Open Archive Toulouse Archive Ouverte (OATAO)

OATAO is an open access repository that collects the work of Toulouse researchers and makes it freely available over the web where possible.

This is an author-deposited version published in: http://oatao.univ-toulouse.fr/
Eprints ID: 3898

To link to this article: DOI:10.1088/0031-8949/2009/T136/014023
URL: http://dx.doi.org/10.1088/0031-8949/2009/T136/014023

To cite this version: MIGNOT Rémi, HELIE Thomas, MATIGNON Denis. On the singularities of fractional differential systems, using a mathematical limiting process based on physical grounds, Physica Scripta, 2009, vol. T136, n° 014023.
ISSN 0031-8949

Any correspondence concerning this service should be sent to the repository administrator:
staff-oatao@inp-toulouse.fr
On the singularities of fractional differential systems, using a mathematical limiting process based on physical grounds

R Mignot, T Hélie and D Matignon

1 IRCAM and CNRS, UMR 9912, 1, place Igor Stravinsky, 75004 Paris, France
2 ISAE, Applied Mathematics Training Unit, Université de Toulouse, 10, avenue Edouard Belin, BP 54032, F-31055 Toulouse Cedex 4, France
E-mail: mignot@ircam.fr, helie@ircam.fr and denis.matignon@isae.fr

Abstract
Fractional systems are associated with irrational transfer functions for which nonunique analytic continuations are available (from some right-half Laplace plane to a maximal domain). They involve continuous sets of singularities, namely cuts, which link fixed branching points with an arbitrary path. In this paper, an academic example of the 1D heat equation and a realistic model of an acoustic pipe on bounded domains are considered. Both involve a transfer function with a unique analytic continuation and singularities of pole type. The set of singularities degenerates into uniquely defined cuts when the length of the physical domain becomes infinite. From a mathematical point of view, both the convergence in Hardy spaces of some right-half complex plane and the pointwise convergence are studied and proved.

PACS numbers: 43.20.Mv, 44.05.+e

(Some figures in this article are in colour only in the electronic version.)

1. Introduction

This paper is devoted to throwing light on links between some fractional differential systems and physical phenomena modeled by partial differential equations in unbounded domains. Two problems are considered: firstly, in section 2, an academic example of the 1D heat equation; and secondly, in section 3, a realistic model of an acoustic pipe including visco-thermal losses at the walls and a varying cross-section with constant curvature.

On the one hand, a complex analysis of the transfer functions related to these problems reveals that singularities involve cuts between fixed branching points. On the other hand, the same problems considered on a bounded domain give rise to a countable set of poles. These are standard results.

The main point of this paper is to exhibit in which mathematical sense the cuts of the fractional systems under consideration can be viewed as the limit of a densification of the set of poles when the boundary of the domain goes towards infinity.

As a result, whereas the cuts of the transfer function of fractional systems can be chosen arbitrarily (starting from and ending at fixed branching points), the cuts defined from the limit of the countable set of poles are uniquely defined.

From a mathematical point of view, two types of convergence are examined: firstly, the convergence of the transfer functions in the Hardy space of some left half-complex plane is proved; secondly, the pointwise convergence of some analytic continuation of the transfer functions is obtained over the whole Laplace plane except on these cuts.

Rémi Mignot is a PhD student at Télécom ParisTech/TSI.

1 Rémi Mignot is a PhD student at Télécom ParisTech/TSI.
2. A toy model: the heat equation

2.1. Physical model

Consider the following adimensional heat conduction problem on $I = (0, 1/e)$, described by

$$\forall x \in I, \quad \forall t > 0, \quad \partial_t T(x, t) - \partial_x^2 T(x, t) = 0,$$

with null initial conditions,

$$\forall x \in I, \quad T(x, t = 0) = 0,$$

a controlled Neumann boundary condition at $x = 0$,

$$\forall t > 0, \quad -\partial_x T(x = 0, t) = u(t),$$

a free Dirichlet boundary condition at $x = 1/e$,

$$\forall t > 0, \quad T(x = 1/e, t) = 0,$$

and a Dirichlet observation at $x = 0$ as output,

$$\forall t > 0, \quad y(t) := T(x = 0, t).$$

This models a finite bar, at rest before $t = 0$, controlled by a heat flux $u$ and observed by the temperature at the left end at $x = 0$, and for which the temperature is kept equal to zero at the right end $x = 1/e$.

2.2. Transfer function and Hardy spaces

Following e.g. [1], this problem can be solved in the Laplace domain and yields the transfer function $H_e(s) := \hat{y}(s)/\hat{u}(s)$ given by

$$\forall s \in C_0^+, \quad H_e(s) := \frac{\tanh(\sqrt{s}/e)}{\sqrt{s}},$$

where $\forall \alpha \in \mathbb{R}, \quad C_0^+ := \{s \in \mathbb{C} | Re(s) > \alpha\}$, and where $\hat{T}$ and $\hat{u}$, respectively, denote the one-sided Laplace transforms of $T$ and $u$ with respect to the time variable.

In (6), the square root means the analytic continuation of the positive square root on $\mathbb{R}^+$ on a domain compatible with the one-sided Laplace transform, namely $C_0^+$.

$$\sqrt{s} := s = \rho \exp(i\theta) \mapsto \sqrt{\rho} \exp(i\theta/2),$$

with $(\rho, \theta) \in \mathbb{R}^+ \times (-\pi/2, \pi/2)$.

Following e.g. [2], let us introduce:

**Definition 1.** For $\alpha > 0$ and $m > 0$, $H^m(C_0^\alpha)$ denotes the Hardy space defined by

$$H^m(C_0^\alpha) = \left\{H : C_0^\alpha \to \mathbb{C} \mid H \text{ is holomorphic in } C_0^\alpha, \text{ and } \sup_{\xi > 0} \int_{(2\pi)^m} |H(\xi + i\omega)|^m d\omega < \infty \right\}$$

(8)

The norm of $H \in H^m(C_0^\alpha)$ is then defined by

$$\|H\|_{H^m(C_0^\alpha)} := \sup_{\xi > 0} \left[\frac{1}{2\pi} \int_{(2\pi)^m} |H(\xi + i\omega)|^m d\omega\right]^{1/m}.$$

Then, the following theorem holds.

**Theorem 2.** Let $\varepsilon > 0$, then

$$\forall \alpha > 0, \quad \forall m > 2, \quad H_e \in H^m(C_0^\alpha).$$

**Proof.** Let $\varepsilon > 0, \, \alpha > 0$ and $m > 2$. Since

$$\forall s \in C_0^+, \quad H_e(s) = \frac{1 - \exp(-2\sqrt{s}/\varepsilon)}{\sqrt{s}} + \exp(-2\sqrt{s}/\varepsilon),$$

with (7), it follows that

$$|H_e(s)|^m \sim |s|^{-m/2} \text{ as } |s| \to \infty,$$

$$|H_e(s)|^m \sim \varepsilon^{-m} \text{ as } s \to 0.$$ (12)

Hence, $\int_{C_0^\alpha} |H_e(s)|^m \, ds$ is a finite integral for $m > 2$ (due to (12)) and $\alpha > 0$. □

**Theorem 3.** The function $H_0$ defined by

$$H_0 : C_0^\alpha \to \mathbb{C}, \quad s \mapsto 1/\sqrt{s},$$

is analytic over $C_0^\alpha$.

$$\forall \alpha > 0, \quad \forall m > 2, \quad H_0 \in H^m(C_0^\alpha).$$

Moreover,

$$H_e \converges \text{ on } H^m(C_0^\alpha) \text{ as } \varepsilon \to 0^+.\quad (16)$$

**Proof.** Using (7), $H_0$ is well defined, and analytic in $C_0^\alpha$.

**Proving** (15) is analogous to proving (10), but conversely to theorem 2, the case $\alpha = 0$ cannot be included here: the condition $\alpha > 0$ ensures the convergence of the integral in (9) for $\alpha \to 0$.

Now, as for (16), the behavior of $\frac{1}{\sqrt{s}} \exp(-2\sqrt{s}/\varepsilon)$ has to be studied as $\varepsilon \to 0^+$. Denoting $z = \sqrt{s}/\varepsilon$, we get $|z|^{-1/2} 2^{-3/4} \leq 2|s|^{-1/2} e^{-2|\Re(z)|} \leq 1 + e^{-2|\Re(z)|}$. Then, for $s = \zeta + io$ and $\zeta > \alpha$, we have $e^{-2|\Re(z)|} \leq e^{-2\alpha/|\Re(z)|}$. Raising the latter bound to power $m$ and using (15) yields the desired result. □

2.3. Complex analysis and analytic continuations

The transformation $\sqrt{s} \mapsto -\sqrt{s}$ keeps $H_e$ invariant, so that $H_e$ is a function of only $s$. More precisely, $H_e$ can be analytically continued on the domain $D_e$ given by

$$D_e = \mathbb{C} \setminus P_e,$$

(17)

where $P_e$ is the countable set of poles of $H_e$. Note that $0 \notin P_e$ and $H_e(0) = 1$. The set of zeros of $H_e$ is given by

$$Z_e = \{z_n = -e_n^2 n^2 \pi^2 | n \in \mathbb{N}\}.$$ (19)

Using the formula $\tanh(z)/z = \sum_{n \in \mathbb{N}} (z^2 + (n + \frac{1}{2})^2 \pi^2)^{-1}$ from e.g. [3], $H_e$ proves to be a meromorphic function that can
be expanded into
\[ H_\epsilon : \mathbb{C} \setminus \mathcal{P}_\epsilon \to \mathbb{C}, \]
\[ s \mapsto \sum_{n \in \mathbb{N}} \frac{\epsilon}{s + \epsilon^2(n + \frac{1}{2})^2 \pi^2}. \] (20)

Note the difference between (6) and (20): the latter is the unique maximal analytic continuation of the former.

**Remark 4.** Poles \( \mathcal{P}_\epsilon \) and zeroes \( Z_\epsilon \) are intertwined on the negative real axis \( \mathbb{R}^- \), as already noted in e.g. [4]. Moreover, in a mathematically rigorous way,
\[ \bigcup_{\epsilon > 0} \mathcal{P}_\epsilon = \mathbb{R}^- . \] (21)

This is the reason why we now define \( D_0 := \mathbb{C} \setminus \mathbb{R}^- \) and \( H_0 : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C}, \)
\[ s \mapsto \int_0^\infty \frac{1}{\pi \sqrt{s + \xi}} \frac{1}{\xi} \, d\xi. \] (22)

Using the links between fractional calculus and diffusive representations, as in [5], it can be proved\(^4\) that \( H_0(s) \) as defined by (22) also has the value \( s^{-1/2} \), as expected, once definition (7) has been analytically continued to \( (\rho, \theta) \in \mathbb{R}^+ \times (-\pi, \pi) \).

Note the difference between (14) and (22): the latter is a maximal analytic continuation of the former, but it is certainly not unique! It is well known that any branch cut between the branchpoints \( s = 0 \) and another branchpoint at infinity in \( \Re(s) < 0 \) would also do (see e.g. [5, 6]). Among these analytic continuations, (22) defines the unique limit of \( H_\epsilon \) on \( D_\epsilon \), for \( \epsilon \to 0^+ \), as stated in the following theorem.

**Theorem 5.** Let \( \epsilon > 0 \), then \( D_\epsilon \subset D_{\epsilon'} \), and
\[ \forall s \in \mathbb{C} \setminus \mathbb{R}^- , \quad \lim_{\epsilon \to 0^+} H_\epsilon(s) = H_0(s). \] (23)

**Proof.** Let \( s \) be fixed in \( \mathbb{C} \setminus \mathbb{R}^- \). Then, \( \forall \epsilon > 0 \), \( H_\epsilon \) is analytic at \( s \), and from the extended definition of (7) above, \( \Re(\sqrt{s}) > 0 \), so that \( \lim_{\epsilon \to 0^+} \exp(-2\sqrt{s}/\epsilon) = 0 \). The limit of (11) for \( \epsilon \to 0^+ \) yields the desired result. \( \square \)

**Remark 6.** The cut \( C \), which appears as the limit set of singularities of physical problems on bounded domains, is characterized by
\[ \Re(\sqrt{s}) = 0 \Leftrightarrow s \in C := \mathbb{R}^- . \] (24)

2.4. Integral representations and interpretations
Well-posed integral representations of both \( H_\epsilon \) and \( H_0 \) are given by
\[ H_\epsilon(s) = \int_0^\infty \frac{1}{s + \xi} \, d\mu_\epsilon(\xi). \] (25)
From e.g. [7], the well-posedness condition reads \( \int_0^\infty \frac{1}{s + \xi} \, d\mu(\xi) < \infty \); it is fulfilled by the associated measures \( \mu_\epsilon \) and \( \mu_0 \), defined as follows:
\[ \mu_\epsilon = \sum_{n \in \mathbb{N}} 2\epsilon \delta_{\epsilon n}(\xi), \quad \text{for } \epsilon > 0 \],
\[ d\mu_0(\xi) = \frac{1}{\pi \sqrt{\xi}} \, d\xi. \] (27)

For \( \epsilon > 0 \), \( \mu_\epsilon \) is a discretely supported measure at points \( \xi_n = -\epsilon n = \epsilon^2(n + \frac{1}{2})^2 \pi^2 \), \( n \in \mathbb{N} \); whereas \( \mu_0 \) is absolutely continuous w.r.t. the Lebesgue measure on \( \mathbb{R}^+ \). Both these systems are presented in examples 2.1 and 2.2 of [8], and fully analyzed as well-posed systems in examples 3.2 and 3.4 of [7].

We have the following convergence theorem for the associated measures:

**Theorem 7.** The weak convergence of measures holds:
\[ \mu_\epsilon \wto \mu_0, \quad \text{as } \epsilon \to 0^+. \] (28)

Hence, for \( s \in D_0 \) and with \( \varphi_\epsilon(\xi) := \frac{1}{\sqrt{\pi \xi}} \) as test function in \( C_0(\mathbb{R}^+ \setminus C) \), we recover \( H_\epsilon(s) \to H_0(s) \), as \( \epsilon \to 0^+ \).

**Proof.** Let \( \varphi \in C_C(\mathbb{R}^+ \setminus C) \); we compute \( (\mu_0, \varphi) = \int_0^\infty \frac{1}{\pi \sqrt{\xi}} \varphi(\xi) \, d\xi \) on the one hand, and \( (\mu_\epsilon, \varphi) = \sum_{n=0}^{\infty} 2\epsilon \varphi(\xi_n) \) on the other hand. With the change of variables \( \xi = x^2 \pi^2 \), the test function reads \( \psi(x) := \varphi(\xi) \) and still belongs to \( C_C(\mathbb{R}^+ \setminus C) \). The only thing to prove is then
\[ 2\epsilon \sum_{n=0}^{\infty} \psi \left( \frac{x}{\sqrt{\pi n + 1/2}} \right) \to 2 \int_0^\infty \psi(x) \, dx, \]
as \( \epsilon \to 0^+ \), which is nothing but the limit of a Riemann sum. One can also try to extend the previous result to \( \psi \in C_0(\mathbb{R}^+ \setminus C) \), and not only \( C_C(\mathbb{R}^+ \setminus C) \).

Nevertheless, the well-posedness conditions help prove the last item, even if \( \varphi_\epsilon \notin C_C(\mathbb{R}^+ \setminus C) \), but \( \varphi_\epsilon \in C_0(\mathbb{R}^+ \setminus C) \). \( \square \)

3. A more involved model in acoustics

3.1. Acoustic model of a piece of pipe

3.1.1. Acoustic model. Consider a mono-dimensional model of linear acoustic propagation in axisymmetric pipes, which takes into account the visco-thermal losses and the varying cross section. The acoustic pressure \( p \) and the velocity \( v \) are governed by the Webster–Lokshin equation (see [9, 10] referred to in [11], and more recently [12] for the Webster component) and the Euler equation, given by, in the Laplace domain,
\[ \left[ \left( \frac{s}{c_0^2} \right)^2 + 2\eta(\ell) \left( \frac{s}{c_0} \right)^{3/2} + \gamma(\ell) \right] - \partial_t^2 \right] p(\ell, s) = 0, \] (29)
\[ \rho_0 s v(\ell, s) + \partial_t p(\ell, s) = 0, \] (30)
where \( s \) is the Laplace variable, \( \ell \) is the curvilinear abscissa of the wall, \( c_0 \) is the speed of sound, \( \rho_0 \) is the mass density and \( r(\ell) \) is the radius of the pipe. \( \eta \) quantifies the effect of
the visco-thermal losses and $\Upsilon = r^\prime / r$ is the curvature of the horn.

Note that the symbol $s^{3/2}$ is the Laplace transform of the fractional time derivative $\partial_t^{3/2}$, as introduced in e.g. [13]. The role of the parameter $\eta$ alone, when $\Upsilon = 0$, has been fully understood in [14], in which three closed-form solutions of this problem have been obtained. The diffusive phenomenon in which we are interested in this section is actually due to the curvature $\Upsilon (\ell)$ and requires special treatment.

Let $\psi^+ (\ell, s)$ and $\psi^- (\ell, s)$ be defined by

$$
\begin{bmatrix}
\psi^+ \\
\psi^-
\end{bmatrix} = \frac{r}{2} \begin{bmatrix}
1 & \rho_0 c_0 \\
1 & -\rho_0 c_0
\end{bmatrix} \begin{bmatrix}
p \\
\sqrt{s^2 + 2 \beta s^{3/2}}
\end{bmatrix} + \frac{c_0 r'}{2 s} \begin{bmatrix}
-1 \\
0
\end{bmatrix} p.
$$

(31)

This alternative acoustic state defines traveling waves $\psi^+$ and $\psi^-$, which extend the usual decoupled ingoing and outgoing planar or spherical waves propagating in straight or conical lossless pipes, respectively ($\eta = 0$, $\Upsilon = 0$).

### 3.1.2. Adimensional problem for a piece of pipe

Consider a section of horn with length $L$, constant positive curvature $\Upsilon > 0$, and constant losses coefficient $\eta$. Let us define the adimensional variables and coefficients for this horn:

$$\ell = \frac{\ell}{L}, \quad s = \frac{s}{c_0 \sqrt{\Upsilon}}, \quad \varepsilon = \frac{1}{L \sqrt{\Upsilon}} \quad \text{and} \quad \beta = \frac{\eta}{\sqrt{\Upsilon}},$$

and for any dimensional function $F$, let us define its adimensional version $\tilde{F}$ such as $F(\ell, s, \varepsilon, \beta) = \tilde{F}(\tilde{\ell}, \tilde{s}, \varepsilon, \beta)$.

$\Upsilon$ becomes $\overline{\Upsilon} = 1$ and equations (29)–(31) become, for all $\ell \in (0, 1)$,

$$
\left( s^2 + 2 \beta s^{3/2} + 1 - \varepsilon^2 \overline{\Upsilon}^2 \right) \begin{bmatrix}
p \\
\sqrt{s^2 + 2 \beta s^{3/2}}
\end{bmatrix} = 0,
$$

(32)

$$
\rho_0 c_0 \psi^+ + \varepsilon \overline{\Upsilon} \psi^+ = 0,
$$

(33)

$$
\begin{bmatrix}
\psi^+ \\
\psi^-
\end{bmatrix} = \frac{r}{2} \begin{bmatrix}
1 & \rho_0 c_0 \\
1 & -\rho_0 c_0
\end{bmatrix} \begin{bmatrix}
p \\
\sqrt{s^2 + 2 \beta s^{3/2}}
\end{bmatrix} + \frac{c_0 r'}{2 s} \begin{bmatrix}
-1 \\
0
\end{bmatrix} p.
$$

(34)

Note that $s = 1/(L \sqrt{\Upsilon})$ is inversely proportional to $L$: it will play the same role as $\varepsilon$ in section 2 to compute some limit simulation transfer function when $s \to 0^+$.

In the following we only consider adimensional problems; the notation $\overline{X}$ is re-notated $X$ for the sake of legibility.

### 3.2. Transfer functions and Hardy spaces

#### 3.2.1. Two-port representation and reflection function

Solving (32)–(34) for $\ell \in (0, 1)$ with zero initial conditions, controlled boundary conditions $\psi_0^0 (s) := \psi^+ (\ell = 0, s)$ (ingoing wave at $\ell = 0$) and $\psi_1^0 (s) := \psi^- (\ell = 1, s)$ (ingoing wave at $\ell = 1$) and observing the outgoing traveling waves $(\psi_0^+, \psi_0^-)$ lead to the solution $(\psi_1^+, \psi_0^+)' = Q_s (s) [\psi_0^0, \psi_1^0]'$. The scattering matrix $Q_s$ is given by (cf [15, 16])

$$Q_s (s) = \begin{bmatrix}
T_s (s) & R_s (s) \\
R_s (s) & T_s (s)
\end{bmatrix}, \quad \psi_0^+, \psi_1^+ = Q_s (s) [\psi_0^0, \psi_1^0]'.
$$

Both $T_s$ and $R_s$ are intricate transfer functions that are not convenient to use for simulation purposes in the time domain. Moreover, everything depends on $\varepsilon$, which mixes the effects and makes things difficult to analyze.

In the following we are only interested in the transfer function $R_s (s) = \psi_0^0 (s)/\psi_0^0 (s)$, which represents the global reflection on the traveling waves of the horn at the left end ($\ell = 0$).

The function $R_s$ is given by, $\forall s \in C_0^+$,

$$R_s (s) = \frac{1}{2} \frac{s}{\sinh\left(\frac{1}{2} \varepsilon \overline{\Upsilon} s\right)} + \frac{1}{2} \frac{1}{\cosh\left(\frac{1}{2} \varepsilon \overline{\Upsilon} s\right)},
$$

(35)

$$R_s (s) = \frac{1}{2} \frac{1}{\cosh\left(\frac{1}{2} \varepsilon \overline{\Upsilon} s\right)} + \frac{1}{2} \frac{1}{\tanh\left(\frac{1}{2} \varepsilon \overline{\Upsilon} s\right)},
$$

(36)

where, as for equation (7) of section 2, in (35), $\Gamma (s)$ denotes the analytic continuation of the positive square root of

$$\Gamma (s)^2 = s^2 + 2 \beta s^{3/2} + 1,
$$

(37)

on the domain $C_0^+$ which is compatible with the one-sided Laplace transform.

**Remark 8.** Now, in the part of the complex plane for which $\Re(\Gamma (s)) > 0$, letting $z := \Gamma (s)/\varepsilon$, working on formula (36), we find as in section 2 that, as $\varepsilon \to 0^+$,

$$R_s (z) = \frac{1}{2} \left( \frac{s}{\sinh\left(\frac{1}{2} \varepsilon \overline{\Upsilon} s\right)} + \frac{1}{\cosh\left(\frac{1}{2} \varepsilon \overline{\Upsilon} s\right)} \right) = \frac{s - \Gamma (s)}{s + \Gamma (s)} := R_0 (s).
$$

(38)

#### 3.2.2. Physical interpretation

To reduce the simulation cost, a decomposition of the two-port $Q_s$ into elementary transfer functions can be looked for. Using the method presented in [17], we get the framework of figure 1.

Here, $R_0 (s)$, already defined by (38), represents the wave reflection at the interfaces of the horn and

$$W_s (s) := e^{-\Gamma (s)/s},
$$

(39)

represents the propagation through the horn. For the present work, this framework is of interest because the parameter $\varepsilon$ is now clearly isolated only in $W_s$. We obtain another algebraic expression for $R_s$, namely:

$$R_s = R_0 \frac{1 - W_s^2}{1 - R_0^2 W_s^2},
$$

(40)
which helps prove the pointwise convergence result below. Note that, whereas the functions \( R_0 \) and \( W_\varepsilon \) of the decomposition depend on \( \Gamma(s) \), \( R_\varepsilon \) is a function of only \( \Gamma(s)^2 \) and \( s \), see (36).

Since \( L = 1/(\varepsilon \sqrt{\Gamma}) \), \( L \to \infty \) as \( \varepsilon \to 0 \). The pointwise convergence allows us to interpret the function \( R_0 \) as the waves reflection of a semi-infinite horn, which is anechoic (\( W_\varepsilon(s) \to 0 \) as \( \varepsilon \to 0^+ \), as soon as \( \Re(\Gamma(s)) > 0 \)).

We are now in a position to perform a detailed complex analysis of both \( R_\varepsilon \) and \( R_0 \) in terms of poles, zeroes, branching points and cuts, and analyze their evolution with respect to \( \varepsilon \).

### 3.2.2.1. Hardy spaces

Now, we give some properties of the transfer functions in \( \mathbb{C}_0^+ \). In [16], the following results have been proved for \( \varepsilon > 0 \):

\[
\forall s \in \mathbb{C}_0^+, \quad \Re(\Gamma(s) - s) > 0, \tag{41}
\]

\[
\forall s \in \mathbb{C}_0^+, \quad |R_0(s)| < 1 \quad \text{and} \quad |W_\varepsilon(s)| < 1, \tag{42}
\]

\[
\forall m > 0, \quad \forall W_\varepsilon \in \mathbb{H}^m(\mathbb{C}_0^+), \tag{43}
\]

\[
\forall m > 2, \quad R_0 \in \mathbb{H}^m(\mathbb{C}_0^+) \quad \text{and} \quad R_\varepsilon \in \mathbb{H}^m(\mathbb{C}_0^+). \tag{44}
\]

Moreover, the following result holds:

**Theorem 9.** Let \( \varepsilon > 0 \),

\[
\forall \alpha > 0, \quad \forall m > 2, \quad R_\varepsilon \xrightarrow{\mathbb{H}^m(\mathbb{C}_0^+)} R_0 \quad \text{as} \quad \varepsilon \to 0^+. \tag{45}
\]

**Proof.** Let \( s \in \mathbb{C}_0^+ \). From (41), \( |W_\varepsilon(s)| < e^{-\varepsilon \Re(s)/\varepsilon} < e^{-\alpha/\varepsilon} \), then with (42), we get \( \frac{1-|R_0|^2}{1-|R_\varepsilon|^2} < \frac{1}{1-\varepsilon^2}. \) Consequently

\[
|R_0 - R_\varepsilon| = |R_0 - R_\varepsilon| \frac{1-|R_\varepsilon|^2}{1-|R_\varepsilon|^2} < |R_0| \frac{\varepsilon^{2\alpha/\varepsilon}}{1-\varepsilon^2}. \]

Now, from (44), \( R_0 \in \mathbb{H}^m(\mathbb{C}_0^+) \subset \mathbb{H}^m(\mathbb{C}_0^+) \). Finally, \( \|R_0 - R_\varepsilon\|_{\mathbb{H}^m(\mathbb{C}_0^+)} < \|R_0\|_{\mathbb{H}^m(\mathbb{C}_0^+)} \to 0, \) when \( \varepsilon \to 0^+ \) for \( m > 2 \). \( \square \)

### 3.3. Branching points and cuts of \( \Gamma(s) \)

In this section, we discuss the possible analytic continuations of function \( \Gamma \) in the whole Laplace domain.

#### 3.3.1. Cut on \( \mathbb{R}^- \)

From (37) and because of \( \varepsilon^{3/2} \), \( \Gamma^2 \) has a cut that links \( s = 0 \) and \( s = \infty \) in \( \mathbb{C}_0^+ \). As it has been done in section 2, we choose the cut on \( \mathbb{R}^- \). Note that this choice is required to ensure the hermitian symmetry, causality and stability of the transfer functions.

#### 3.3.2. Symmetric cut

Function \( \Gamma(s) \) has some other branching points, which are solutions of \( \Gamma(s)^2 = 0 \). In the appendix of [15], it has been checked that there are two conjugate solutions with a negative real part. Now we must choose a cut that links these branching points.

To ensure the hermitian symmetry, causality and stability of transfer functions, the cut must satisfy two constraints:

- (C1) the cut must be symmetric w.r.t. \( \mathbb{R} \),
- (C2) the cut must lie only in \( \mathbb{C}_0^+ \).

Figure 2 shows two different choices of this kind.

#### 3.3.2.1. Positive square root

In the sequel, we will make use of a special choice of the symmetric cut, given by

\[
\Gamma(s) := \sqrt{s^2 + 2\beta s^{3/2} + 1}, \tag{46}
\]

where \( \sqrt{\cdot} \) stands for the holomorphic extension to \( (\rho, \theta) \in \mathbb{R}^* \times (\pi, \pi) \) of the square root defined by (7).

Defining \( \Gamma(s) \) by (46), function \( \Gamma \) is holomorphic in \( \mathbb{C} \setminus (\mathbb{R}^- \cup \mathcal{C}) \), with \( \mathcal{C} := \{ s \in \mathbb{C} / \Gamma(s)^2 \in \mathbb{R}^- \} \). The cuts are \( \mathbb{R}^- \) and \( \mathcal{C} \) (cf figure 3). Note that \( \mathcal{C} \) has been proved to be included in \( \mathbb{C}_0^+ \), and its geometry only depends on the coefficient \( \beta > 0 \) (for the adimensional problem).

The important property of definition (46) of \( \Gamma \) is that

\[
\Re(\Gamma(s)) > 0, \quad \forall s \in \mathbb{C} \setminus \mathcal{C}. \tag{47}
\]

### 3.4. Poles, zeroes and convergence

#### 3.4.1. Poles and Zeros of \( R_\varepsilon \)

Recall that the transformation \( \Gamma \mapsto -\Gamma \) keeps \( R_\varepsilon \) invariant, so that \( R_\varepsilon \) is a function of only \( \Gamma^2 \); more precisely, \( R_\varepsilon \) can be analytically continued on the domain \( \mathcal{D}_\varepsilon \) given by

\[
\mathcal{D}_\varepsilon = \mathbb{C} \setminus (\mathbb{R}^- \cup \mathcal{P}_\varepsilon), \tag{48}
\]

\[
\mathcal{P}_\varepsilon = \left\{ s \in \mathbb{C} / \frac{\tanh(\Gamma(s)/s)}{\Gamma(s)} = \frac{-2s}{s^2 + \Gamma(s)^2} \right\}. \tag{49}
\]

\( \mathcal{P}_\varepsilon \) is a set of singularities of \( R_\varepsilon \). Unfortunately, it is difficult to study them explicitly; however, numerical
Conjecture 10. Elements of $P_\varepsilon$ are isolated singularities, and there are an infinite number of such poles. As a corollary, there is no accumulation point. Let $P_\varepsilon$ now denote the set of poles of $R_\varepsilon$.

From (35), the set of zeros of $R_\varepsilon$ is $\mathbb{Z}_e \cup \{\zeta_0, \overline{\zeta_0}\}$, where

$$\mathbb{Z}_e = \{\zeta_0\} \quad \text{and} \quad \overline{\zeta_0} \in \mathbb{C} / \Gamma((\zeta_0)^2 = -\varepsilon^2 n^2 \pi^2 \quad |n \in \mathbb{N}^*\},$$

$$\zeta_0 = (2\varepsilon)^{-3/2} e^{i\pi/3}.$$  \hfill (50)

$\zeta_0$ is the solution of $\Gamma(s) + s = 0$, when $\Gamma$ is defined by (46), and we note that elements of $\mathbb{Z}_e$ lie on $\mathbb{C}$ (i.e. $\mathbb{Z}_e \subseteq \mathbb{C}$).

3.4.2. Pointwise convergence. As already discussed in section 3.3 (section 2), the analytic continuation of $\Gamma$ is not unique, and so for $R_\varepsilon$. Nevertheless, similarly to section 2, $R_\varepsilon$ with $\Gamma$ defined by (46) corresponds to the unique limit of $R_\varepsilon$ defined in $D_1$, for $\varepsilon \to 0^+$, as stated in the following theorem:

Theorem 11. Let the open set $D_0$ and the analytic function $R_\varepsilon$ be defined by

$$D_0 := \mathbb{C} \setminus (\mathbb{R}^+ \cup \mathbb{C} \cup \{\zeta_0, \overline{\zeta_0}\}), \quad \text{(51)}$$

$$R_\varepsilon : D_0 \to \mathbb{C}, \quad \varepsilon > 0, \quad \text{(52)}$$

with the function $\Gamma$, as defined by (46).

Then, $\forall s \in D_0$, and $\Gamma(s)$ defined by (46); from (47), $\Re\{\Gamma(s)\} > 0$, so that $e^{-2\varepsilon n^2 \pi^2} \to 0$ when $\varepsilon \to 0^+$, as first noted in section 3.2 (section 3).

Proof. Let $s \in D_0$, and $\Gamma(s)$ defined by (46); from (47), $\Re\{\Gamma(s)\} > 0$, so that $e^{-2\varepsilon n^2 \pi^2} \to 0$ when $\varepsilon \to 0^+$, as first noted in section 3.2 (section 3). \hspace{1cm} \Box

Note that $\zeta_0$ and $\overline{\zeta_0}$, two zeros of $R_\varepsilon$, are actually the two unique poles of $R_\varepsilon$.

Remark 12. The cut $\mathcal{C}$, which appears as the limit set of singularities of physical problems on bounded domains, is characterized by

$$\Re\{\Gamma(s)\} = 0 \Leftrightarrow s \in \mathcal{C}.$$  \hfill (54)

Unfortunately, we have not succeeded in proving the convergence of poles of $R_\varepsilon$ to the cut $\mathcal{C}$ and $\{\zeta_0, \overline{\zeta_0}\}$ (poles of $R_\varepsilon$). However, numerical simulations illustrate this phenomenon: see figure 4 where poles and zeros are represented by white and black dots, respectively.

Proposition 13. The poles of $R_\varepsilon$ move continuously towards the cut $\mathcal{C}$ when parameter $\varepsilon$ varies continuously.

Proof. Using the analytical continuation of (40) defined by (46), we get an equivalent definition of $P_\varepsilon$, which is $s \in \mathbb{C} / R_\varepsilon(s)W_\varepsilon(s) = 1$. Since $W_\varepsilon(s) \to 0$ when $\varepsilon \to 0^+$, for $s \in \mathbb{C} \setminus \mathcal{C}$, the poles of $R_\varepsilon$ move to $\mathcal{C}$ or to the solutions of $|R_\varepsilon(s)| = \infty$ ($\zeta_0$ and $\overline{\zeta_0}$).

4. Conclusion

Some irrational transfer functions $H_\varepsilon$ have been derived as the limit of solutions $H_\varepsilon$ of physical boundary problems, on domains $(0, 1/\varepsilon)$. The convergence of $H_\varepsilon$ towards $H_0$ in Hardy spaces for some Laplace right half-plane has been proved.

On the one hand, the limit transfer functions $H_\varepsilon$ are those of causal fractional systems, for which an infinite number of analytic continuations are available. Indeed, the set of singularities involve cuts that can be arbitrarily chosen between fixed branching points, in $\Re\{\varepsilon(s)\} \leq 0$.

On the other hand, the maximal analytic continuations of $H_\varepsilon$ and their singularities of pole type are unique. As the main result of this paper (theoretically for model 1 and numerically for model 2), their pointwise limit uniquely defines a particular maximal analytic continuation of $H_\varepsilon$ for which the cuts are hermitian symmetrical and described by a characteristic equation (see (24) and (54)). Moreover, integral representations of $H_\varepsilon$ and of $H_0$ are available, and their corresponding measures are such that $\mu_\varepsilon$ converges towards $\mu_0$, in a weak sense.

However, open questions arise from this work. Firstly, we still have to prove the conjecture (numerically observed) stating that, for $\varepsilon > 0$, the singularities of the acoustic model define an infinite countable set of isolated poles. Secondly, estimating these poles and their residues should be studied to define discrete measures $\mu_\varepsilon$ and analyze their weak convergence towards a limit measure. A technical difficulty is that, contrary to the heat conduction example, the poles approach the cut defined from the limit set but they do not belong to this cut. Hence, the test functions will have to be carefully chosen. Thirdly, for both examples, we observe that the cuts correspond to a limit set of poles but also to a limit set of zeros (which are intertwined with poles for example 1). This matches with widely used approximations of fractional
operators that use placement of intertwined poles and zeros, as, for example, in [4]. A question that arises then is: is this property generally satisfied, or are there some cases for which cuts correspond to limit sets of singularities exclusively (but not zeros)? Fourthly, the unique limit sets of singularities are obtained from the sequence of physically meaningful problems. Questions are then: can distinct sequences of physically meaningful causal problems lead to the same transfer function in \( C^+ \) but different limit sets of singularities in \( C^- \)? If not, how can this limit set be characterized? It should be noted that once a state-space representation \( \dot{X} = A_\varepsilon X \) has been chosen for a sequence of physically meaningful PDE problems, then all the singularities of any transfer functions built from a system with a control operator \( B_\varepsilon \) and an observation operator \( C_\varepsilon \) do belong to \( \text{spec}(A_\varepsilon) \): only point spectrum for \( \varepsilon > 0 \) and continuous spectrum for \( \varepsilon = 0 \). Hence, \( \text{spec}(A_\varepsilon) \) fixes the general location of singularities, see e.g. [6]: this last remark should help obtain relevant information for our questions.

Acknowledgment
This work was supported by the CONSONNES project, ANR-05-BLAN-0097-01.

References