The Adaptive Coherence Estimator is the Generalized Likelihood Ratio Test for a Class of Heterogeneous Environments

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***Abstract*—The adaptive coherence estimator (ACE) is known to be the generalized likelihood ratio test (GLRT) in partially homogeneous environments, i.e., when the covariance matrix M_s of the secondary data is proportional to the covariance matrix M_p of the vector under test (or $M_s = \gamma M_p$). In this letter, we show that ACE is indeed the GLRT for a broader class of nonhomogeneous environments, more precisely when M_s is a random matrix, with inverse complex Wishart prior distribution whose mean only is proportional to M_p. Furthermore, we prove that, for this class of heterogeneous environments, the ACE detector satisfies the constant false alarm rate (CFAR) property with respect to γ and M_p.**

***Index Terms*—Adaptive coherence estimator, adaptive detection, generalized likelihood ratio test, nonhomogeneous environments.**

I. Introduction

In many applications such as sonar and radar, it is desired to detect the presence of a signal of interest embedded in colored noise. The standard detection problem is formulated as a binary hypothesis test

$$H_0 \quad \begin{cases} \boldsymbol{x} \sim \tilde{\mathcal{N}}_m(\boldsymbol{0}, \boldsymbol{M}_p) \\ \boldsymbol{z}_k \sim \tilde{\mathcal{N}}_m(\boldsymbol{0}, \boldsymbol{M}_s)\ k = 1, \ldots, K \end{cases} \tag{1a}$$

$$H_1 \quad \begin{cases} \boldsymbol{x} \sim \tilde{\mathcal{N}}_m(\alpha \boldsymbol{a}, \boldsymbol{M}_p) \\ \boldsymbol{z}_k \sim \tilde{\mathcal{N}}_m(\boldsymbol{0}, \boldsymbol{M}_s)\ k = 1, \ldots, K \end{cases} \tag{1b}$$

where α and $\boldsymbol{a}$ stand for the amplitude and signature of the target in the cell under test, respectively. In (1), $\tilde{\mathcal{N}}_m(\bar{\boldsymbol{x}}, \boldsymbol{M})$ denotes the complex normal distribution of a m-dimensional vector with mean $\bar{\boldsymbol{x}}$ and covariance matrix $\boldsymbol{M}$, and $\boldsymbol{x}$ is the primary data vector with covariance matrix $\boldsymbol{M}_p$. The training samples $\{\boldsymbol{z}_k\}_{k=1,\ldots,K}$ are assumed to be independent with covariance matrix $\boldsymbol{M}_s$. The primary data vector $\boldsymbol{x}$ is also independent from the training samples.

When the signature $\boldsymbol{a}$ and the structure of the covariance matrix $\boldsymbol{M}_p$ are both known (i.e., $\boldsymbol{M}_p$ is known up to a scaling factor), the generalized likelihood ratio test (GLRT) was shown to be the constant false alarm rate (CFAR) matched subspace detector [1], [2]. The detector makes no use of the secondary data but requires prior knowledge on the noise statistics. To circumvent this drawback, an adaptive version of the detector was introduced in [3] and referred to as the adaptive coherence estimator. It consists in replacing the primary covariance matrix of the CFAR matched subspace detector by the estimate $\boldsymbol{S} = \boldsymbol{Z}\boldsymbol{Z}^H$, which is the sample covariance matrix (SCM), and $\boldsymbol{Z} = [\boldsymbol{z}_1 \ \ldots \ \boldsymbol{z}_K]$. The detection test then reduces to

$$t = \frac{|\boldsymbol{a}^H \boldsymbol{S}^{-1} \boldsymbol{x}|^2}{(\boldsymbol{a}^H \boldsymbol{S}^{-1} \boldsymbol{a})(\boldsymbol{x}^H \boldsymbol{S}^{-1} \boldsymbol{x})} \underset{H_1}{\overset{H_0}{\lessgtr}} \eta. \tag{2}$$

The test (2) appeared to be very attractive for radar, sonar, and communication applications as it is invariant to data scaling. Later the adaptive coherence estimator (ACE) was shown to be the GLRT in a partially homogeneous environment [4], i.e., when the secondary covariance matrix is proportional to the primary covariance matrix

$$\boldsymbol{M}_s = \gamma \boldsymbol{M}_p \tag{3}$$

with $\boldsymbol{M}_p$ and γ unknown. Additionally, in [5], the authors proved the ACE to be the uniformly most powerful invariant (UMPI) detection test. References [6]–[8] provide a performance analysis of the ACE under homogeneous and nonhomogeneous environment. The ACE test was also presented independently for detecting targets embedded in compound-Gaussian clutter [9]. It emerges naturally as the adaptive counterpart of an asymptotic approximation of the GLRT when the structure of the clutter covariance matrix is known.

However, real-world environments lead to a large amount of heterogeneities which cannot be all embraced by the model (3). In the context of space time adaptive processing, Melvin proposed in [10] models of amplitude and spectral clutter heterogeneities (e.g., due to clutter edges or intrinsic clutter motion) that cause profound structural mismatches between primary and secondary covariance matrices. In a previous work [11], we proposed a new model of heterogeneous environments in a Bayesian framework. Our aim was to have a model that allows one to keep $\boldsymbol{M}_s$ around $\boldsymbol{M}_p$ while having mathematical tractability to derive new detectors. More precisely, the secondary covariance matrix $\boldsymbol{M}_s$ was assumed to be a random matrix distributed according to a complex inverse Wishart distribution with mean $\boldsymbol{M}_p$, i.e.,

$$\boldsymbol{M}_s | \boldsymbol{M}_p \sim \tilde{\mathcal{W}}_m^{-1}\left((\nu - m)\boldsymbol{M}_p, \nu\right). \tag{4}$$

This model ensures that $\boldsymbol{M}_s \neq \boldsymbol{M}_p$ with probability one. Furthermore, the parameter ν scales the distance between $\boldsymbol{M}_s$ and $\boldsymbol{M}_p$: the larger ν, the more homogeneous the environment [11].

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We now extend this Bayesian model by introducing a power scaling factor γ

$$\boldsymbol{M}_s|\gamma, \boldsymbol{M}_p \sim \tilde{\mathcal{W}}_m^{-1}\left(\gamma(\nu-m)\boldsymbol{M}_p, \nu\right). \quad (5)$$

In this way, the heterogeneity level is increased compared to (4). However, on average, we recognize the partially homogenous deterministic environment

$$\mathcal{E}\{\boldsymbol{M}_s|\boldsymbol{M}_p\} = \gamma\boldsymbol{M}_p. \quad (6)$$

In this letter, we show that the GLRT for the detection problem of (1) *with the environment described by* (5) *is also the ACE.* The test is shown to be independent of the primary covariance matrix $\boldsymbol{M}_p$ and the power ratio γ and hence possesses the CFAR property.

II. Generalized Likelihood Ratio Test

In this section, we show that the GLRT for the partially homogeneous Bayesian environment described by (5) is the ACE. The GLRT is classically defined as follows:

$$\frac{\max\limits_{\alpha,\gamma,\boldsymbol{M}_p} f_1(\boldsymbol{x}, \boldsymbol{Z}|\alpha, \gamma, \boldsymbol{M}_p)}{\max\limits_{\gamma,\boldsymbol{M}_p} f_0(\boldsymbol{x}, \boldsymbol{Z}|\gamma, \boldsymbol{M}_p)} \underset{H_1}{\overset{H_0}{\lessgtr}} \eta \quad (7)$$

where $f_\mu(.)$ stands for the joint distribution of $\boldsymbol{x}$ and $\boldsymbol{Z}$ under hypothesis H_μ, $\mu \in \{0, 1\}$.

A. Distributions

This section derives the distributions required in (7). Let us denote $\boldsymbol{x}_\mu = \boldsymbol{x} - \mu\alpha\boldsymbol{a}$ the centered primary snapshot. Since $\boldsymbol{x}_\mu$ and $\boldsymbol{Z}$ are independent, the joint distribution of $(\boldsymbol{x}_\mu, \boldsymbol{Z})$ is

$$f_\mu(\boldsymbol{x}_\mu, \boldsymbol{Z}|\gamma, \boldsymbol{M}_p) = f_\mu(\boldsymbol{x}_\mu|\gamma, \boldsymbol{M}_p) f(\boldsymbol{Z}|\gamma, \boldsymbol{M}_p). \quad (8)$$

The density of $\boldsymbol{x}_\mu$ conditional to γ and $\boldsymbol{M}_p$ was set to $\boldsymbol{x}_\mu \sim \tilde{\mathcal{N}}_m(\boldsymbol{0}, \boldsymbol{M}_p)$ and thus can be written as

$$f_\mu(\boldsymbol{x}_\mu|\gamma, \boldsymbol{M}_p) = \pi^{-m}|\boldsymbol{M}_p|^{-1}\mathrm{etr}\left\{\boldsymbol{M}_p^{-1}\boldsymbol{x}_\mu{\boldsymbol{x}_\mu}^H\right\} \quad (9)$$

where $|.|$ and $\mathrm{etr}\{.\}$ stand for the determinant and the exponential of the trace of a matrix, respectively. In order to derive the density of $\boldsymbol{Z}$ conditioned on γ and $\boldsymbol{M}_p$, note that

$$f(\boldsymbol{Z}|\gamma, \boldsymbol{M}_p) = \int_{\boldsymbol{M}_s} f(\boldsymbol{Z}|\boldsymbol{M}_s) f(\boldsymbol{M}_s|\gamma, \boldsymbol{M}_p) d\boldsymbol{M}_s. \quad (10)$$

Using the independence of the $\boldsymbol{z}_k$'s, the conditional density of $\boldsymbol{Z}$ is given by

$$f(\boldsymbol{Z}|\boldsymbol{M}_s) = \pi^{-mK}|\boldsymbol{M}_s|^{-K}\mathrm{etr}\left\{-\boldsymbol{M}_s^{-1}\boldsymbol{Z}\boldsymbol{Z}^H\right\}. \quad (11)$$

The conditional distribution of $\boldsymbol{M}_s$ given $(\boldsymbol{M}_p, \gamma)$ is defined by the heterogeneity model (5), i.e.,

$$f(\boldsymbol{M}_s|\gamma, \boldsymbol{M}_p) = \frac{|\gamma(\nu-m)\boldsymbol{M}_p|^\nu}{\tilde{\Gamma}_m(\nu)|\boldsymbol{M}_s|^{\nu+m}}\mathrm{etr}\left\{-\gamma(\nu-m)\boldsymbol{M}_s^{-1}\boldsymbol{M}_p\right\} \quad (12)$$

with

$$\tilde{\Gamma}_m(\nu) = \pi^{\frac{m(m-1)}{2}}\prod_{k=1}^{m}\Gamma(\nu - m + k). \quad (13)$$

Using (11) and (12), the distribution of $\boldsymbol{Z}$ can thus be expressed as (see [11] for similar derivations)

$$f(\boldsymbol{Z}|\gamma, \boldsymbol{M}_p) = \pi^{-mK}\frac{\tilde{\Gamma}_m(K+\nu)}{\tilde{\Gamma}_m(\nu)}\frac{|\gamma(\nu-m)\boldsymbol{M}_p|^\nu}{|\gamma(\nu-m)\boldsymbol{M}_p + \boldsymbol{S}|^{K+\nu}}. \quad (14)$$

Finally, the joint distribution of $(\boldsymbol{x}_\mu, \boldsymbol{Z})$ conditioned on γ, $\boldsymbol{M}_p$ is given by

$$f_\mu(\boldsymbol{x}_\mu, \boldsymbol{Z}|\gamma, \boldsymbol{M}_p) = \pi^{-m(K+1)}\frac{\tilde{\Gamma}_m(K+\nu)}{\tilde{\Gamma}_m(\nu)} \times \frac{|\gamma(\nu-m)\boldsymbol{M}_p|^\nu}{|\gamma(\nu-m)\boldsymbol{M}_p+\boldsymbol{S}|^{K+\nu}}|\boldsymbol{M}_p|^{-1}\mathrm{etr}\left\{-\boldsymbol{M}_p^{-1}\boldsymbol{x}_\mu\boldsymbol{x}_\mu^H\right\}. \quad (15)$$

B. Maximum Likelihood Estimate (MLE) of $\boldsymbol{M}_p$

Differentiating the logarithm of (15) and setting the derivative to zero implies that the MLE of $\boldsymbol{M}_p$ verifies

$$\left[(\nu-1)\boldsymbol{M}_p^{-1} + \boldsymbol{M}_p^{-1}\boldsymbol{x}_\mu\boldsymbol{x}_\mu^H\boldsymbol{M}_p^{-1}\right][\gamma(\nu-m)\boldsymbol{M}_p + \boldsymbol{S}] - (\nu+K)\gamma(\nu-m)\boldsymbol{I} = \boldsymbol{0}. \quad (16)$$

By multiplying this equation by $\boldsymbol{S}^{-1/2}\boldsymbol{M}_p$ on the left-hand side and by $\boldsymbol{M}_p^{-1}\boldsymbol{S}^{1/2}$ on the right-hand side, one recognizes a quadratic matrix equation

$$\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H\boldsymbol{Y}^2 + \left[(\nu-1)\boldsymbol{I} + \gamma(\nu-m)\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H\right]\boldsymbol{Y} - (K+1)\gamma(\nu-m)\boldsymbol{I} = 0 \quad (17)$$

with

$$\widetilde{\boldsymbol{x}}_\mu = \boldsymbol{S}^{-1/2}\boldsymbol{x}_\mu \quad (18)$$

$$\boldsymbol{Y} = \boldsymbol{S}^{1/2}\boldsymbol{M}_p^{-1}\boldsymbol{S}^{1/2}. \quad (19)$$

We proceed as in [11] to solve (17). Note that the matrix $\boldsymbol{Y}$ is Hermitian positive definite. Consider one of its eigenvectors $\boldsymbol{u}$ associated with the eigenvalue $\lambda > 0$. Multiplying (17) by $\boldsymbol{u}$, we obtain

$$\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H\boldsymbol{u} = \frac{\gamma(\nu-m)(K+1) - \lambda(\nu-1)}{\lambda^2 + \lambda\gamma(\nu-m)}\boldsymbol{u}. \quad (20)$$

Hence, $\boldsymbol{u}$ is necessarily an eigenvector of $\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H$, and $(\gamma(\nu-m)(K+1) - \lambda(\nu-1))/(\lambda^2 + \lambda\gamma(\nu-m))$ is an eigenvalue of $\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H$ associated with $\boldsymbol{u}$. As $\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H$ is a rank one Hermitian positive matrix, there exists a unitary matrix $\boldsymbol{U}$ such that

$$\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H = \boldsymbol{U}\,\mathrm{diag}(\widetilde{\boldsymbol{x}}_\mu^H\widetilde{\boldsymbol{x}}_\mu, 0, \ldots, 0)\,\boldsymbol{U}^H. \quad (21)$$

Thus, $\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H$ has two different eigenvalues, and λ verifies one of the following equations:

$$\frac{\gamma(\nu-m)(K+1) - \lambda(\nu-1)}{\lambda^2 + \lambda\gamma(\nu-m)} = \begin{cases} 0 \\ \widetilde{\boldsymbol{x}}_\mu^H\widetilde{\boldsymbol{x}}_\mu. \end{cases} \quad (22)$$

The first equation yields a value of $\widetilde{\lambda}$ proportional to γ:

$$\tilde{\lambda} = \frac{(\nu - m)(K+1)}{\nu - 1}\gamma. \quad (23)$$

The second equation is a second-order polynomial equation

$$\|\widetilde{\boldsymbol{x}}_\mu\|^2\lambda^2 + \left[\gamma(\nu - m)\|\widetilde{\boldsymbol{x}}_\mu\|^2 - (\nu - 1)\right]\lambda - C = 0 \quad (24)$$

where $\|\widetilde{\boldsymbol{x}}_\mu\|^2 = \widetilde{\boldsymbol{x}}_\mu^H \widetilde{\boldsymbol{x}}_\mu$ and $C = \gamma(\nu - m)(K+1)$. This equation has a unique real positive solution whose explicit form is not required for our analysis. The remaining derivations will use (24) only.

The solution of the quadratic matrix equation (17) is thus given by

$$\boldsymbol{Y} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H \quad (25)$$

with

$$\boldsymbol{\Lambda} = \operatorname{diag}(\lambda, \tilde{\lambda}, \cdots, \tilde{\lambda}). \quad (26)$$

Finally, using (19) and (25), the MLE of $\boldsymbol{M}_p$ is shown to depend on γ only through the matrix $\boldsymbol{\Lambda}$

$$\boldsymbol{M}_p = \boldsymbol{S}^{1/2}\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\boldsymbol{U}^H\boldsymbol{S}^{1/2}. \quad (27)$$

C. MLE of γ

Let us denote

$$g(\gamma) = \max_{\boldsymbol{M}_p} f_\mu(\boldsymbol{x}_\mu, \boldsymbol{Z}|\gamma, \boldsymbol{M}_p). \quad (28)$$

Using the previous expressions (27) and (15), one obtains

$$g(\gamma) \propto \frac{|\gamma\boldsymbol{\Lambda}^{-1}|^\nu |\boldsymbol{\Lambda}|}{\left|\gamma(\nu - m)\boldsymbol{\Lambda}^{-1} + \boldsymbol{I}\right|^{K+\nu}} \operatorname{etr}\left\{-\boldsymbol{\Lambda}\boldsymbol{U}^H\widetilde{\boldsymbol{x}}_\mu\widetilde{\boldsymbol{x}}_\mu^H\boldsymbol{U}\right\}. \quad (29)$$

Then noticing that $\gamma\boldsymbol{\Lambda}^{-1}$ does not depend on $\widetilde{\lambda}$ [use (23) and (26)], the above expression can be simplified to

$$g(\gamma) \propto \gamma^{\nu+m-1}\lambda^{K+1}\left[\gamma(\nu - m) + \lambda\right]^{-(K+\nu)} \exp\left(-\|\widetilde{\boldsymbol{x}}_\mu\|^2\lambda\right). \quad (30)$$

Differentiating the logarithm of g and equating the derivative to zero implies that the MLE of γ verifies

$$[\gamma(\nu - m) + \lambda]\left[\frac{\nu + m - 1}{\gamma} + \frac{K+1}{\lambda}\frac{\partial\lambda}{\partial\gamma} - \|\widetilde{\boldsymbol{x}}_\mu\|^2\frac{\partial\lambda}{\partial\gamma}\right] - (K+\nu)\left[(\nu - m) + \frac{\partial\lambda}{\partial\gamma}\right] = 0. \quad (31)$$

Then, gathering the terms which depend on $\partial\lambda/\partial\gamma$ and multiplying by $\lambda\gamma$, one obtains

$$\begin{aligned}&(\nu - m)(m - K - 1)\gamma\lambda + (\nu + m - 1)\lambda^2 \\ &\quad + \left[(K+1)(\nu - m)\gamma^2 + (1 - \nu)\gamma\lambda \right.\\ &\quad\left. -(\nu - m)\|\widetilde{\boldsymbol{x}}_\mu\|^2\gamma^2\lambda - \|\widetilde{\boldsymbol{x}}_\mu\|^2\gamma\lambda^2\right]\frac{\partial\lambda}{\partial\gamma} = 0. \quad (32)\end{aligned}$$

Using (24), we observe that the coefficient of $\partial\lambda/\partial\gamma$ is equal to zero. So the MLEs of γ and λ are proportional

$$\lambda = -\frac{(\nu - m)(m - K - 1)}{\nu + m - 1}\gamma. \quad (33)$$

Plugging (33) in (24), we obtain the following expression for the MLE of γ:

$$\gamma = -\frac{1}{\|\widetilde{\boldsymbol{x}}_\mu\|^2}\frac{m}{(\nu - m)}\frac{\nu + m - 1}{m - K - 1}. \quad (34)$$

D. MLE of α

Noticing that λ is proportional to γ, see (33), and that the product $\gamma\|\widetilde{\boldsymbol{x}}_\mu\|^2$ is constant [cf (34)], we have

$$\max_{\gamma, \boldsymbol{M}_p} f_\mu(\boldsymbol{x}_\mu, \boldsymbol{Z}|\gamma, \boldsymbol{M}_p) \propto \frac{1}{\|\widetilde{\boldsymbol{x}}_\mu\|^{2m}}. \quad (35)$$

The MLE of α under H_1 amounts to minimizing the quantity

$$\left\|\boldsymbol{S}^{-1/2}(\boldsymbol{x} - \alpha\boldsymbol{a})\right\|^2 = (\boldsymbol{x} - \alpha\boldsymbol{a})^H\boldsymbol{S}^{-1}(\boldsymbol{x} - \alpha\boldsymbol{a}). \quad (36)$$

The minimum is well known to be achieved for [12]

$$\alpha = \frac{\boldsymbol{a}^H\boldsymbol{S}^{-1}\boldsymbol{x}}{\boldsymbol{a}^H\boldsymbol{S}^{-1}\boldsymbol{a}} \quad (37)$$

and is equal to

$$\boldsymbol{x}^H\boldsymbol{S}^{-1}\boldsymbol{x} - \frac{\left|\boldsymbol{a}^H\boldsymbol{S}^{-1}\boldsymbol{x}\right|^2}{\boldsymbol{a}^H\boldsymbol{S}^{-1}\boldsymbol{a}}. \quad (38)$$

E. GLRT Statistic

The mth root of the GLR can be expressed as

$$\frac{\boldsymbol{x}^H\boldsymbol{S}^{-1}\boldsymbol{x}}{\min_\alpha \|\widetilde{\boldsymbol{x}}_1\|^2} \equiv \frac{\boldsymbol{x}^H\boldsymbol{S}^{-1}\boldsymbol{x}}{\boldsymbol{x}^H\boldsymbol{S}^{-1}\boldsymbol{x} - \frac{|\boldsymbol{a}^H\boldsymbol{S}^{-1}\boldsymbol{x}|^2}{\boldsymbol{a}^H\boldsymbol{S}^{-1}\boldsymbol{a}}} \quad (39)$$

and hence the GLRT is the ACE defined in (2).

III. Detector Performances

A. CFAR Behavior

We show in this section that under H_0, the distribution of the test statistic (2) is independent of $\boldsymbol{M}_p$ and γ, and hence that ACE has the enjoyable CFAR property. We proceed as in [12] and consider the unitary matrix $\boldsymbol{V}$ such that

$$\boldsymbol{V}^H\boldsymbol{M}_p^{-1/2}\boldsymbol{a} = \boldsymbol{e} \quad (40)$$

$$\boldsymbol{e} = [1, 0, \ldots, 0]^T. \quad (41)$$

Let us define

$$\breve{\boldsymbol{x}} = \boldsymbol{V}^H\boldsymbol{M}_p^{-1/2}\boldsymbol{x} \quad (42)$$

$$\breve{\boldsymbol{Z}} = \gamma^{-1/2}(\nu - m)^{-1/2}\boldsymbol{V}^H\boldsymbol{M}_p^{-1/2}\boldsymbol{Z} \quad (43)$$

$$\breve{\boldsymbol{S}} = \breve{\boldsymbol{Z}}\breve{\boldsymbol{Z}}^H. \quad (44)$$

Then the test statistic can be rewritten as

$$t = \frac{\left|\boldsymbol{e}^H\breve{\boldsymbol{S}}^{-1}\breve{\boldsymbol{x}}\right|^2}{(\boldsymbol{e}^H\breve{\boldsymbol{S}}^{-1}\boldsymbol{e})(\breve{\boldsymbol{x}}^H\breve{\boldsymbol{S}}^{-1}\breve{\boldsymbol{x}})}. \quad (45)$$

Under H_0, $\breve{\boldsymbol{x}}$ has a complex normal distribution

$$\breve{\boldsymbol{x}} \sim \tilde{\mathcal{N}}_m(0, \boldsymbol{I}). \quad (46)$$

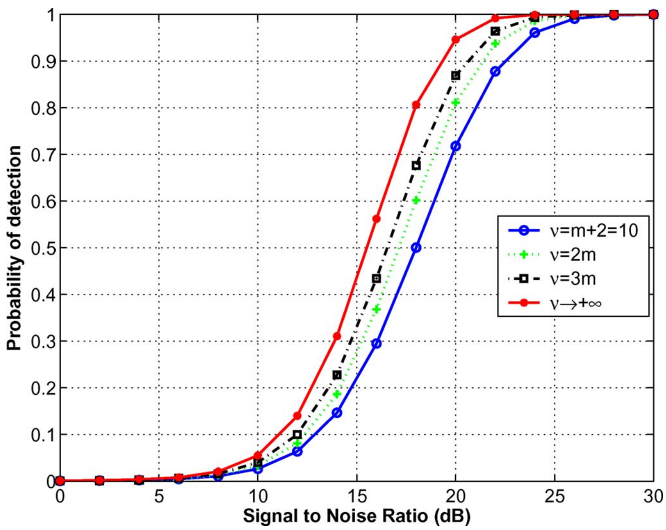

Fig. 1. Probability of detection versus SNR—Influence of ν.

Conditioned on $\boldsymbol{M}_s$, γ, and $\boldsymbol{M}_p$, the matrix $\breve{\boldsymbol{S}}$ has a complex Wishart distribution with K degrees of freedom

$$f(\breve{\boldsymbol{S}}|\boldsymbol{M}_s,\gamma,\boldsymbol{M}_p) = \frac{1}{\tilde{\Gamma}_m(K)} \frac{|\breve{\boldsymbol{S}}|^{K-m}\,|\gamma(\nu-m)\boldsymbol{M}_p|^K}{|\boldsymbol{M}_s|^K} \times \mathrm{etr}\left\{-\boldsymbol{M}_s^{-1}\left(\gamma(\nu-m)\boldsymbol{M}_p^{1/2}\boldsymbol{V}\breve{\boldsymbol{S}}\boldsymbol{V}^H\boldsymbol{M}_p^{1/2}\right)\right\}. \quad (47)$$

Therefore, the distribution of $\breve{\boldsymbol{S}}$ given $(\boldsymbol{M}_p,\gamma)$ is

$$\begin{aligned} f(\breve{\boldsymbol{S}}|\gamma,\boldsymbol{M}_p) &= \int_{\boldsymbol{M}_s} f(\breve{\boldsymbol{S}}|\boldsymbol{M}_s,\gamma\boldsymbol{M}_p) f(\boldsymbol{M}_s|\gamma,\boldsymbol{M}_p) d\boldsymbol{M}_s \\ &= \int_{\boldsymbol{M}_s} \frac{|\breve{\boldsymbol{S}}|^{K-m}\,|\gamma(\nu-m)\boldsymbol{M}_p|^{K+\nu}}{\tilde{\Gamma}_m(K)\tilde{\Gamma}_m(\nu)|\boldsymbol{M}_s|^{K+\nu+m}} \\ &\quad \times \mathrm{etr}\left\{-\boldsymbol{M}_s^{-1}\gamma(\nu-m)\right. \\ &\quad \left.\times\left[\boldsymbol{M}_p^{1/2}\boldsymbol{V}\breve{\boldsymbol{S}}\boldsymbol{V}^H\boldsymbol{M}_p^{1/2}+\boldsymbol{M}_p\right]\right\} d\boldsymbol{M}_s \\ &= \frac{\tilde{\Gamma}_m(K+\nu)}{\tilde{\Gamma}_m(K)\tilde{\Gamma}_m(\nu)} \frac{|\widetilde{\boldsymbol{S}}|^{K-m}}{|\boldsymbol{I}+\widetilde{\boldsymbol{S}}|^{K+\nu}} \end{aligned} \quad (48)$$

which is recognized as a multivariate F-distribution.

We first observe that $\boldsymbol{e}$ in (41) is a fixed vector. Moreover, $\breve{\boldsymbol{x}}$ and $\breve{\boldsymbol{S}}$ are independent —since $\boldsymbol{x}$ and $\boldsymbol{Z}$ are independent— and the distributions of $\breve{\boldsymbol{x}}$ in (46) and $\breve{\boldsymbol{S}}$ in (48) are parameter free. Since the test statistic in (45) depends only on $\boldsymbol{e}$, $\breve{\boldsymbol{x}}$, and $\breve{\boldsymbol{S}}$, it follows that its distribution under H_0 does not depend on $\boldsymbol{M}_p$ or γ, which proves that ACE is CFAR.

B. Numerical Simulations

In this section, we study the influence of ν on the detector performances. The dimension of the observation space is set to $m = 8$, and $K = 2m$ training samples are available. The probability of false alarm is set to $P_{fa} = 10^{-4}$. The detector threshold is computed from $200/P_{fa}$ Monte Carlo runs, with a different value of $\boldsymbol{M}_s$ at each run, drawn from the conditional distribution in (12). The signal-to-noise ratio (SNR) is defined as SNR $= |\alpha|^2 \boldsymbol{a}^H \boldsymbol{M}_p^{-1} \boldsymbol{a}$. Fig. 1 displays the probability of detection P_d, obtained from 10^5 runs, versus SNR for different values of ν.

From inspection of this figure, we observe that, as expected, the more homogenous the environment (i.e., the larger ν), the better the detector performances. There is almost 3 dB difference between the case $\nu = m+2$ and the case $\nu \to +\infty$ which corresponds to the partially homogeneous model (3). This shows that the Bayesian environment (5) models a larger degree of heterogeneity than the deterministic environment. Furthermore, the figure shows that the heterogeneity level does not depend linearly on ν. Indeed when ν is small, a slight variation in ν results in a large variation of the homogeneity level. The trend is inverted for larger values of ν.

IV. Conclusions

The adaptive coherence estimator is a well-known detection scheme which has proved to be effective in a number of nonhomogeneous environments. It is the UMPI test in partially homogeneous environments and is known to perform well also in compound-Gaussian clutter with fully correlated texture. In this letter, we showed that it is also the GLRT in nonhomogeneous environments such that the conditional distribution of $\boldsymbol{M}_s|\boldsymbol{M}_p$ is a complex inverse Wishart distribution with $\mathcal{E}\{\boldsymbol{M}_s\} = \gamma\boldsymbol{M}_p$. This seems to indicate that ACE is (close to) optimum for a large class of nonhomogeneous environments, at least it is rather robust to covariance mismatches characterized by scaling ambiguities. This is to be contrasted with the marked selectivity of ACE with respect to steering vector mismatches. Indeed, ACE is known to possess strong rejection capabilities for signals whose signatures differ from the presumed ones.

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