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Riemannian metrics on 2D manifolds related to the Euler-Poinsot rigid body problem.

Bernard Bonnard, Olivier Cots and Nataliya Shcherbakova

Abstract—The Euler-Poinsot rigid body problem is a well known model of left-invariant metrics on $SO(3)$. In the present paper we discuss the properties of two related reduced 2D models: the sub-Riemannian metric of a system of three coupled spins and the Riemannian metric associated to the Euler-Poinsot problem via the Serret-Andoyer reduction. We explicitly construct Jacobi fields and explain the structure of conjugate loci in the Riemannian case and give the first numerical results for the spin dynamics case.

I. INTRODUCTION

The Euler-Poinsot rigid body problem describes the motions of a rigid body fixed in its center of mass. It is one of the most famous problems in Classical Mechanics having many important applications, for instance, in attitude control of satellites in space engineering or in quantum systems.

The motions of the body are solutions to the Hamiltonian system associated to the Hamiltonian

$$H = \frac{1}{2} \left( \frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3} \right),$$

where $I_i$, $i = 1, 2, 3$ are the principal moments of inertia, and $H_i$ are the components of the angular momentum vector written in the moving frame that coincides with the principal axes of inertia of the body ([11]). This vector is related to the angular velocity vector $(u_1, u_2, u_3)$ via the formula $H_i = u_i I_i$.

The system of equations of the Euler-Poinsot motion is Liouville integrable, and every trajectory is determined by the values of two first integrals: $H$ and $G = \sum_{i=1}^{3} H_i^2$. Thus in the space of momenta $H_i$ the motion of the body can be described by the curves of intersections of the energy ellipsoid $H = \text{const}$ with the sphere of angular momentum $G = \text{const}$ called polhodes in Classical Mechanics. Except the so-called separating polhodes, every trajectory evolves on a 2D torus, where the motion is pseudo-periodic with one frequency equal to zero.

From the optimal control point of view, the motions of the rigid body are the extremals of the following optimal control problem:

$$R(t) = R(t) \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} + \frac{1}{2} \sum_{i=1}^{T} \int_{0}^{u(t)} u_i^2 dt \rightarrow \min,$$

where $R(t)$ is a further canonical transformation.

II. THE SERRET-ANDOYER REDUCTION OF THE EULER-POINSO T PROBLEM

A. The Serret-Andoyer canonical variables

The Serret-Andoyer variables $x, y, w, p_x, p_y, p_w$, where $p_x, p_y, p_w$ denote canonical impulses associated to $x, y,$ and $w$, are defined by $H_x = \sqrt{p_x^2 - p_y^2 \sin^2 y}$, $H_y = \sqrt{p_y^2 - p_z^2 \cos^2 y}$ and $H_z = p_w$. So, $|p_x| = \sqrt{G}$ and the Hamiltonian $H$ takes the form

$$H_a = \frac{1}{2} \left( (p_x^2 - p_y^2) (A \sin^2 y + B \cos^2 y) + C p_z^2 \right),$$

where $y = \arctan(\frac{B}{A})$, $A = I_1^{-1}$, $B = I_2^{-1}$ and $C = I_3^{-1}$. A further canonical transformation is needed to get the standard action-angle representation ([7]). Instead, in this paper we...
use the Serret-Andoyer reduction to transform the Euler-Poinsot problem into a 2D Riemannian problem. Denoting $z(y) = 2(A\sin^2 y + B\cos^2 y)$, $H_a$ takes the form
\[
H_a = \frac{1}{2} \left( \frac{z(y)^2}{2} + \frac{C - z(y)^2}{2} \right). \tag{2}
\]
Observe that if the momenta of inertia of the body are ordered as $A < B < C$, then $z(y) \in [2A, 2B]$ and $2C - z(y) > 0$, therefore $H_a$ is positive and it defines the Riemannian metric
\[
g_a = (z(y)^{-1} dx^2 + (2C - z(y))^{-1} dy^2), \tag{3}
\]
which is referred as the Serret-Andoyer metric in the sequel.

Since $H_a$ does not depend on $w$, $p_w = \text{const}$ is a first integral associated to the cyclic variable $w$. The reduced dynamics on the $(x, y)$ plane is described by the Hamiltonian equations
\[
\begin{align*}
\dot{x} &= p_x (A\sin^2 y + B\cos^2 y), & \dot{p}_x &= 0, \\
\dot{y} &= p_y (C - A\sin^2 y - B\cos^2 y), & \dot{p}_y &= (B - A)(p_x^2 - p_y^2)\sin y \cos y,
\end{align*} \tag{4}
\]
where $x$ is another cyclic variable and $p_x$ is a first integral. Note that the dynamics on the plane $(y, p_y)$ is described by a standard pendulum-type phase portrait. Indeed, $H_a$ is $\pi$-periodic with respect to $y$, it verifies the symmetry relations:
\[
H_a(y, p_y) = H_a(y, -p_y), \quad H_a(y, p_y) = H_a(-y, p_y), \tag{5}
\]
and $p_y = 0$, $y = \frac{\pi}{2}$, $k = 0, 1$ are its points of equilibrium. A standard computation shows that the equilibrium $y = 0$, $p_y = 0$ is a saddle, while $y = \frac{\pi}{2}$, $p_y = 0$ is a stable equilibrium of center type. In particular, it follows that in order to parametrize all phase trajectories on the $(y, p_y)$-plane it would be sufficient to consider trajectories starting at $y(0) = \frac{\pi}{2}$. We summarize our analysis in the following:

**Proposition 1.** The pendulum motion in the $(y, p_y)$ plane can be interpreted on the cylinder $y \in [0, \pi] \text{ mod } \pi$ with the stable equilibrium at $y = \frac{\pi}{2}$ and unstable ones at $y = 0, \pi$. There are two types of periodic trajectories on the cylinder: oscillating trajectories homotopic to zero, and rotating trajectories, while non-periodic trajectories are separatrices joining 0 to $\pi$, which correspond to separating polhodes of the full problem. Moreover, all these trajectories share the reflection symmetry with respect to the axes $y = 0$ and $p_y = 0$.

**B. Polar representation of the Serret-Andoyer metric**

In what follows we assume that the principal momenta of inertia verify $A < B < C$. In order to obtain the Darboux normal form $g_a = d\varphi^2 + \mu(\varphi)d\theta^2$ for the metric (3), we must solve the pair of equations
\[
\frac{dy}{\sqrt{C - (A\sin^2 y + B\cos^2 y)}} = d\varphi, \quad d\theta = dx.
\]
The second equation implies that $\theta = x$ modulo a rotation by a fixed angle $\alpha$, so that in the sequel we assume $x \equiv \theta$. As for $\varphi$, according to Section II, we set $y_0 = \frac{\pi}{2}$. Let $\zeta = \sin y$, then $\zeta(0) = 1$ and
\[
\int_{\theta_0}^{\theta} d\varphi = -\int_{\zeta}^{1} \frac{d\zeta}{\sqrt{(C-A)(1-\zeta^2)(k'+k\zeta^2)}},
\]
where
\[
k = \frac{(B-A)}{(C-A)}, \quad k' = 1 - k = \frac{(C-B)}{(C-A)}.
\]
Denoting $\alpha = \sqrt{C-A}$ and choosing $\theta_0 = 0$, we finally obtain a standard elliptic integral
\[
-\alpha \varphi = \int_{\zeta}^{1} \frac{d\zeta}{\sqrt{(1-\zeta^2)(k'+k\zeta^2)}} = \frac{\alpha^2}{2}.
\]

Thus $\sin y = \zeta = \sin(-\alpha \varphi k) = \sin(\alpha \varphi k)$, which implies $z = 2(A\sin^2 y + B\cos^2 y) = 2(A\sin^2(\alpha \varphi k) + B\sin^2(\alpha \varphi k))$. So, we can now formulate the following

**Proposition 2.** The Serret-Andoyer metric $g_a$ can be put into the Darboux normal form $d\varphi^2 + m(\varphi)d\theta^2$ with $m(\varphi) = (A\sin^2(\alpha \varphi k) + B\sin^2(\alpha \varphi k))^2 - 1 \in [l_2, l_1]$, $\alpha = \sqrt{C-A}$ and $k = \frac{(B-A)}{(C-A)}$.

In what follows $'$ denotes the derivative w.r.t. $\varphi$. The standard Gauss curvature formulae for Darboux-type metrics $G(\varphi) = -\frac{1}{(m(\varphi))'}$ implies

**Corollary 1.** The Gauss curvature of $g_a$ is given by
\[
G(\varphi) = \frac{(A + B + C)(z(\varphi) - z_-)(z(\varphi) - z_+)}{z(\varphi)^2},
\]
where $z(\varphi) = 2(A\sin^2(\alpha \varphi k) + B\sin^2(\alpha \varphi k))$ and
\[
z_{\pm} = 2\frac{AB + AC + BC \pm \sqrt{(AB + AC + BC)^2 - 3ABC(A + B + C)}}{A + B + C}.
\]
In particular, $G$ is $\frac{2\kappa(\varphi)}{\alpha}$-periodic and reflectionally symmetric: $G(\varphi) = G(-\varphi)$. In addition, $z_- \in [2A, 2B]$, while $z_+ > 2\alpha$, thus on $[-\kappa(\varphi)/\alpha, \kappa(\varphi)/\alpha]$ the Gauss curvature change it sign twice at $\pm \varphi_1$, $\varphi_1 = \alpha^{-1}\sin^{-1}(\sqrt{\frac{z_- - z_+}{2l_1}})$. It has local maxima at $\varphi = \pm \frac{\kappa(\varphi)}{\alpha}$ and $\varphi = 0$, and a minimum at the point $\varphi$, such that $z(\varphi) = \frac{2\alpha z_-}{z_+ - z_-}$. More precisely,
\[
\max_{[0, \frac{\alpha l_1}{\kappa(\varphi)}]} G(\varphi) = G(0) = \frac{(B-A)(C-A)}{A},
\]
\[
\min_{[0, \frac{\alpha l_1}{\kappa(\varphi)}]} G(\varphi) = -\frac{(A + B + C)(z_+ - z_-)^2}{4z_+ - z_-}.
\]
These results are illustrated in Fig.1.

**C. Jacobi fields for Riemannian metrics on 2D surfaces of revolution**

In this section, using the Hamiltonian formalism, we construct the Jacobi fields for the Darboux-type metrics. As a by-product, we obtain the conjugate points equation.

Taking a local chart in an open domain $U$, the Riemannian metric on a two-dimensional surface of revolution can be written in polar coordinates in the form $g = d\varphi^2 + m(\varphi)d\theta^2$ where $m(\varphi) > 0$. Let $q = (\varphi, \theta) \in U$ denote the state
variables and $p = (p_\theta, p_\theta)$ the corresponding coordinates on fibers. At each point $z = (p, q) \in T^*U$ we denote by $\partial_p$ and $\partial_q$ the vertical and the horizontal parts of the linear space $T_z(T^*U)$. The Hamiltonian associated to $g$ reads

$$H = \frac{1}{2} \left( p_\phi^2 + p_\theta^2 \mu(\phi) \right), \quad \text{where } \mu(\phi) = \frac{1}{m(\phi)},$$

and the geodesics are projections on $U$ of the extremal curves - solutions to the Hamiltonian system

$$\dot{\phi} = p_\phi, \quad \dot{p}_\phi = -\frac{1}{2} e_0^\ast \mu' \phi, \quad \dot{\theta} = p_\theta \mu(\phi), \quad \dot{p}_\theta = 0, \quad (6)$$

where $p_\theta$ is the first integral: $p_\theta = \text{const}$. System (6) admits two particular types of solutions: meridian curves with $p_\theta = 0$ and $\theta(t) = \delta_0$, and parallel solutions characterized by $\phi(0) = \phi_0(0)$ and $\phi(t) = \phi_0$, which, by (6), is equivalent to $\mu'(\phi) = 0$.

In what follows we assume that

(A1) $\phi = 0$ is a parallel solution (i.e. $\mu'(0) = 0$);

(A2) $\mu(\phi) = \mu(-\phi)$, and $\mu''(0) > 0$.

Then on every energy level set $H = h > 0$ in a neighborhood of $\phi = 0$ there is a family of periodic solutions to the equation $\dot{\phi}^2 = 2h - p_\theta^2 \mu(\phi)$ describing the evolution of $\phi(t)$ along geodesics. Note that $\phi(t) \in [-\delta_0, \delta_0]$, where $\delta_0 : p_\theta^2 \mu(\delta_0) = 2h$.

The Hamiltonian function $H$ defines a quadratic form on the cotangent bundle $T^*U$, we denote by $\tilde{H}$ the Hamiltonian vector field and by $e^\tilde{H}$ the Hamiltonian flow generated by $\tilde{H}$.

**Definition 1.** Let $t \to y(t)$, $t \in [0, T]$, be a solution of (6). The variational system associated to (6) along $y(t)$ is called the Jacobi equation. The Jacobi field $J : [0, T] \to T_{y(t)}(T^*U)$ is a non-trivial solution of the Jacobi equation $J(t) = e^{\tilde{H}t}J(0)$.

In practice, the computation of Jacobi fields reduces to the integration of the linearized Hamiltonian system $w(t) = \frac{dH(\tilde{H}t)}{dt}w(t)$, where $w = \delta z$. Setting $w_1 = \delta p_\phi$, $w_2 = \delta p_\theta$, $w_3 = \delta \phi$ and $w_4 = \delta \theta$ from (6) we get

$$\dot{w}_1 = -p_\theta \mu'(\phi)w_2 - \frac{p_\theta^2 \mu''(\phi)}{2}w_3, \quad \dot{w}_2 = 0, \quad \dot{w}_3 = w_1, \quad \dot{w}_4 = \mu(\phi)w_2 + p_\theta \mu'(\phi)w_3.$$

(7)

It follows that $w_2 = \text{const}$, and moreover, (7) reduces to the second order equation

$$\ddot{w}_3 + \frac{p_\theta^2 \mu''(\phi)}{2}w_3 = -p_\theta \mu'(\phi)w_2.$$

(8)

where $\varphi = \phi(t)$ is a known function along $\gamma(t)$, while $p_\theta$ and $w_2$ are constant. Once $w_3(t)$ is found, we immediately get $w_1 = \dot{w}_3$, and $w_4(t) = w_4(0) + \int_0^t (w_2\mu(\phi(\tau)) + p_\theta \mu'(\phi(\tau))w_3(\tau))d\tau$.

**D. Solving equation (8).**

Let us rewrite (8) in the form $\dot{x} + \Phi(t)x = \Psi(t)$, where $x = w_3$. Setting $\Phi(t) = \frac{1}{2}p_\theta^2 \mu'(\phi(t))$, and $\Psi(t) = -p_\theta \mu'(\phi(t))w_2$. This equation can be integrated in a rather standard way if we know a solution to the homogeneous equation $\tilde{w}(t) + \Phi(t)\tilde{w}(t) = 0$. In our case, in view of (6), $\tilde{w}(t) = p_\theta(t)$ satisfies this condition. Then (8) can be solved via the following procedure:

1. **Step 1.** Let $y$ be such that $x = w_3$, and denote $\xi = \dot{y}$. Then

$$\ddot{\xi} + a(t)\xi = b(t), \quad a(t) = \frac{2w(t)}{w(t)}, \quad b(t) = \frac{G(t)}{w(t)}$$

for all $t$ such that $w(t) \neq 0$. Denote $M(t) = \exp\left(\int_0^t a(\tau)d\tau\right)$. Then $(\xi'M(t))' = b(t)M(t)$, which yields $\xi(t) - \xi(0) = M(t)^{-1}\int_0^t b(\tau)M(\tau)d\tau$ provided $M(0) = 1$.

2. **Step 2.** Taking $w(t) = p_\theta(t)$ and using the definitions of $\Phi$ and $\Psi$, we get

$$a(t) = -\frac{2p_\theta^2 \mu'(\phi(t))}{p_\theta^2(\phi)}, \quad M(t) = \frac{p_\theta^2(t)}{p_\theta^2(0)},$$

and hence

$$\xi(t) = \frac{p_\theta^2(0)}{p_\theta^2(t)} \left( \frac{p_\theta w_2}{p_\theta^2(0)}(\mu(\phi(t)) - \mu(\phi_0)) + \xi(0) \right).$$

(9)

The final expression for $y(t)$ can be now computed via the quadrature: $y(t) = y(0) + \int_0^t \xi(\tau)d\tau$.

The whole basis of Jacobi vector fields can be now written by varying the initial conditions $w_3(0), i = 1 \ldots 4$. Observe that the first three Jacobi fields can be easily recovered from the well known properties of the Hamiltonian systems associated to Riemannian metrics. Indeed, (8) admits two obvious solutions with $w_2 = 0$: the Hamiltonian vector fields $J_1 = \tilde{H}$ and $J_2 = p_\theta = \partial_\theta$, which are both invariant with respect to the tangent flow $e_t\tilde{H}$. These two fields emanate from the horizontal part of the basis at $t = 0$.

The remaining two fields can be obtained by taking vertical initial conditions $w_3(0) = w_3(0) = 0, w_2 \neq 0$. Observe that if $p_\theta(0) \neq 0$ and $w_3(0) \neq 0$, then $\xi(0) = 1$. Setting $\xi(0) = 1$ and $w_2 = p_\theta$ is equivalent to take the Euler field $\dot{E} = p_\theta \partial_\phi + p_\theta \partial_\theta$ as the generator of the solution. Since $[\tilde{H}, \tilde{E}] = -\tilde{H}$, and all Lie brackets of higher order are zero, we get $J_3(t) = \tilde{E} + t\tilde{H}$.

The computation of the only non-trivial Jacobi field $J_4(t)$ can be simplified by an appropriate choice of $\xi(0)$. For instance, we can take $y(0) = 0$ and $y'(0) = \xi(0) = -\frac{p_\theta w_2}{p_\theta^2(0)}$. Then, setting $w_2 = 1$, taking into account that $w_3 = \frac{p_\theta \mu'}{\mu}$ and $w_1 = w_3$, after all necessary simplifications, we get

$$w_3(t) = -p_\theta p_\theta \Phi(\phi(t)), \quad w_4 = \frac{p_\theta^2(t)}{p_\theta}(\Phi(\phi(t))),$$
Consider a Darboux type metric $g$ verifying the period. This fact, in view of (10), implies the second statement.

By re-parameterizing $\bar{\theta}$ denotes the conjugate time is a solution to $\Lambda(\bar{\theta}) = 0$, where $\Lambda(\bar{\theta})$ are the extremities of the variation of $\varphi$. Let us recall that $\Lambda(\varphi)$ is $\Lambda(\bar{\theta})$. The function $\Lambda(\bar{\theta})$ is bounded as $t \to \bar{t}$.

We resume our computation in the following

**Proposition 3.** The vector fields $J_1, J_2, J_3(t)$ and $J_4(t)$ form a basis of Jacobi fields along a given solution $(\theta(t), \varphi(t), p_\varphi(t), p_\theta(t))$ of (6).

**Corollary 2.** Consider a Darboux type metric $g$ verifying (A1), (A2). Then the conjugate $(\theta(t), \varphi(t))$ times along any periodic trajectory issued from the point $(\bar{\theta}_0, \varphi_0)$ with $p_\varphi(0) \neq 0$ arc-length parametrized geodesics $(h) = 1/2$.

Then the classical Poincaré result ([9]) on the minimal distance to the conjugate locus implies that along arc-length parametrized geodesics $(h) = 1/2$.

**Theorem 1.** The first conjugate locus to a point $(\varphi_0, \theta_0)$ of the Serret-Andoyer metric $g_a$ is formed by the conjugate points along oscillating trajectories only. It consists of two components, symmetric with respect to the vertical line $\theta = 0$. Each component of the locus is formed by two smooth branches, which asymptotically tend to the horizontal lines $\varphi = \pm \frac{K(k)}{\alpha}$, and form a unique horizontal cusp on the line $\varphi = -\varphi_0$.

Denote now by $(\varphi_0(\cdot; \theta_0), \theta_0(\cdot; \varphi_0))$ a smooth parametrization of the conjugate locus to the point $(\varphi_0, \theta_0)$. Then the classical Poincaré result ([9]) on the minimal distance to the conjugate locus implies that along arc-length parametrized geodesics $(h) = 1/2$.

**III. OPTIMAL CONTROL MODEL OF A SYSTEM OF THREE COUPLED SPIN-1/2 PARTICLES**

**A. Problem statement**

In [11] and [12] the authors proposed the method that reduces the minimal time optimal control problem for a system of three spin-1/2 particles with unequal Ising couplings to the sub-Riemannian problem on $S O(3)$:

$$R(t) = R(t) \left( \begin{array}{ccc} 0 & -u_3 & 0 \\ u_3 & 0 & -u_1 \\ 0 & u_1 & 0 \end{array} \right), \quad u_1^2 + u_3^2 = \min, \quad u_i(0)$$

where $T > 0$ is fixed. It turns out that this problem can be transformed into a certain singular Riemannian problem on the sphere $S^2$. Indeed, denote $r_i = R_{ii}$, then since $||r|| = 1$, we obtain the metric

$$g = \frac{d\theta_1^2 + d\theta_2^2}{r_2^2}, \quad k^2 = \frac{1}{r_3},$$
having singularity at \( r_2 = 0 \). In the spherical coordinates \( r_1 = \sin \varphi \cos \theta \), \( r_2 = \cos \varphi \), \( r_3 = \sin \varphi \sin \theta \), the Hamiltonian of this metric takes the form

\[
H = \frac{1}{4k^2} \left( r_\varphi^2 + r_\theta^2 \cot^2 \varphi + (k^2 - 1)(r_\varphi \cos \theta - r_\theta \cot \varphi \sin \theta)^2 \right).
\]

**Proposition 4.** If \( k = 1 \), then \( H = \frac{1}{2} \left( p_\varphi^2 + p_\theta^2 \cot^2 \varphi \right) \) defines the standard Grushin metric on \( S^2 \).

**Proposition 5.** The family of metrics \( g \) depending upon the parameter \( k \) have a fixed singularity on the equator \( \varphi = \frac{\pi}{2} \).

Proposition 4. The family of metrics \( g \) depending upon the parameter \( k \) have a fixed singularity on the equator \( \varphi = \frac{\pi}{2} \) and a discrete symmetry group defined by

\[
H(\varphi, p_\varphi) = H(\pi - \varphi, -p_\varphi), \quad H(\theta, p_\theta) = H(-\theta, -p_\theta).
\]

Let us now consider the following two sub-manifolds of \( SO(3) \):

\[
M_0 = \{ R \in SO(3) : R^t = (r(0), \ldots, r(0)) = (1, 0, 0) \},
\]

\[
M_1 = \{ R \in SO(3) : R^t = (r(T), \ldots, r(T)) = (0, 0, 1) \}.
\]

Denoting \( M^\perp \) the symplectic lift of a sub-manifold \( M \in SO(3) \), one can prove the following result:

**Proposition 6.** The extremals of the Riemannian metric \( g \) on \( S^2 \) with boundary conditions \( r(0), r(T) \) are extremals of the sub-Riemannian problem on \( SO(3) \) with parameter \( k^2 = I_1/I_3 \), satisfying the boundary conditions \( (R(0), \lambda(0)) \in M_0^\perp \), \( (R(T), \lambda(T)) \in M_1^\perp \), where \( \lambda \) denotes the adjoint vector.

This identification allows to parametrize the extremal on \( SO(3) \) directly. The Hamiltonian takes the form

\[
H_0 = \frac{1}{4} \left( \frac{H_1^2}{I_1} + \frac{H_3^2}{I_3} \right).
\]

Setting \( \cos \vartheta = \frac{H_1}{\sqrt{H_1^2 + H_3^2}} \), \( \sin \vartheta = \frac{H_3}{\sqrt{H_1^2 + H_3^2}} \), we get

\[
\frac{d^2 \vartheta}{dt^2} = \frac{\sin 2\vartheta (k^2 - 1)}{2I_1}, \quad k^2 = \frac{I_1}{I_3},
\]

in which we recognize the pendulum equation. To complete the integration, one can use a suitable parametrization of \( SO(3) \), for instance, Euler’s angles. Another method is to let \( I_2 \to \infty \) in the formulae describing the extremal solutions of the rigid body.

**B. Evolution of the conjugate loci: numerical results**

We now present the results of numerical computations of conjugate loci of the spin problem using the HamPath code [5]. The initial conditions were taken at \( \varphi(0) = \pi/2 \), \( \theta(0) = 0 \), and the evolution of the conjugate lobe is represented via the deformation of the parameter \( k \) starting from \( k = 1 \). There are two different cases to be analyzed: \( k > 1 \) and \( k < 1 \). Since we start from the axis of symmetry, the Hamiltonian \( h = p_\varphi^2/4 \) at \( t = 0 \). Restricting the extremals to \( h = 1 \), we can parameterize geodesics by \( p_\varphi = \pm 2 \), \( p_\theta \in \mathbb{R} \). By symmetry, we can fix \( p_\varphi = -2 \) and consider \( p_\theta \geq 0 \). For any \( k \), the conjugate locus has a contact of order two at the initial point, as \( p_\theta \to \infty \).

1) \( \text{Fig. 4: } k \geq 1 \): Observe that \( \vartheta \) is not monotonous for all the trajectories. This is true even for small values of \( k \), like \( k = 1.01 \), taking \( p_\theta = 0.1 \) and \( T > 14 \). Denote \( t_1^k(p_0, k) \) the first conjugate time and \( q_1^k(p_0, k) = (\theta^*, \varphi^*) \) the corresponding conjugate point. In Fig. 4.a), we represent the map \( k \in [1, 1.5] \rightarrow t_1^k \) for \( p_\theta = 10^{-4} \). Observe that \( \theta^*(t_1^k) \) switches between 0 and \( \pi \) three times at \( 1 < k_1 < k_2 < k_3 < 1.5 \), and \( k_2 - k_1 \neq k_3 - k_2 \). We then restrict our analysis to \( k \leq 3 \).

The subplots b)-d) of Fig. 4 show the deformation of one branch \( (p_\varphi = -2 \) and \( p_\theta \geq 0 \) of the conjugate locus for \( k \) in \( [1, k_1], [k_1, k_2] \) and \( [k_2, k_3] \) respectively. For any \( k \in [1, k_1] \), the branch is located in the half-plane \( \theta \geq 0 \). If we denote by \( k_1 < k < k_2 \), the parameter value such that \( \varphi(t_1^k) = \pi/2 \), then \( \approx 1.155 \) and the branch form a loop for \( k \leq k_3 \). The different character of the deformation of the conjugate locus can be seen also analyzing the behavior of the trajectories. For \( k \leq k_1 \) and \( p_\theta \geq 0 \) we detected four types of trajectories represented in Fig. 5.

2) \( \text{Fig. 6: } k \leq 1 \): This case is easier to interpret. We give on Fig. 6 the conjugate locus for \( k \in \{0.8, 0.5, 0.2, 0.1\} \) with 15 chosen trajectories. The key point is the non-monotony of the \( \theta \)-variable for \( k < 1 \).

We conclude this section by Fig. 7 showing the deformation of the conjugate locus on the sphere\(^1\). Observe that the cusp moves along the meridian with respect to the parameter \( k \). It does not cross the equator for \( k < 1 \) while for \( k > 1 \) it first crosses the North pole \( k = k_1 \), then the equator \( k = \bar{\pi} \). For \( k \geq \bar{\pi} \), the conjugate locus has self-intersections. Then, it crosses poles again for \( k = k_2 \) and \( k_3 \). This is repeated for greater values of \( k \) making the loops smaller and smaller.

**IV. Conclusion**

We presented here two 2D models, intrinsically related to the Euler-Poinsot rigid body motion. The new theoretical result of this paper is the explicit construction of the basis of Jacobi vector fields for Darboux-type metrics on 2D surfaces of revolution. The consequences of this computation go beyond the content of this paper. From one hand, it gives an alternative and simplified proof of the conjugate locus equation that we used to describe the structure of the conjugate locus of the Serret-Andoyer metric. On the other hand, even if this result gives no information on the conjugate

\(^1\)only the half \( p_\varphi(0) = -2 \), \( p_\theta(0) \in \mathbb{R} \) is plotted to clarify the figures
locus of the original 3D problem since the Serret-Adoyer transformation mixes the state and the co-state variables, it can be used to describe the dynamics of the Jacobi fields basis in of the Euler-Poinsot problem in 3D.

The numerical results on spin dynamics, presented in Section III, significantly improve the results of [12]. They also give an idea about the complexity of the conjugate locus of the Euler-Poinsot problem in 3D, since this problem can be seen as a limit case $I_2 \to \infty$.

REFERENCES


