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Asymptotic stability of the linearised Euler equations with long-memory impedance boundary condition

Florian Monteghetti¹*, Denis Matignon², Estelle Piot¹, Lucas Pascal¹

¹ONERA – The French Aerospace Lab, Toulouse, France
²Department of Applied Mathematics, ISAE-SUPAERO, Toulouse, France
*Email: florian.monteghetti@onera.fr

Abstract

This work focuses on the well-posedness and stability of the linearised Euler equations (1) with impedance boundary condition (2,3). The first part covers the acoustical case (u₀ = 0), where the complexity lies solely in the chosen impedance model. The existence of an asymptotically stable C₀-semigroup of contractions is shown when the passive impedance admits a dissipative realisation; the only source of instability is the time-delay τ. The second part discusses the more challenging aeroacoustical case (u₀ ≠ 0), which is the subject of ongoing research. A discontinuous Galerkin discretisation is used to investigate both cases.

Keywords: impedance boundary condition, diffusive representation, stability, discontinuous Galerkin

Introduction

This work focuses on the (dimensionless) homentropic linearised Euler equations (LEEs)

\[
\begin{align*}
\partial_t p + \nabla \cdot u + u_0 \cdot \nabla p + \gamma p \nabla \cdot u_0 &= 0 \\
\partial_t u + u_0 \cdot \nabla u + [u \cdot \nabla] u_0 + [u \cdot \nabla] u_0 &= 0,
\end{align*}
\]

(1)
defined on \( (0, \infty) \times \Omega \), where \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz open subset, \( p \) (u) is the acoustic pressure (velocity), \( u_0 \in C^\infty(\overline{\Omega})^n \) is the (given) base flow, and \( \gamma > 1 \) is the specific heat ratio. On the boundary \( \Gamma := \partial \Omega \) (with outward normal \( n \)), a so-called acoustical impedance boundary condition is prescribed:

\[
p(t, x) = [z \ast u \cdot n(\cdot, x)](t) \quad (x \in \Gamma := \partial \Omega),
\]

(2)

where the impedance \( z \in \mathcal{D}'_+(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R}) \), causal convolution kernel) models a mono-dimensional medium as a continuous linear time-invariant system.

A recent analysis of acoustical models in the time domain [6] has shown that a wide range of sound absorbing materials and ground layers, assumed locally-reacting, can be modelled by kernels such as \( (\cdot')^{\tau} \) is the weak derivative, \( a_0, a_1 \geq 0 \):

\[
z = a_0 \delta + a_1 \delta' + D_2 + D_3(\cdot - \tau),
\]

(3)

where \( \tau \geq 0 \) and \( D_i \in L^1_{\text{loc}}(0, \infty) \) is a causal oscillatory-diffusive kernel \( (I \subset \mathbb{Z} \text{ countable, poles } \Re[s_{n,i}] < 0, r_{n,i} > 0, \mu_i \text{ positive Borel measure}) \):

\[
D_i(t) = \sum_{n \in I_i} r_{n,i} e^{s_{n,i} t} + \int_0^\infty e^{-\xi t} d\mu_i(\xi),
\]

(4)

which models resonances and visco-thermal losses (e.g. fractional kernel \( D_2 \propto t^{-1/2} \)). A key feature of such positive real kernels is that they can be realised (in the sense of systems theory) by a diagonal, dissipative, infinite-dimensional dynamical system. Note that, if \( \tau > 0 \) in (3), then (2) is a delayed boundary condition, which models wave reflections. The two sources of instability in (1,2) are the base flow \( u_0 \) and the impedance \( z \).

1 Acoustical case

The acoustical assumption \( u_0 = 0 \) removes hydrodynamic instabilities, but leaves room for purely acoustical ones triggered by the impedance boundary condition (2,3). Below, the delayed \( (\tau = 0) \) and undelayed \( (\tau > 0) \) cases are successively investigated by recasting the PDE (1,2) into a Cauchy problem on a Hilbert space \( \mathcal{H} \):

\[
\dot{X}(t) = \mathcal{A} X(t), \quad X(0) = X_0 \in \mathcal{D}(\mathcal{A}).
\]

(5)

To express \( \mathcal{A} \), a time-domain realisation of \( z \) in a state-space \( \Theta \) is needed. The given asymptotic stability results (see Thms. 3 and 5), crucially rely on the dissipativity of this realisation.

1.1 Undelayed impedance \( (\tau = 0) \)

Impedances \( z \) of increasing complexity can be considered, with \( \Theta \) either finite or infinite-dimensional: proportional \( (z = a_0 \delta) \), for which no realisation is required; derivative \( (z = a_1 \delta') \), for which \( \Theta = \mathbb{C} \). For the sake of brevity and clarity, only two simplified examples (compared with (3)) are given below before the statement of the general result.

Example 1. Let \( \check{z}(s) \) be a real rational function, bounded for \( \Re[z] \geq 0 \). If \( \Re[\check{z}(s)] \geq 0 \) (passivity), then it can be realised by a dissipative ODE
in $\Theta = \mathbb{R}^N$, with a suitable energy norm (positive real lemma, see [4, § 3.1]). Eq. (5) is then defined on $\mathcal{H} = L^2(\Omega) \times (L^2(\Omega))^n \times L^2(\Gamma; \Theta)$.

Example 2. Let $z = a_0 \delta + D_2$ (not $D'_2$), and define the weighted spaces $\Phi_{a(\xi)} = L^2(0, \infty; \alpha(\xi) \, d\mu_2)$. The diagonal, dissipative, infinite-dimensional realisation of $D_2$ in $\Theta = \Phi_1$ leads to $\mathcal{H} = L^2(\Omega) \times (L^2(\Omega))^n \times L^2(\Gamma; \Phi_1)$, and (5) then reads:

$$AX = A \left( \begin{array}{c} p \\ u \\ \varphi \end{array} \right) = \left( \begin{array}{c} -\nabla \cdot u \\ -\nabla p \\ -B \varphi + A \varphi + B u \cdot n \end{array} \right)$$

$$V = H^1 \times (H^1(\text{div}) \cap (H^{1/2})^n) \times L^2(\Gamma; \Phi_{1+\xi})$$

$$\mathcal{D}(A) = \left\{ X \in V : \left[ A \varphi + B u \cdot n \right] \in L^2(\Gamma; \Phi_1) \wedge p|_{\Gamma} = a_0 u \cdot n + C \varphi \right\}$$

where, formally, $(A \varphi)(x, \xi) = -\xi \varphi(x, \xi)$ (state operator), $(B u \cdot n)(x, \xi) = 1(\xi) u \cdot n(x)$ (control), and $(C \varphi)(x) = \int_{\Gamma} \varphi(x, \xi) \, d\mu_2(\xi)$ (observation).

Theorem 3. Assume that $\tau = 0$ in (3). If $\Re\{a_0\} > 0$, $a_1 > 0$, $\Re[s_{n,i}] < 0$, $r_{n,i} > 0$ and $\mu_i$ is a positive Borel measure, then $z$ admits a dissipative realisation, and (5) has a unique strong solution $X$, such that $\|X(t)\| \leq \|X_0\|$ for $t \geq 0$ and $\|X(t)\| \rightarrow 0$.

Proof (Sketch). Similar to Thm. 3. The energy norm on the hyperbolic variables, $\|\cdot\|_{L^2(\Gamma; L^2(0, \tau; \mathcal{C}))}$ (here, $\Theta = \mathcal{C}$), is tuned so that $\mathcal{A}$ is dissipative. [5]

2 Aeroacoustical case

The aeroacoustical assumption is $u_0 \neq 0$ in (1). In the case of a subsonic base flow ($\|u_0\| < 1$), and under stringent assumptions on $u$ and $u_0$ (which must be, in particular, potential), the energy functional of Cantrell and Hart [1, Eq. (64)] can be used to construct a contraction $C_0$-semigroup. Without these assumptions, however, there is no energy balance, and the dissipativity of $\mathcal{A}$ is lost: well-posedness can only be achieved in a space like $e^{-\mu t}L^2(\Omega)$”, for some $\mu > \omega_0(A) > 0$, where $\omega_0(A)$ is the growth rate of $\mathcal{A}$. (This constitutes a difficulty of the LEEs, compared to e.g. the Galbrun equation, see [11].) Current research focuses on the identification of instabilities with (2,3), see e.g. [3].

3 Numerical method

Insights into the stability of (5) can be gained by a numerical approximation of the temporal growth rate $\omega_0(A)$. A nodal discontinuous Galerkin method [2] is used to formulate $\dot{X}^h = \mathcal{A}^h X^h + B^h X(-\tau)$, with $X^h = (\rho^h, u^h, \varphi^h)$. The time-domain impedance boundary condition (2,3) is enforced through a centred numerical flux that couples the acoustical unknowns $(\rho^h, u^h)$ with the memory variables $\varphi^h$. If $\tau > 0$, finite-dimensional criteria, which rely on e.g. linear matrix inequalities (LMIs) or spectral conditions, are used to assess stability.

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References