An Improved Adaptive Sidelobe Blanker
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Abstract—We propose a two-stage detector consisting of a subspace detector followed by the whitened adaptive beamformer orthogonal rejection test. The performance analysis shows that it possesses the constant false alarm rate property with respect to the unknown covariance matrix of the noise and that it can guarantee a wider range of directivity values with respect to previously proposed two-stage detectors. The probability of false alarm and the probability of detection (for both matched and mismatched signals) have been evaluated by means of numerical integration techniques.

Index Terms—Adaptive radar detection, adaptive sidelobe blanker, constant false alarm rate (CFAR), mismatched signals.

I. INTRODUCTION

In the last decades several papers have addressed adaptive radar detection of targets embedded in Gaussian or non-Gaussian disturbance. Most of these papers follow the lead of the seminal paper by Kelly [1], where the generalized likelihood ratio test (GLRT) is used to conceive an adaptive decision scheme capable of detecting coherent pulse trains in presence of Gaussian disturbance with unknown spectral properties. In [2], it is shown that an ad hoc detector, based on the two-step GLRT-based design procedure and referred to in the following as adaptive matched filter (AMF), can also achieve better probability of detection than Kelly’s detector for high signal-to-noise ratios (SNRs). The tools of invariance have been used in [3] to address adaptive detection in presence of structured and unstructured disturbance. Detection of point-like targets in non-Gaussian noise has been dealt with in [4] and references therein). The case of point-like targets assumed to belong to a known subspace of the observables has been addressed in [5] and [6]. The case of rank-one waveforms that are only partially known has been dealt with in [7]; therein, the invariance properties of the decision problem are studied and the corresponding maximal invariant is derived. Finally, the authors design the GLRT-based detector for the problem at hand. In addition, several detection algorithms for point-like or extended targets embedded in Gaussian disturbance are encompassed as special cases of the amazingly general framework and derivation described in [8].

All of the above papers assume that returns from the target can be modeled in terms of (a linear combination of) deterministic signals known up to multiplicative, possibly complex, constants, taking into account both target and channel effects. Moreover, those detectors rely on the assumption that a set of secondary data, namely returns free of signals components, but sharing the spectral properties of the noise in the data under test, is available. Such secondary data are used to come up with fully adaptive detection schemes.

It is also important to stress that most detectors have not been designed to distinguish between mainlobe and sidelobe mismatched signals. As a matter of fact, the actual signal backscattered from a target (or target’s scattering centers) can be different from the nominal one and the mismatch can have different reasons as, for example [9], [10], the following:

• coherent scattering from a direction different to that in which the radar system is steered (sidelobe target);
• imperfect modeling of the mainlobe target by the nominal steering vector, where the mismatch may be due to multipath propagation, array calibration uncertainties, beam-pointing errors, etc.

Thus, it might be important to trade detection performance of mainlobe targets for rejection capabilities of sidelobe ones. In order to face with this dilemma, the adaptive beamformer orthogonal rejection test (ABORT) detector was introduced in [9]. The idea of ABORT is to modify the null hypothesis, which usually states that the vector under test contains noise only, so that it possibly contains a vector which, in some way, is orthogonal to the assumed target’s signature. Doing so, if a signal with actual steering vector different from the nominal one is present, the detector will be less inclined to declare a detection, as the null hypothesis will be more plausible. For instance, if a sidelobe target is present, the ABORT detector will exhibit less false alarms than detectors, which are rather sensitive to mismatched signals, as the AMF. As customary, it is assumed that a set of secondary data is available at the receiver. The directivity of such detector is in between that of Kelly’s detector (which, in turn, is more directive than the AMF) and the one of the adaptive coherence estimator (ACE) [4], [5].

However, in the original ABORT formulation, the fictitious signal under the null hypothesis was assumed to be orthogonal to the nominal one, in the quasi-whitened space, i.e., after whitening by the sample covariance matrix of the training samples. In [11], such an assumption was modified to address adaptive detection of distributed targets embedded in homogeneous disturbance, by resorting to the GLRT with the useful and the
fictitious signals orthogonal in the whitened space, i.e., after whitening with the true covariance matrix. This seemingly minor modification led to significant differences compared to ABORT and, more particularly, to an enhanced rejection of sidelobe signals. Furthermore, in [11], it is shown that such detector, referred to in the following as whitened ABORT (W-ABORT), may become even more selective than the ACE.

On the other hand, increased robustness can be achieved by resorting to the tools of subspace detection, namely assuming that the target belongs to a known subspace of the observables, or also constraining the possible useful signals to belong to a proper cone with axis the whitened nominal steering vector [12], [13]. Moreover, extending such a cone idea by modeling the null hypothesis as the complement of a cone to the entire space of the observables leads to a family of detectors capable of trading detection performance of mainlobe targets for rejection capabilities of sidelobe ones. A way to combine the ABORT rationale with the subpace idea has been dealt with in [10] as a possible means to retain an acceptable detection loss for slightly mismatched mainlobe targets.

Another class of detectors that combines the statistics of Kelly’s detector and of the AMF has been proposed by Kalson [14]: the detector statistic depends on a design parameter \( \alpha \in [0,1] \) that allows to obtain receiver operating characteristics, actually the probability of detection (\( P_d \)) versus the SNR, in between that of Kelly’s detector and the one of the AMF.

Unfortunately, though, it seems difficult to find a decision scheme capable of providing at the same time good capabilities to reject sidelobe targets and high power in case of slightly mismatched mainlobe targets. In order to face with this problem, the so-called two-stage detectors have been proposed; such schemes are formed by cascading two detectors (usually with opposite behaviors): the overall one declares the presence of a target in the data under test only when data under test survive both detection thresholds. A rather famous two-stage detector is the adaptive sidelobe blanker (ASB). The ASB has been proposed as a means for mitigating the high number of false alarms of the AMF in the presence of undetected interferences [15]–[17]. It can be seen as the cascade of the AMF and the ACE. Remarkably, it can adjust directivity by proper selection of the two thresholds in order to trade good rejection capabilities of sidelobe targets for an acceptable detection loss of matched signals [18], [19]. Richmond has also provided analytical expressions for the \( P_d \) and the probability of false alarm (\( P_{fa} \)) of the ASB and demonstrated that, in homogeneous environment and with matched signals, it has higher or commensurate \( P_{fa} \) for a given \( P_d \), than both the AMF and the ACE and an overall performance that is commensurate with Kelly’s detector [20]. A further two-stage detector, consisting of the cascade of the AMF and Kelly’s detector, has been proposed as a computationally efficient implementation of the latter [19]. Moreover, in [21], a two-stage detector obtained cascading a RAO test to the AMF has been proposed and assessed. Note that the above two-stage detectors are invariant to the group of transformations given in [3].

More recently, another two-stage detector, the so-called subspace-based ASB (S-ASB), has been proposed [22]. Specifically, the S-ASB is obtained cascading a subspace GLRT-based detector [referred to in the following as subspace detector (SD)] and the ACE; the performance assessment has shown that such solution can increase the robustness of the composite detector while retaining the same selectivity of the ASB. However, it is not invariant to the group of transformations given in [3]. Based upon the experience of [22], herein we propose a two-stage detector aimed at increasing also the selectivity of the S-ASB, i.e., the capability to reject mismatched signals. This is accomplished by cascading the SD and the W-ABORT. The performance assessment confirms that its directivity varies in a wider range than its competitors, at least for the considered class of mismatched signals, when we constrain the maximum loss with respect to Kelly’s detector for matched signals, given \( P_{fa} \) and \( P_d \).

The remainder of the paper is organized as follows. The next section is devoted to the problem formulation and to the description of the proposed detector while Section III contains its performance assessment; to this end, we derive analytical expressions for \( P_{fa} \) and \( P_d \) for matched and mismatched signals. In Section IV, we report some illustrative examples in which we compare the performance of the proposed detector with those of the S-ASB, ASB, and of the two-stage detector proposed in [21]. Concluding remarks are given in Section V. Finally, Appendixes A and B contain some analytical derivations.

II. PROBLEM FORMULATION

Assume that a linear array formed by \( N_a \) antennas senses the cell under test and that each antenna collects \( N_f \) samples. Denote by \( z \in \mathbb{C}^{N_f \times 1} \) the \( N \)-dimensional column vector, with \( N = N_a N_f \), containing returns from the cell under test. We want to test whether or not \( z \) contains useful target echoes. As customary, we assume that a set of \( K \) secondary data, \( z_k \in \mathbb{C}^{N_f \times 1}, k = 1, \ldots, K, K \geq N \), namely data free of signal components but sharing the same statistical properties of the noise in the cell under test, is available.

The detection problem can be recast as

\[
\begin{align*}
H_0 : & \quad z = n_k, \quad k = 1, \ldots, K \\
H_1 : & \quad z = \alpha p + n_k, \quad z_k = n_k, \quad k = 1, \ldots, K
\end{align*}
\]

where the following holds:

- \( n_k \in \mathbb{C}^{N_f \times 1}, k = 1, \ldots, K, \) are independent and identically distributed complex normal vectors with zero-mean and unknown, positive-definite covariance matrix \( R \in \mathbb{C}^{N_f \times N_f} \), i.e., \( n_k \sim \mathcal{CN}(0, R) \), \( k = 1, \ldots, K \);
- \( p \in \mathbb{C}^{N_f \times 1} \) is the direction of the (possible) mainlobe target echo, possibly different from that of the nominal steering vector \( p_0 \in \mathbb{C}^{N_f \times 1} \);
- \( \alpha \in \mathbb{C} \) is an unknown (deterministic) factor which accounts for both target and channel effects.
In the following we propose and assess a two-stage detector obtained by cascading the SD \[8\], whose statistic is given by

\[
t_{SD} = \frac{z^\dagger S^{-1} H (H^\dagger S^{-1} H)^{-1} H^\dagger S^{-1} z}{1 + z^\dagger S^{-1} z}
\]

and the W-ABORT, whose statistic is given by \[11\]

\[
t_{WA} = \frac{1}{\left(1 + z^\dagger S^{-1} z\right)(v_0^\dagger S^{-1} v_0) - 1} \left(1 + z^\dagger S^{-1} z\right)
\]

where the following holds:

- \( S \in \mathbb{C}^{N \times N} \) is \( K \) times the sample covariance matrix of the secondary data, i.e., \( S = ZZ^\dagger \) with \( Z = [z_1 \cdots z_K] \in \mathbb{C}^{N \times K} \);
- \( H \in \mathbb{C}^{N \times r} \) is a full-column-rank matrix (and, hence, \( r \geq 1 \) is the rank of \( H \)).
- \( v_0 \) is the well-known decision statistic of Kelly’s detector \[1\].

Summarizing, the operation of the newly-proposed detector, referred to in the following as WAS-ASB, can be pictorially described as follows:

\[
t_{SD} \overset{\eta}{\geq} H_0, \quad t_{WA} \overset{\xi}{\geq} H_1
\]

where \( \eta \) and \( \xi \) form the threshold pair to be set in order to guarantee the overall desired \( P_{fa} \).

### III. PERFORMANCE ASSESSMENT

In this section, we derive analytical expressions for \( P^d \) and \( P_{fa} \) of the WAS-ASB; to this end, we replace \( t_{SD} \) with the equivalent decision statistic \( \tilde{t}_{SD} = 1/(1 - t_{SD}) \). Based upon results contained in \[22\], it is possible to show that \( \tilde{t}_{SD} \) and \( t_{WA} \) admit the following stochastic representations:

\[
\tilde{t}_{SD} = (\tilde{t}_K + 1)(1 + c)
\]

\[
t_{WA} = \frac{(\tilde{t}_K + 1)}{(1 + b)(1 + c)}
\]

where \( \tilde{t}_K = t_K/(1 - t_K) \) (see Appendix A).

Moreover, under the \( H_0 \) hypothesis:

- \( \tilde{t}_K \), given \( b \) and \( c \), is ruled by the complex central F-distribution with 1, \( K - N + 1 \) degrees of freedom \[8\] (for a definition of complex normal related statistics, see also \[20\, App. A\]);
- \( b \) is a complex central F-distributed random variable (rv) with \( N - r, K - N + r + 1 \) degrees of freedom, i.e., \( b \sim CF_{N-r,K-N+r+1} \) (see \[22\, App. B\]);
- \( c \sim CF_{r-1,K-N+2} \) (see \[22\, App. B\]);
- \( b \) and \( c \) are statistically independent rv’s (see \[22\, App. B\]).

Now, the \( P_{fa} \) of the two-stage detector can be expressed as

\[
P_{fa}(\eta, \xi) = P[t_{SD} > \tilde{t}_K, t_{WA} > \xi; H_0]
\]

\[
= P[t_{SD} > \tilde{t}_K, t_{WA} > \xi; H_0]
\]

\[
= P\left[ (\tilde{t}_K + 1)(1 + c) > \tilde{\eta}, \frac{\tilde{t}_K + 1}{(1 + c)(1 + b)} > \xi; H_0 \right]
\]

\[
= 1 - P \left[ \tilde{t}_K \leq \max \left( \tilde{\eta}, \frac{\xi}{1 + \gamma} - 1, \frac{\xi(1 + b)(1 + c) - 1}{1 + \gamma}; H_0 \right) \right]
\]

\[
= 1 - \int_{0}^{\infty} \int_{0}^{\infty} P \left[ \tilde{t}_K \leq \max \left( \tilde{\eta}, \frac{\xi}{1 + \gamma} - 1, \frac{\xi(1 + b)(1 + c) - 1}{1 + \gamma}; H_0 \right) \right]
\]

\[
\times \psi(b, c) d\beta d\gamma
\]

\[
= 1 - \int_{0}^{\infty} \int_{0}^{\infty} P_{00} \left( \max \left( \frac{\tilde{\eta}}{1 + \gamma} - 1, \frac{\xi(1 + b)(1 + c) - 1}{1 + \gamma} \right) \right)
\]

\[
\times \psi(b, c) d\beta d\gamma
\]

where \( \tilde{\eta} = 1/(1 - \eta), \psi (\cdot) \) is the probability density function (pdf) of the rv \( b \sim CF_{N-r,K-N+r+1}, \psi (\cdot) \) is the pdf of the rv \( c \sim CF_{r-1,K-N+2}, \) and \( P_{00} (\cdot) \) is the cumulative distribution function (CDF) of the rv \( \tilde{t}_K \), given \( b \) and \( c \) (and under \( H_0 \)), i.e., the CDF of a rv \( b \) by the \( CF_{1,K-N+1} \) distribution.

The following remarks are now in order. First, it is apparent that the WAS-ASB possesses the constant false alarm rate (CFAR) property with respect to \( R \). Second, note that the \( P_{fa} \) of the two-stage detector depends on two thresholds; as a consequence, there exist infinite threshold pairs that guarantee the same value of \( P_{fa} \). Third, observe that the \( P_{fa} \) is independent of \( H \) but for the value of \( r \) (and provided that the first column of \( H \) is \( v_0 \)). For the reader’s ease, Fig. 1 shows the contour plots for the WAS-ASB corresponding to different values of \( P_{fa} \), as functions of the threshold pairs \((\xi, \tilde{\eta})\). \( N = 16, K = 32, \) and \( r = 2 \). All curves have been obtained by means of numerical integration techniques. Note that, for a preassigned \( P_{fa} \), the threshold of the SD does not approximately change for values of the threshold of the W-ABORT ranging from 0 to 0.2.

Under the \( H_1 \) hypothesis, we assume a misalignment between the actual steering vector \( p \) and the nominal one \( v_0 \), i.e., \( p \neq v_0 \). In this case, the rv’s \( b \) and \( c \) depend on \( \cos^2 \theta \) defined through \( (5) \) of Appendix B; for this reason, in the following, we will denote these rv’s by \( b_\theta \) and \( c_\theta \). Due to the useful signal components, the distributions of \( \tilde{t}_K, b_\theta \), and \( c_\theta \) change, more precisely (see Appendix B):

- \( \tilde{t}_K \), given \( b_\theta \) and \( c_\theta \), is ruled by the complex noncentral F-distribution with \( 1, K - N + 1 \) degrees of freedom and noncentrality parameter

\[
e_\theta = \frac{\text{SNR} \sin^2 \theta}{(1 + b_\theta)(1 + c_\theta)}
\]

where

\[
\text{SNR} = |\mathbf{p}^\dagger \mathbf{M}^{-1} \mathbf{p}|
\]

is the total available signal-to-noise ratio;
In the case of perfect match between \( \mathbf{b}_d \) and \( \mathbf{p} \), i.e., \( \theta = 0 \), \( \delta_{b_d} \) and \( \delta_{c_0} \) are equal to zero, thus rv’s \( c_0 \) and \( b_0 \) obey to the complex central F-distributions with \( N-r, K-N+r+1 \) and \( r-1, K-N+2 \) degrees of freedom, respectively. On the other hand, \( \tilde{\gamma}_k \) is still subject to the complex noncentral F-distribution with \( 1, K-N+1 \) degrees of freedom and noncentrality parameter given by

\[
\delta^2_0 = \frac{\text{SNR}}{1 + b_0(1 + c_0)}
\]

and (in this case) the \( P_d \) does not depend on the choice of \( \mathbf{H} \) but for the value of \( r \) (and provided that the first column of \( \mathbf{H} \) is \( \mathbf{v}_0 \)).

IV. ILLUSTRATIVE EXAMPLES AND DISCUSSION

In this section, we present some numerical examples to show the effectiveness of the WAS-ASB, also in comparison to the ASB, the S-ASB, and the two-stage detector proposed in [21] and referred to in the following as AMF-RAO. All curves have been obtained by means of numerical integration techniques\(^3\).

In all examples, the noise is modeled as an exponentially-correlated complex normal vector with one-lag correlation coefficient \( \rho \), namely the \( (i,j) \)th element of the covariance matrix \( \mathbf{R} \) is given by \( \rho^{i-j}, i,j = 1, \ldots, N, \) with \( \rho = 0.95 \). The probability of false alarm is set to \( 10^{-4} \). Moreover, we set \( N_t = 1 \), \( N_a = N \), and choose \( \mathbf{v}_0 = \mathbf{s}(0) \) with

\[
\mathbf{s}(\phi) = \frac{1}{\sqrt{N}} \left[ e^{i2\pi\lambda \sin \phi} \ldots e^{i(N-1)2\pi \sin \phi} \right]^T
\]

where \( \lambda \) is the interelement spacing, \( \mathbf{H} \) is the radar operating wavelength, and \( T \) denotes transpose. Moreover, we will denote by \( \phi_T \) the azimuthal angle of the impinging useful target echo, i.e., \( \mathbf{p} = \mathbf{s}(\phi_T) \).

Until now, we have left aside the problem of how to choose \( r \) and the remaining columns of \( \mathbf{H} \). First, remember that, under matching conditions, \( P_d \) does not depend on the choice of \( \mathbf{H} \) but for the value of \( r \). However, the design of \( \mathbf{H} \) will impact on the performance of the detector in presence of mismatched signals. In fact, values of \( r \) greater than one are necessary to increase its robustness. The design of \( \mathbf{H} \) is a challenging problem: simulation results not reported herein indicate that values of \( r > 2 \) or \( r = 2 \) and a vector \( \mathbf{v}_1 \) significantly mismatched with respect to \( \mathbf{v}_0 \) may produce curves of \( P_d \) versus \( \phi_T \) whose transition from the “bandpass” to the “stopband” is not sharp; see [22] for a more detailed discussion about this issue. For this reason, in the following we set \( r = 2 \) and \( \mathbf{v}_1 = \mathbf{s}(\phi_1) \), with \( \phi_1 = \pi/300 \), and discuss the behavior of the corresponding detector in order to prove its effectiveness.

In Figs. 2–4, we plot \( P_d \) versus \( \text{SNR} \) for the S-ASB, the WAS-ASB, and the AMF-RAO, respectively, as they compare to Kelly’s detector [1], for the case of a matched target, \( N = 16, K = 32 \); in these figures, we show two curves (in addition to the curve of Kelly’s detector) for each of them: such curves correspond to the limiting behaviors of the two-stage detectors for

\[
\delta_0^2 = \frac{\text{SNR}}{1 + b_0(1 + c_0)}
\]

\( \mathbf{p} \) is the pdf of a rv ruled by the \( \mathcal{CF}_{K-N+2}(\tilde{\gamma}_0) \) distribution. \( p_{b_0}(\cdot) \) is the pdf of a rv ruled by the \( \mathcal{CF}_{K-N+2}(\tilde{\gamma}_0) \), and \( p_{c_0}|_{b_0}(\cdot) \) is the pdf of a rv ruled by the \( \mathcal{CF}_{K-N+2}(\tilde{\gamma}_0) \).

\(^3\) For the WAS-ASB, we also show results obtained through Monte Carlo simulation. To this end, we resorted to \( 100/P_{\text{IN}} \) and \( 10^4 \) independent trials in order to evaluate the thresholds necessary to ensure a preassigned value of \( P_{\text{IN}} \), and the \( P_d \)’s, respectively. Simulated results are presented in Figs. 3 and 7 and are shown as cross markers.
threshold settings which guarantee \( P_{fb} = 10^{-4} \). Fig. 2 shows that the maximum loss of the S-ASB with respect to Kelly’s detector is less than (about) 1.2 dB (at \( P_d = 0.9 \)); such a maximum loss increases to (about) 2 dB in Figs. 3 and 4 for the WAS-ASB and the AMF-RAO.

In Figs. 5–8, we plot \( P_d \) versus \( \phi_T \) (measured in degrees) for the ASB, the S-ASB, the WAS-ASB, and the AMF-RAO, respectively. \( N = 16, K = 32 \); all plotted curves refer to those threshold pairs which ensure the best rejection capabilities of sidelobe targets under the constraint that the loss with respect to Kelly’s detector and for matched signals is less than (about) 1 dB at \( P_d = 0.9 \) and \( P_{fb} = 10^{-4} \). Observe from Figs. 5 and 6 that the S-ASB can ensure better robustness with respect to the ASB, due to the first stage (the SD), which is less sensitive than the AMF to mismatched signals. However, S-ASB and ASB exhibit the same capability to reject sidelobe targets, according to the fact that the second stage (the ACE) is the same. As it can be seen from Figs. 6 and 7, instead, the WAS-ASB guarantees the same robustness of the S-ASB, but better rejection capabilities than the latter (and, consequently, better rejection capabilities than the ASB), due to the fact that the ACE has been replaced by the W-ABORT. Other simulation results not reported here, in order not to burden too much the analysis, have shown that the WAS-ASB is generally more selective than the ASB and the S-ASB and more robust than the ASB. Finally, from Figs. 7 and 8, we see that the AMF-RAO is slightly more selective than the WAS-ASB for the considered parameter values.

In Figs. 9 and 10, we plot contours of constant \( P_d \), as functions of SNR and \( \phi \); as for the curves of Figs. 5–8, thresholds are such that the loss with respect to Kelly’s detector is less than (about) 1 dB for the perfectly matched case. More precisely, in Fig. 9, we analyze the robustness with respect to mismatched signals and compare the WAS-ASB and the ASB;
Fig. 6. $P_d$ versus target azimuthal angle for the S-ASB with $N = 16$, $K = 32$, $r = 2$, and SNR = 19 dB.

Fig. 8. $P_d$ versus target azimuthal angle for the AMF-RAO with $N = 16$, $K = 32$, and SNR = 19 dB. Cross markers denote results obtained by Monte Carlo simulation.

Fig. 7. $P_d$ versus target azimuthal angle for the WAS-ASB with $N = 16$, $K = 32$, $r = 2$, and SNR = 19 dB. Cross markers denote results obtained by Monte Carlo simulation.

Fig. 9. Contours of constant $P_d$ for the WAS-ASB and the ASB with $N = 16$, $K = 32$, $r = 2$, and threshold pairs corresponding to the most robust case.

As a final comment, observe that the WAS-ASB is more time consuming than the ASB and the AMF-RAO; note though that resorting to the usual Landau notation all of them are $O(N^2)$, namely computation of their decision statistics requires a number of flops proportional to $K N^2$ (remember that $K \geq N$) which is in fact the number of flops required to evaluate $S$.

V. CONCLUSION

We have proposed a two-stage detector consisting of a GLRT-based subspace detector followed by the W-ABORT. The performance analysis has been conducted analytically for both matched and mismatched signals. Although the proposed detector is not invariant to a meaningful group of transformations, it possesses the CFAR property with respect to the unknown covariance matrix of the noise and it can guarantee a wider range of directivity values with respect to its natural competitors. Thus, it appears as a viable means to trade detection performance of mainlobe targets for rejection capabilities.
of sidelobe ones. However, the design of flexible detection algorithms is still an open problem. Following the lead of [23], where the theory of convex optimization is used to provide flexibility for array beampattern synthesis, we are currently investigating the design of tunable receivers taking advantage of a stage aimed at adaptive weighting of the received signals.

APPENDIX A
STOCHASTIC REPRESENTATION OF THE SD AND THE W-ABORT

In this section, we come up with suitable stochastic representations for $\tilde{k}_{\text{SD}}$ and $t_{\text{WA}}$. As to the subspace detector, we refer to [22, App. A], where it is shown that

$$\tilde{k}_{\text{SD}} = (\tilde{k} + 1)(1 + c).$$

Now, let us concentrate on $t_{\text{WA}}$ and note that

$$t_{\text{WA}} = \frac{1}{(\tilde{k}_K - 1)^2(1 + z^* S^{-1} z)}$$

$$= \frac{|z^* S^{-1} v_0|^2}{\langle v_0 | S^{-1} v_0 \rangle (1 + z^* S^{-1} z) |z^* S^{-1} v_0|^2} \frac{1}{(\tilde{k}_K - 1)^2}$$

$$= t_{\text{AMF}} (\tilde{k}_K - 1)^2,$$

where

$$t_{\text{AMF}} = \frac{|z^* S^{-1} v_0|^2}{\langle v_0 | S^{-1} v_0 \rangle}.$$
and observe that

\[ t_{\text{AMP}} = \frac{\hat{t}_K}{\beta}. \]

It follows that \( t_{\text{WA}} \) can be recast as

\[ t_{\text{WA}} = (1 + \hat{t}_K) \beta. \]

The above stochastic representation is useful to characterize the W-ABORT [24], but it has to be further modified in order to derive the statistical characterization of the overall two-stage detector. More precisely, we need to write the term \( \beta \) in a different form. To this end, applying the whitening transformation \( R^{-1/2} \) to the vectors \( z \) and \( z_k \), \( k = 1, \ldots, K \), yields

\[ \beta = \left( 1 + z_w^\dagger S_w^{-1} z_w - \frac{|z_w^\dagger S_w^{-1} v_w|^2}{v_w^\dagger S_w^{-1} v_w} \right)^{-1} \]

where \( z_w = R^{-1/2} z \), \( v_w = R^{-1/2} v \), and \( S_w = R^{-1/2} S R^{-1/2} \); then, define a unitary matrix \( U \in \mathbb{C}^{N \times N} \) that rotates \( H_0 \), which is a slice of unitary matrix obtained by means of QR factorization of the matrix \( R^{-1/2} H \), into the first \( r \) elementary vectors, i.e.,

\[ U H_0 = \begin{bmatrix} I_r & 0_{(N-r) \times r} \end{bmatrix} \]

and, in particular,

\[ U R^{-1/2} v_0 = \sqrt{v_0^\dagger R^{-1/2} v_0} e_1. \]

It follows that \( \beta \) can be recast as

\[ \beta = \left( 1 + x^\dagger S_1^{-1} x - \frac{|x^\dagger S_1^{-1} e_1|^2}{e_1^\dagger S_1^{-1} e_1} \right)^{-1} \tag{4} \]

where \( x = U z_w \) and \( S_1 = U S_w U^\dagger \). Following the lead of [1] and [8], we decompose all vectors into two components: an \( A \)-component consisting of the first element only and a \( B \)-component consisting of the rest of the vector, namely as

\[ x = \begin{bmatrix} x_A \\ x_B \end{bmatrix}, \quad S_1 = \begin{bmatrix} S_{1,AA} & S_{1,AB} \\ S_{1,BA} & S_{1,BB} \end{bmatrix}, \]

and

\[ S_1^{-1} = \begin{bmatrix} C_{AA} & C_{AB} \\ C_{BA} & C_{BB} \end{bmatrix}; \]

it follows that

\[ x^\dagger S_1^{-1} x = (C_{AA} x_A + C_{AB} x_B)^\dagger C_{AA}^{-1} \]

\[ \times (C_{AA} x_A + C_{AB} x_B) + x_B^\dagger S_1^{-1} x_B, \]

\[ e_1^\dagger S_1^{-1} e_1 = C_{AA}, \]

\[ x^\dagger S_1^{-1} e_1 = (C_{AA} x_A + C_{AB} x_B)^\dagger. \]

Substituting the above equations into (4) yields

\[ \beta = \frac{1}{1 + x_B^\dagger S_1^{-1} x_B}. \]

Again, let us partition the \( B \)-components into two subcomponents, a \( B_0 \)-component containing the first \( r - 1 \) elements and a \( B_1 \)-component containing the rest of the \( B \)-vector

\[ x_B = \begin{bmatrix} x_{B_0} \\ x_{B_1} \end{bmatrix}, \]

and

\[ S_{1,BB} = \begin{bmatrix} S_{1,BB_{B_0}} & S_{1,BB_{B_1}} \\ S_{1,BB_{B_0}} & S_{1,BB_{B_1}} \end{bmatrix}. \]

Now, by using equation A1-9 of [8], it is possible to show that 1 + \( x_B^\dagger S_1^{-1} x_B \) can be rewritten as

\[ \left( x_{B_0} - S_{1,BB_{B_0}}^{-1} S_{1,BB_{B_1}} x_{B_1} \right)^\dagger \]

\[ \times \left( S_{1,BB_{B_0}} - S_{1,BB_{B_1}} S_{1,BB_{B_1}}^{-1} S_{1,BB_{B_1}} \right)^{-1} \]

\[ \times \left( x_{B_0} - S_{1,BB_{B_0}}^{-1} S_{1,BB_{B_1}} x_{B_1} \right) \]

\[ + 1 + x_{B_1}^\dagger S_{1,BB_{B_1}}^{-1} x_{B_1} \]

\[ = (1 + b)(1 + c), \]

where [see the equation shown at the bottom of the page]. Summarizing, we have found that \( t_{\text{WA}} \) can be written as

\[ t_{\text{WA}} = \frac{\hat{t}_K + c}{(1 + b)(1 + c)}. \]

**APPENDIX B**

**STATISTICAL CHARACTERIZATION OF \( b \) AND \( c \)**

**IN CASE OF MISMATCH**

This section is devoted to the statistical characterization of the rv’s \( \hat{t}_K \) and \( c_{QP} \), i.e., the rv’s \( b \) and \( c \) when the actual steering
vector $\mathbf{p}$ and the nominal one $\mathbf{n}_0$ are not aligned. Under this hypothesis, the random vector $\mathbf{x}$ is distributed as [25]

$$\mathbf{x} \sim \mathcal{CN}_N\left(\alpha \sqrt{\mathbf{p}^\dagger \mathbf{R}^{-1} \mathbf{p}} \mathbf{h}_{\mathbf{B}_0}, \sin \theta \mathbf{I}_N\right)$$

where $\mathbf{h}_{\mathbf{B}_0} \in \mathbb{C}^{(r-1) \times 1}$ and $\mathbf{h}_{\mathbf{B}_1} \in \mathbb{C}^{(N-r) \times 1}$ are such that:

$$||\mathbf{h}_{\mathbf{B}_0}||^2 + ||\mathbf{h}_{\mathbf{B}_1}||^2 = 1$$

and

$$e^{j\phi} \cos \theta = \frac{\mathbf{v}_0^\dagger \mathbf{R}^{-1} \mathbf{p}}{\sqrt{\mathbf{v}_0^\dagger \mathbf{R}^{-1} \mathbf{R}^{-1} \mathbf{v}_0}}.$$  (5)

Now, recast $\mathbf{q}_0$ as follows:

$$\mathbf{q}_0 = \mathbf{v}' \mathbf{G}^{-1} \mathbf{v}$$

where

$$\mathbf{v} = \frac{\mathbf{x}_{\mathbf{B}_0} - \mathbf{S}_{1 \mathbf{h}_0} \mathbf{S}_{1 \mathbf{h}_0}^{-1} \mathbf{x}_{\mathbf{B}_1}}{\sqrt{1 + b_0}}$$

and

$$\mathbf{G} = \mathbf{S}_{1 \mathbf{h}_0} - \mathbf{S}_{1 \mathbf{h}_1} \mathbf{S}_{1 \mathbf{h}_1}^{-1} \mathbf{S}_{1 \mathbf{h}_0}.$$  

following the rationale of the proof of Proposition 2 in [22], the following can be readily shown:

- $\mathbf{v}$ and $\mathbf{G}$ are each other independent;
- $\mathbf{G}$ is distributed according to the complex Wishart distribution with parameters $r-1$, $K - (N-1) + (r-1)$, and $\mathbf{I}_{r-1}$ (also conditionally on $\mathbf{S}_{1 \mathbf{h}_1}$ and $\mathbf{x}_{\mathbf{B}_1}$);
- given the $\mathbf{B}_1$-components,

$$\mathbf{v} \sim \mathcal{CN}_{r-1}\left(\alpha \sqrt{\frac{\mathbf{p}^\dagger \mathbf{R}^{-1} \mathbf{p}}{1 + b_0}} \mathbf{h}_{\mathbf{B}_0}, \sin \theta \mathbf{I}_{r-1}\right).$$

Therefore, by Theorem 1 in [22] we have that $c_0 = \frac{\mathbf{v}_0 \mathbf{v}}{\mathbf{v}_0 \mathbf{G}^{-1} \mathbf{v}_0}$, given $b_0$, is ruled by the complex noncentral F-distribution with $r-1$, $K - N + 2$ degrees of freedom and noncentrality parameter given by

$$\delta^2_{c_0} = \frac{||E[\mathbf{x}_{\mathbf{B}_1}]||^2}{\text{SNR} ||\mathbf{h}_{\mathbf{B}_1}||^2 \sin^2 \theta}.$$

where $E[.]$ denotes statistical expectation. Theorem 1 in [22] comes in handy also to find the distribution of the rv $b_0$; more precisely, note that, since $\mathbf{x}_{\mathbf{B}_1} \sim \mathcal{CN}_{N-r}(\alpha \sqrt{\frac{\mathbf{p}^\dagger \mathbf{R}^{-1} \mathbf{p}}{1 + b_0}} \mathbf{h}_{\mathbf{B}_0}, \sin \theta \mathbf{I}_{N-r})$ and $\mathbf{S}_{1 \mathbf{h}_1} \sim \mathcal{CN}_{N-r}(\mathbf{K}, \mathbf{I}_{N-r})$, random variable $b_0$ is ruled by the complex noncentral F-distribution with $N - r$, $K - N + r + 1$ degrees of freedom and noncentrality parameter given by

$$\delta^2_{b_0} = \frac{||E[\mathbf{x}_{\mathbf{B}_1}]||^2}{\text{SNR} ||\mathbf{h}_{\mathbf{B}_1}||^2 \sin^2 \theta}.$$

REFERENCES


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