alpha \in (0, 1)$ such that

$$\left\| z_n^-(0) - z_0 \right\| \leq \alpha^n \left\| z_0^+ - z_0 \right\|, \quad \forall\, n \leq 1.$$

The convergence of the algorithm by back and forth observers can be summarize by this illustration:



Partial reconstruction of z_0

Note that systems (4)–(5) are always well defined without the exact observability assumption. Then two questions arise naturally:

1. **Given arbitrary C and $\tau > 0$, does the algorithm converge ?**
2. **If it does, what is the limit of $z_n^-(0)$, and how is it related to z_0 ?**

Using the operator Ψ_τ, we have the orthogonal decomposition

$$X = \operatorname{Ker} \Psi_\tau \oplus \overline{\operatorname{Ran} (\Psi_\tau)^*}. \tag{7}$$

Theorem . *With the previous definitions and notation, and denoting by Π the orthogonal projector from X onto $\overline{\operatorname{Ran} (\Psi_\tau)^*}$, the following statements hold true:*

1. *We have for all $z_0, z_0^+ \in X$*

$$\left\| (I - \Pi) \left(z_n^-(0) - z_0 \right) \right\| = \left\| (I - \Pi) \left(z_0^+ - z_0 \right) \right\|, \quad \forall\, n \geq 1.$$

2. *The sequence $\left(\left\| \Pi \left(z_n^-(0) - z_0 \right) \right\| \right)_{n \geq 1}$ is strictly decreasing and verifies*

$$\left\| \Pi \left(z_n^-(0) - z_0 \right) \right\| \longrightarrow 0, \quad n \to \infty.$$

3. *The rate of convergence is exponential, i.e. there exists a constant $\alpha \in (0, 1)$, independent of z_0 and z_0^+, such that*

$$\left\| \Pi \left(z_n^-(0) - z_0 \right) \right\| \leq \alpha^n \left\| \Pi \left(z_0^+ - z_0 \right) \right\|, \quad \forall\, n \geq 1,$$

if and only if $\operatorname{Ran} (\Psi_\tau)^*$ *is closed in X.*

In general, it is difficult to characterise the projector Π. However, if the initial guess z_0^+ belongs to $\overline{\operatorname{Ran} (\Psi_\tau)^*}$ (for instance, $z_0^+ = 0$), then all iterative approximations $z_n^-(0)$ belong to $\overline{\operatorname{Ran} (\Psi_\tau)^*}$.

Corollary . *Under the assumptions of the previous Theorem, if $z_0^+ \in \overline{\operatorname{Ran} (\Psi_\tau)^*}$, then*

$$\left\| z_n^-(0) - \Pi z_0 \right\| \longrightarrow 0, \quad n \to \infty.$$

Furthermore, the decay rate is exponential if and only if $\operatorname{Ran} (\Psi_\tau)^*$ *is closed in X.*

Generalization and example

Using the framework of well-posed linear systems, we can use a result of Curtain and Weiss [3] to handle the case of (some) unbounded observation operators and derive a result similar to the main Theorem (formally, we also take $A^\pm = \pm A - \gamma C^*C$, with a suitably chosen $\gamma > 0$).

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_0 and Γ_1 being relatively open in $\partial\Omega$. Denote by ν the unit normal vector of Γ_1 pointing towards the exterior of Ω. Consider the following wave system

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0, & \forall x \in \Omega, t > 0, \\ w(x,t) = 0, & \forall x \in \Gamma_0, t > 0, \\ w(x,t) = u(x,t), & \forall x \in \Gamma_1, t > 0, \\ w(x,0) = w_0(x), \ \dot{w}(x,0) = w_1(x), & \forall x \in \Omega, \end{cases} \tag{8}$$

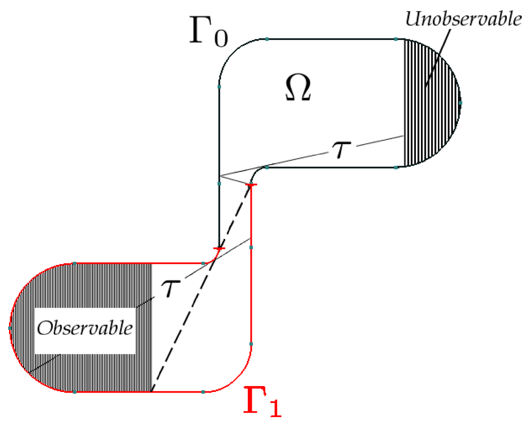
with u the input function (the control), and (w_0, w_1) the initial state. We observe this system on Γ_1, leading to

$$y(x,t) = -\frac{\partial (-\Delta)^{-1} \dot{w}(x,t)}{\partial \nu}, \quad \forall x \in \Gamma_1, t > 0. \tag{9}$$

Using a result of Guo and Zhang [6], we can show that the system (8)–(9) fits into the framework of well-posed linear systems and we can thus apply the generalization of the main Theorem to recover the observable part of the initial data (w_0, w_1).

For instance, let us consider the following configuration.

We can easily obtain two subdomains of Ω (the striped ones), using the geometric optic condition of Bardos, Lebeau and Rauch [2], such that all initial data with support in the left (resp. right) one are observable (resp. unobservable).



Simulations

We used Gmsh[a] to mesh our domain, and GetDP[b] to discretize (4)–(5) by finite elements in space and a Newmark scheme in time.

We choose a suitable initial data to bring out these inclusions (in particular $w_1 \equiv 0$). We perform some simulations (using GMSH and GetDP) and obtain 6% of relative error (in $L^2(\Omega)$) on the reconstruction of the observable part of the data after three iterations.

The initial position and its reconstruction after 3 iterations

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[a] http://geuz.org/gmsh/

[b] http://www.geuz.org/getdp/