In many areas of science, we need to recover the initial data of a physical system from partial observation over some finite time interval. In seismology and meteorology, this problem is known as data assimilation. It also arises in medical imaging, for instance in thermoacoustic tomography where the problem is to recover an initial data for a 3D wave type equation from surface measurement [6].

In the last decade, new algorithms based on time reversal (see Fink [4, 5]) appeared to answer this question. We can mention, for instance, the Back and Forth Nudging proposed by M. Fink and Huan [1], the Time Reversal Focusing by Thong and Zhang [10], the algorithm proposed by T. Ramdani and Tucsnak [7] and finally, the one we will consider here, the forward-backward observers algorithm proposed by T. Ramdani and Weiss [11].

Under some assumptions, we prove that this algorithm allows to recover the observable part of the initial state, i.e. the part which contributes to the measurement, and give necessary and sufficient conditions to get exponential decay.

THE DYNAMICAL SYSTEM

Let X and Y be two Hilbert spaces, and \( A : D(A) \to X \) a skew-adjoint operator (possibly unbounded), and \( C \in \mathcal{L}(X, Y) \). We consider the system

\[
\dot{x}(t) = A x(t), \quad y(t) = C x(t), \quad \forall t \geq 0,
\]

Such a system is often used to model vibrating system (acoustic or elastic waves) or quantum system (Schrödinger equation).

We observe system (1) via the operator \( C \) during a finite time interval \((0, \tau)\), with \( \tau > 0 \), leading to the measurement

\[
y(t) = C x(t), \quad \forall t \in (0, \tau).
\]

THE INVERSE PROBLEM

Is it possible to recover \( z_0 \) of (1) from the measurement \( y \) given by (2) \( \tau \)?

Considering systems (1) and (2), a natural question arises

\[
\text{If we denote } \Phi : X \to L_2(0, \tau; X), \text{ the continuous linear operator which associates } z_0 \text{ to the measurement } y, \text{ i.e. } y = \Phi z_0, \text{ it is clear that the problem is well-posed if } \Phi \text{ is left-invertible. In other words, it is well-posed if there exists a constant } k_1 > 0 \text{ such that }
\]

\[
\|\Phi z_0\| \geq k_1 \|z_0\|, \quad \forall z_0 \in X.
\]

(3)

If the above inequality (3) holds, we say that (1) (or (A,C)) is exactly observable.

THE ITERATIVE ALGORITHM

Under some assumptions, Ramdani, Tucsnak and Weiss [11] proposed the iterative use of the following back and forth observers (4)–(5) to reconstruct \( z_0 \). In our case, these assumptions are equivalent to the exact observability assumption (3) (see [11, Proposition 3.7]).

Let \( A^* = A \) and \( A = -C^* \), and \( A = -A^* \), \( C = C^* \). The generator of the two exponentially stable C-semigroups (denoted \( T^t \) and \( T^{-t} \) respectively) for all \( t > 0 \) (see Liu [9] or [11, Proposition 3.7]) and let \( z_0 \in X \) be the initial state (usually \( z_0 = 0 \)). Then the algorithm reads as for all \( n \in \mathbb{N}^* \)

\[
\begin{align}
\tilde{z}_n(t) & = -A^* \tilde{z}_n(t) + A \tilde{z}_{n+1}(t), \quad \forall t \in (0, \tau), \quad (4) \\
\tilde{z}_{n+1}(0) & = z_0, \quad \forall n \geq 2, \\
\tilde{z}_{n+1}(t) & = -A^* \tilde{z}_{n+1}(t) + C \tilde{y}(t), \quad \forall t \in (0, \tau), \quad (5)
\end{align}
\]

The convergence of the algorithm by back and forth observers can be summarize by this illustration:

REFERENCES