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Flexible Multibody System Linear Modeling for Control Using Component Modes Synthesis and Double-Port Approach

The main objective of this study is to propose a methodology for building a parametric linear model of flexible multibody systems (FMS) for control design. This new method uses a combined finite element (FE)–state-space approach based on component mode synthesis and a double-port approach. The proposed scheme offers the advantage of an automatic assembly of substructures, preserving the elastic dynamic behavior of the whole system. Substructures are connected following the double-port approach for considering the dynamic coupling among them, i.e., dynamic coupling is expressed through the transfer of accelerations and loads at the connection points. The proposed model allows the evaluation of arbitrary boundary conditions among substructures. In addition, parametric variations can be included in the model for integrated control/structure design purposes. The method can be applied to combinations of chainlike or/and starlike flexible systems, and it has been validated through its comparison with the assumed modes method (AMM) in the case of a rotary spacecraft and with a nonlinear model of a two-link flexible arm. [DOI: 10.1115/1.4034149]

1 Introduction

The modeling of flexible structures has been a major concern in control applications for the last 30 years due to the development of larger and lighter structures [1]. Large size and reduced mass imply a reduction of the rigidity of the structure, lowering the associated natural frequencies [2]. Low natural frequencies may interfere with the controller’s bandwidth, affecting the dynamic behavior of the system. Simple and accurate models for large flexible structures which predict these characteristics are indispensable for designing, optimizing, and controlling engineering systems.

The study of large flexible structures implies dealing with large dynamic models because of the amount of degrees-of-freedom (DOF) involved, making their analysis complex and time-consuming. In order to reduce this complexity and facilitate structural understanding, a substructuring technique is used, which consists of the macrodiscretization of the large system into a set of subsystems known as substructures or appendages. Each substructure is separately modeled and, in this way, the large flexible structure is considered to be an FMS. The dynamics of FMS are significantly important in the design, optimization, and control of many systems, such as space vehicles [3,4] or robot manipulators [5,6]. Control modeling techniques must deal with FMS substructures, including the necessary control inputs/outputs in order to recover the dynamic behavior of the whole system for different boundary conditions. Indeed, these boundary conditions, which depend upon the controller, are unknown during the open-loop modeling phase.

Furthermore, modeling techniques must be able to handle the information provided by the FE models of the different substructures, since they are the most widely used tool in structural analysis [7,8]. However, including FE data in a model for control purposes is not straightforward. Several techniques have been studied to take into account the FE data provided for the different substructures of FMS. Among these techniques, two methods have drawn the researchers’ attention: the “FE-transfer matrix” (FE-TM) method and the “component modes synthesis (CMS) Method.” The FE-TM method [9] has been proven to be a powerful tool for the analysis [8,10] and the control of flexible structures [11,12]. The CMS method [13–16] has received significant attention in the aerospace industry since its idea of matrix condensation lends itself particularly well to the concept of substructuring. Among the available dynamic substructuring methods, CMS stands the most systematic and efficient procedure for developing a satisfactory decomposed model [16,17].

Several research articles have studied how to obtain representations of substructures modeled with FE. Young [18] substructured the complete FE model of a two-dimensional truss in order to synthesize a distributed control law. Nevertheless, here the substructure assembly process was mainly based on the overlapping between the inertia and stiffness matrices provided by the substructures’ FE model. This required deep knowledge of the FE theory and impeded straightforward concatenation of complex structures. Later, Su et al. [19] proposed a CMS-based method to decompose a structure into a collection of substructures to synthesize decentralized controllers and called it substructural controller synthesis. However, the overlapping between systems was done at the matrix level, presenting the same practical disadvantages as in Young [18]. Guy et al. [4], based on Alazard et al. [20] and Cumer and Chretien [21], came up with a state-space representation from the substructure’s FE model in order to represent the linear dynamics of a spacecraft with flexible appendages in starlike structure. A particular application of CMS (modal decomposition) was used in order to transfer the influence of flexible appendages to the spacecraft main hub. However, it did not allow for the
representing of flexible substructures in chainlike assembly, and the results were not compared with other methods. Lately, Alazard et al. [22] stated that the double-port approach could be an effective method to link flexible substructures in chainlike assembly. Subsequently, Perez et al. [23] used Craig–Bampton decomposition [24,25] together with the double-port approach in order to model chainlike FE substructures and successfully applied it to a modular spacecraft [26]. In spite of the results, these studies did not take into account the fact that Craig–Bampton decomposition is a particular case of CMS and thus did not take advantage of a more general formulation in CMS form.

This paper proposes a general framework for linear modeling of FMS substructures, with emphasis on chainlike assembly. The main contribution is to establish a general framework, in CMS formulation, for transforming and assembling substructure FE models into proper linear state-space representations which can be used for control purposes, called the two-input two-output port (TITOP) model. In contrast to other studies, in which CMS transformation is only used to overlap the different substructural contributions at a matrix level, the TITOP model uses CMS decomposition and simplifies it to derive the one-connection-point and two-connection-point state-space models of the substructures forming the FMS. Another contribution is the double-port form provided by the TITOP model, which has never been done before for a FE model transformed with CMS. With this, substructure interactions are reduced to a transfer of loads–accelerations through their boundaries. Each substructure is described by a TITOP block diagram allowing arbitrary boundary conditions to be taken into account without any modification of the substructure block-diagram or overlapping manipulations in the mass and stiffness matrices. This makes the TITOP model less sensitive than other classical methods (such the AMM [27]), to changes in boundary conditions. Furthermore, the TITOP model allows interaction with several structural parameters inside the model, which is useful for integrated control/structure design in preliminary study phases. The last contribution is the comparison made between the TITOP modeling technique and linear/nonlinear models based on the AMM.

The paper is structured as follows: In Sec. 2, the highlights of CMS are explained. In Sec. 3, CMS is applied to develop linear models in state-space form in order to model FMSs. One-connection-point and two-connection-points models are developed, as well as the technique of assembly and the possibilities to parameterize the models. In Sec. 4, the modeling technique is applied to a rotary spacecraft, and the results are compared with the AMM to demonstrate the robustness of the TITOP model to changes in boundary conditions. Section 5 shows how to perform the TITOP technique for modeling a two-link flexible arm and compares the obtained dynamics to the nonlinear model proposed in Ref. [28]. Section 6 concludes the paper.

2 Introduction to Component Modes Synthesis

When an FE modeling technique is applied to a given substructure, the equations of motion have the following matrix form in terms of generalized coordinates:

\[
\begin{align*}
\mathbf{M}\{\ddot{u}\} + \mathbf{D}\{\dot{u}\} + \mathbf{K}\{u\} &= \{F\} \\
\end{align*}
\]

It is generally assumed that the existence of damping does not cause coupling of the undamped natural modes of vibration [13]. Therefore, the following undamped equation can be used in order to determine the substructure’s natural modes:

\[
\begin{align*}
\mathbf{M}\{\ddot{u}\} + \mathbf{K}\{u\} &= \{F\} \\
\end{align*}
\]

The aim is to obtain suitable models for control theory application from Eq. (2). This implies a formulation of the equations of motion that establish the correct relation between applied forces and accelerations to the substructure under study in a linearized manner. In this work, the reformulation of the equations of motion is addressed through component-mode transformation [13,14,16]. This method allows to separate substructure displacement sources into three categories: rigid-body displacements, redundant boundary displacements, and natural vibration displacements.

As it can be seen in Fig. 1, the most general type of substructure presents three displacement categories [13,16]: rigid displacements (Fig. 1(a)), redundant boundary displacements (Fig. 1(b)), and natural vibration displacements (Fig. 1(c)). If the substructure is not constrained, six independent rigid-body displacements modes exist, corresponding to three translations and three rotations with respect to a set of fixed orthogonal coordinate axes (the set \(\mathcal{R} = \{\mathbf{r}_i\}\)). The modes produced in this way are called rigid-body modes. Fewer than six rigid modes may exist if the substructure is partially or totally constrained. The constraint system is statically indeterminate with the redundant constraints (denoted by the set \(\mathcal{C} = \{\mathbf{c}_i\}\)). These constraints are the cause of the attachment to other substructures of the system, and they produce the called constraint modes. Finally, the displacements of other points of the structure relative to the constraints are given by a set of

---

**Fig. 1** Substructure displacements decomposition
independent modes in which all the constraints are fixed, called fixed-constraint natural modes of vibration of the structure (set \( I = \{ i \} \)). For example, a flexible segment of a robotic arm has three types of displacements; the movement of its attachment point at its root, the rigid-body modes; the movement of the opposed attachment point, the constraint modes; and the movement of the middle-point of the segment when everything is fixed, the fixed-constraint natural modes.

Therefore, an arbitrary displacement of the constraints can be divided into rigid-body, constrained, and fixed-constraint displacements. Generally speaking, the displacement of any point \( P(x, y, z) \) is given by the superposition of these three displacements

\[
\mathbf{u}(x, y, z) = \mathbf{u}_I(x, y, z) + \mathbf{u}_C(x, y, z) + \mathbf{u}_N(x, y, z)
\]

(3)

When the equations of motion are obtained with FE analysis, the substructure is discretized so that the displacements are defined at only a set of points. In this case, the displacement at each point can be written as a component of a column vector, and Eq. (3) becomes

\[
\{ \mathbf{u} \} = \{ \mathbf{u}_I \} + \{ \mathbf{u}_C \} + \{ \mathbf{u}_N \}
\]

(4)

In component-mode synthesis, each of these displacements is expressed as a superposition of discretized mode functions in the form of modal matrices \([\phi]\) and a set of generalized coordinates \(\eta\). Thus, the term \(\phi_i\eta_i\) represents the displacement at point \(i\) in the \(j\)th mode. Consequently, the three types of displacements take the following form:

\[
\begin{align*}
\{ \mathbf{u}_I \} &= \left[ \phi^R \right] \{ \eta \} \\
\{ \mathbf{u}_C \} &= \left[ \phi^C \right] \{ \eta \} \\
\{ \mathbf{u}_N \} &= \left[ \phi^N \right] \{ \eta \}
\end{align*}
\]

(5)

Substituting Eq. (5) into Eq. (4), the total displacement may be written as

\[
\{ \mathbf{u} \} = \left[ \phi \right] \{ \eta \}
\]

(6)

where the complete transformation matrix reads as follows:

\[
\left[ \phi \right] = \left[ \phi^R \right] \left[ \phi^C \right] \left[ \phi^N \right]
\]

(7)

The computation of submatrices in Eq. (7) is explained in Appendix A. As a consequence of classifying the modes in three categories, namely, rigid-body modes, constraint modes, and normal modes, Eq. (2) can be partitioned as follows:

\[
\begin{bmatrix}
M_{nn} & M_{nc} & M_{nr} \\
M_{cn} & M_{cc} & M_{cr} \\
M_{rn} & M_{rc} & M_{rr}
\end{bmatrix}
\begin{bmatrix}
\eta_n \\
\eta_c \\
\eta_r
\end{bmatrix}
+ \begin{bmatrix}
K_{nn} & K_{nc} & K_{nr} \\
K_{cn} & K_{cc} & K_{cr} \\
K_{rn} & K_{rc} & K_{rr}
\end{bmatrix}
\begin{bmatrix}
\eta_n \\
\eta_c \\
\eta_r
\end{bmatrix}
= \begin{bmatrix}
F_n \\
F_c + \tilde{F}_c \\
F_r + \tilde{F}_r
\end{bmatrix}
\]

(8)

The “tilde” load term denotes the force resulting from the connection to adjacent structures at the boundary points [16]. Applying the modal transformation given in Eq. (A1) in Appendix A and premultiplying by \([\phi^T]\) and considering that neither interior forces nor external forces are applied \(F_n = F_c = F_r = 0\) Eq. (8) yield

\[
\begin{bmatrix}
\tilde{M}_{nn} & \tilde{M}_{nc} & \tilde{M}_{nr} \\
\tilde{M}_{cn} & \tilde{M}_{cc} & \tilde{M}_{cr} \\
\tilde{M}_{rn} & \tilde{M}_{rc} & \tilde{M}_{rr}
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_n \\
\tilde{\eta}_c \\
\tilde{\eta}_r
\end{bmatrix}
= \begin{bmatrix}
\tilde{K}_{nn} & \tilde{K}_{nc} & \tilde{K}_{nr} \\
\tilde{K}_{cn} & \tilde{K}_{cc} & \tilde{K}_{cr} \\
\tilde{K}_{rn} & \tilde{K}_{rc} & \tilde{K}_{rr}
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_n \\
\tilde{\eta}_c \\
\tilde{\eta}_r
\end{bmatrix}
\begin{bmatrix}
0 \\
\tilde{F}_c \\
\tilde{F}_r + \phi_0 \tilde{F}_c
\end{bmatrix}
\]

(9)

Equation (9) is the partitioned transformed form of the equations of motion. It should be noted that for the transformed stiffness matrix, several submatrices are null matrices. The submatrix \([\tilde{K}_{nn}]\) is null since the work done by a self-equilibrating force system on a rigid-body displacement is zero [13]. The same occurs to the submatrix \([\tilde{K}_{cr}]\) since the work done by the constraint forces on a normal mode displacement is zero because in normal mode, the constraints are fixed. In the same way, submatrix \([\tilde{K}_{cr}]\) is a null matrix.

In consequence of the foregoing results, the partitioned transformed equation of motion takes a simpler form

\[
\begin{bmatrix}
\tilde{M}_{nn} & \tilde{M}_{nc} & \tilde{M}_{nr} \\
\tilde{M}_{cn} & \tilde{M}_{cc} & \tilde{M}_{cr} \\
\tilde{M}_{rn} & \tilde{M}_{rc} & \tilde{M}_{rr}
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_n \\
\tilde{\eta}_c \\
\tilde{\eta}_r
\end{bmatrix}
+ \begin{bmatrix}
\tilde{K}_{nn} & 0 & 0 \\
0 & \tilde{K}_{cc} & 0 \\
0 & 0 & \tilde{K}_{rr}
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_n \\
\tilde{\eta}_c \\
\tilde{\eta}_r
\end{bmatrix}
= \begin{bmatrix}
0 \\
\tilde{F}_c \\
\tilde{F}_r + \phi_0 \tilde{F}_c
\end{bmatrix}
\]

(10)

Equation (10) presents then submatrices which are more attractive for modeling purposes. Physical interpretations can be derived from several submatrices. The square submatrix \([\tilde{K}_{nn}]\) is a diagonal matrix containing the fixed-constraint natural vibration modes, and related with \([M_{nn}]\) by the relationship of Eq. (A2) in Appendix A. The square submatrix \([\tilde{K}_{cr}]\) is the stiffness matrix associated with the redundant constraints, and its order is equal to the number of redundant constraints. The square matrix \([M_{nn}]\) is the rigid-body matrix, i.e., the mass matrix if the substructure is considered as rigid. It contains the whole mass of the system, gravity center position, and rotatory inertia with respect to the rigid-body boundaries. The submatrices \([M_{nn}]\) and \([M_{cr}]\) are the modal participation matrices of the natural modes and constraint boundaries on the rigid-body motion, i.e., how the natural modes and constraint boundaries affect the rigid dynamics.

If damping is taken into account, the damping matrix \([D]\) may be partitioned in the same way as the mass and stiffness matrices

\[
[D] = \begin{bmatrix}
\tilde{D}_{nn} & \tilde{D}_{nc} & \tilde{D}_{nr} \\
\tilde{D}_{cn} & \tilde{D}_{cc} & \tilde{D}_{cr} \\
\tilde{D}_{rn} & \tilde{D}_{rc} & \tilde{D}_{rr}
\end{bmatrix}
\]

(11)

In general, all of the submatrices are not null as in the case of the mass matrix. However, if all the damping forces are internal, then rigid-body motions are not damped and in this case, the third row and the third column of Eq. (11) are null matrices [13]. In this case, Eq. (10) is written with viscous damping as

\[
\begin{bmatrix}
\tilde{M}_{nn} & \tilde{M}_{nc} & \tilde{M}_{nr} \\
\tilde{M}_{cn} & \tilde{M}_{cc} & \tilde{M}_{cr} \\
\tilde{M}_{rn} & \tilde{M}_{rc} & \tilde{M}_{rr}
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_n \\
\tilde{\eta}_c \\
\tilde{\eta}_r
\end{bmatrix}
+ \begin{bmatrix}
\tilde{K}_{nn} & 0 & 0 \\
0 & \tilde{K}_{cc} & 0 \\
0 & 0 & \tilde{K}_{rr}
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_n \\
\tilde{\eta}_c \\
\tilde{\eta}_r
\end{bmatrix}
= \begin{bmatrix}
0 \\
\tilde{F}_c \\
\tilde{F}_r + \phi_0 \tilde{F}_c
\end{bmatrix}
\]

(12)

3 Substructure’s Two-Input Two-Output Port Model Derivation

The properties of the partitioned equations of motion in Eq. (9), obtained in Sec. 2, can be used for simple, accurate, and intuitive modeling of FMS. The advantages of this transformation are maximized when they are applied to one connection point or two connection points. More connection points are possible to model as well, but in the control domain this is not really advantageous. Beyond that, connection complexity obliges to manipulate
the FE model itself, and the problem becomes rather a structural problem than a control modeling problem.

Therefore, two uses of CMS equations are explained in this section. First, the case of one connection point is explained. Second, the case of two connection points is addressed. Next, the modeling of a revolute joint is described. To conclude the section, the assembly technique with both models is explained, and some guidelines for parameterization with TTTPP and superelement techniques are described.

3.1 One Connection Point. In this section, the modeling of a flexible substructure connected to another structure through one connection point is explained. As shown in Fig. 2, a flexible body (substructure) \( A \) is linked to the parent structure \( P \) at the point \( P \). It is assumed that the only external loads applied to \( A \) are the interactions with \( P \) at point \( P \).

The problem is thus how to consider the coupling between \( P \) and \( A \). Several authors opted for overlapping stiffness and mass matrices at a matrix level [18,19] or transferring boundary conditions with the TM method [8,29]. However, other approaches [4,20,22,23] took advantage of other particular transformations (modal decomposition and Craig-Bampton decomposition) and expressed the coupling as a transfer between loads and accelerations through the connection points, i.e., the overlapping between substructures is expressed as an acceleration-load transfer through the common boundaries. In this study, a generalization is presented for the general formulation in CMS.

Therefore, the coupling transfer between \( P \) and \( A \) is expressed as an acceleration-load transfer through the connection point \( P \). Equation (12) offers the advantage of casting the FE model of substructure \( A \) in the state-space representation using accelerations and loads as inputs and outputs through the boundaries. This is possible, thanks to the decoupling of the stiffness matrix when CMS transformation is performed. In the case of one connection point, there are no redundant constraint displacements besides the rigid-body displacements. This implies that the rigid-body displacements (translations and rotations) are directly associated with point \( P \), which constrains the substructure \( A \) to be always fixed to \( P \), sharing the rigid-body motions of the ensemble. As there are no redundant constraint displacements, second row and second column of Eq. (12) can be removed leading to

\[
\begin{bmatrix}
M_{rr} & M_{rs} \\
M_{sr} & M_{ss}
\end{bmatrix}
\begin{bmatrix}
\dot{\hat{\eta}}_r \\
\dot{\hat{\eta}}_s
\end{bmatrix}
+
\begin{bmatrix}
D_{rr} & 0 \\
D_{rs} & 0
\end{bmatrix}
\begin{bmatrix}
\hat{\eta}_r \\
\hat{\eta}_s
\end{bmatrix}
+
\begin{bmatrix}
K_{rr} & 0 \\
K_{rs} & 0
\end{bmatrix}
\begin{bmatrix}
\eta_r \\
\eta_s
\end{bmatrix}
=
\begin{bmatrix}
0 \\
F_r
\end{bmatrix}
\tag{13}
\]

The coupling is established as an exchange acceleration–load through the connection point

\[
\{ F_{A/P,P} \} = \begin{bmatrix} G_{AP}^s \end{bmatrix} \{ \ddot{u}_P \}
\tag{14}
\]

where \( F_{A/P,P} \) is the load transmitted to the structure \( P \) by the appendage \( A \). \( G_{AP}^s(s) \) is the linear model of the appendage \( A \) when connected at point \( P \), and \( \{ \ddot{u}_P \} \) the acceleration of the displacements at point \( P \). In the three-dimensional case, where 6DOF are needed to describe rigid-body motion, \( G_{AP}^s(s) \) is a \( 6 \times 6 \) TM (i.e., \( r = 6 \)). It is trivial that the loads experienced by \( A \) due to adjacent connections, \( \tilde{F}_r \), are in the opposite direction of the loads experienced by \( P \); \( F_{A/P,P} \).

To conclude the section, the case of two connection points is addressed. Next, the modeling of a revolute joint is described. To conclude the section, the assembly technique with both models is explained, and some guidelines for parameterization with TTTPP and superelement techniques are described.

In the case of one connection point, normalized rigid-body accelerations are equal to the acceleration at point \( P \). The matrix \( L_P \) is the modal participation matrix of natural modes at point \( P \), i.e., it expresses how the motion of \( P \) is affected by the natural modes of vibration and vice versa. The square matrix \( J_{AP}^s \) is the direct dynamic model, at point \( P \), of the substructure \( A \) that is assumed rigid [22] and takes the following form for \( r = 6 \):

\[
J_{AP}^s = J_{AP}^s + \begin{bmatrix} I_3 & (sAP) \\
0 & I_3
\end{bmatrix} \tag{16}
\]

where \( sAP \) is the kinematic model between the mass center of substructure \( A \), point \( A \), and the connection point \( P \), written as

\[
t_{AP} = \begin{bmatrix} I_3 & (sAP) \\
0 & I_3
\end{bmatrix}
\tag{17}
\]

with \((sAP)\) being the skew-symmetric matrix associated to the vector \( A \). Considering that the natural vibration modes are normalized with respect to the mass matrix, the submatrix \( [M_{ss}] \) becomes the identity matrix. \([K_{ss}]\) is a diagonal matrix containing the natural modes (fixed-constraint natural modes, \( \omega_n^2 \)), and \([D_{ss}]\) is a diagonal matrix expressed with a damping ratio \( \xi_n \). Consequently, the linear model of the appendage \( G_{AP}^s(s) \) reads

\[
\begin{bmatrix}
\dot{\eta}_n \\
\dot{\eta}_s
\end{bmatrix}
=
\begin{bmatrix}
0_n & I_n \\
-\omega_n^2 I_n & -2\omega_n \xi_n I_n
\end{bmatrix}
\begin{bmatrix}
\eta_n \\
\eta_s
\end{bmatrix}
+
\begin{bmatrix}
0 \\
-L_P
\end{bmatrix}
\{ \ddot{u}_P \}
\tag{18}
\]

\[
\begin{bmatrix}
F_{A/P,P} = L_P^s \begin{bmatrix} -\omega_n^2 I_n & -2\omega_n \xi_n I_n \\
0 & 0
\end{bmatrix} \{ \ddot{u}_P \}
\end{bmatrix}
\]

The physics lying within Eq. (18) can be interpreted from the control domain point of view. The rigid-body displacements of the appendage \( A \) are transmitted by its connection point \( P \) through the whole of the appendage, and this excites the fixed-boundary natural modes (the modes obtained when clamping the appendage at point \( P \)) through the modal participation matrix \( L_P \). This natural modes produce a load transmitted to substructure \( P \) modifying the load that appendage \( A \) will induce to \( P \), which is the residual mass of the appendage \( J_{AP}^s \) times the acceleration at point \( P \). This can be seen schematically as the rigid-body displacement of appendage \( A \) perturbed with a feedback of its own natural vibration modes as shown in Fig. 3.

The model in Eq. (18) is commonly used in space engineering to connect a flexible appendage to a rigid body considered as the main hub [20]. Nevertheless, the model does not take into account what happens if substructure \( A \) is connected to another substructure at the opposite end, since there is no information about its displacement. In Sec. 3.2, an extension of this approach is proposed for the case of two connection points, which is sufficient for modeling chainlike substructures.

![Fig. 2 Substructure A linked to structure P](image-url)
3.2 Two Connection Points. In this section, the modeling of a substructure connected to two different structures through two connection points, one for each structure, is explained. As shown in Fig. 4, the flexible body (substructure) \( A \) is linked to the parent structure \( P \) at the point \( P \) and to a child substructure \( Q \) at the point \( Q \). It is assumed that the only external loads applied to \( A \) are the interactions with \( P \) at point \( P \) and with \( Q \) at point \( Q \).

As seen in Sec. 3.1, the main problem is how to consider the coupling between substructures \( P \), \( A \), and \( Q \). Again, the overlapping between substructures is expressed as an acceleration–load transfer through the common boundaries. A generalization of the double-port approach, proposed by Alazard et al. [22] and used by Perez et al. [23], is presented in this study for the general CMS formulation. In this case, both points, \( P \) and \( Q \), suffer an acceleration–load transfer, in such a way that the acceleration is transferred to the next substructure in the chain (\( Q \) in this case), and the load is transmitted to the previous substructure in the chain (the parent \( P \) structure). Therefore, the objective is to build a double-port model of the substructure \( A \) such that

\[
\begin{bmatrix}
\ddot{u}_Q \\
F_{A/P,P}
\end{bmatrix} = \left[ G_{F,Q}(s) \right] \begin{bmatrix}
F_{Q/A,Q} \\
\ddot{u}_P 
\end{bmatrix}
\tag{19}
\]

As there are only two connection points, the assignment of DOF is simple: rigid-body displacements to connection point \( P \) and the redundant constraint displacements to connection point \( Q \). Thus, the accelerations read

\[
\{\ddot{u}_P\} = \{\ddot{\eta}_P\}; \quad \{\ddot{u}_Q\} = \{\ddot{\eta}_Q\} + \{\phi_{cr}\} \{\dot{\eta}_Q\}
\tag{20}
\]

where \( \{\phi_{cr}\} \) is described in Appendix A, and it has the same form as the kinematic model between connection point \( P \) and connection point \( Q \). Equation (20) implies that the rigid motion is supported by point \( P \), and the constrained motion of connection point \( Q \) is a result of the rigid-body motion in \( P \) transported to point \( Q \) (\( \{\phi_{cr}\} \{\dot{\eta}_Q\} \) plus the constrained motion due to flexibility \( \{\ddot{\eta}_Q\} \)). In the same way, loads are received and transmitted by substructure \( A \) with the following directions:

\[
F_{A/P,P} = -\ddot{F}_P \\
F_{Q/A,Q} = \ddot{F}_Q
\tag{21}
\]

Using the relations given in Eqs. (20) and (21) in combination with Eq. (12), a state-space representation can be obtained for the substructure \( A \)

\[
G_{F,Q}(s)
\left\{ \begin{array}{c}
\ddot{u}_Q \\
F_{A/P,P}
\end{array} \right\} = \left[ G_{F,Q}(s) \right] \left\{ \begin{array}{c}
F_{Q/A,Q} \\
\ddot{u}_P
\end{array} \right\}
\tag{22}
\]

where \( A, B, C, D, \) and \( D_b \) are the short hand notation of the following state-space matrices:

\[
A = \begin{bmatrix}
0 & I_{n_{cr}} \\
-M_{Q}^{-1}M_{Q} & -M_{Q}^{-1}D_{Q}
\end{bmatrix}
\tag{23}
\]

\[
B = \begin{bmatrix}
0_{n_{cr},v} & 0_{n_{cr},w} & 0_{n_{cr},x} \\
M_{Q}^{-1} & -M_{Q} & -M_{Q} \\
0 & I_{cc} & -M_{cc}
\end{bmatrix}
\tag{24}
\]

\[
C = \begin{bmatrix}
0_{n_v} & I_{v} & -M_{Q}^{-1}M_{Q} & -M_{Q}^{-1}D_{Q} \\
M_{Q} & \dot{M}_{Q} & -M_{Q}^{-1}D_{Q} & -M_{Q}^{-1}D_{Q}
\end{bmatrix}
\tag{25}
\]

\[
D = \begin{bmatrix}
0_{n_v} & I_{v} & -M_{Q}^{-1}M_{Q} & -M_{Q}^{-1}D_{Q} \\
-M_{Q} & \dot{M}_{Q} & -M_{Q}^{-1}D_{Q} & -M_{Q}^{-1}D_{Q}
\end{bmatrix}
\tag{26}
\]

with

\[
M_{Q} = \begin{bmatrix}
\dot{M}_{Q} & \dot{M}_{Q} & \dot{M}_{Q} \\
M_{Q} & M_{Q} & \dot{M}_{Q} \\
M_{Q} & \dot{M}_{Q} & \dot{M}_{Q}
\end{bmatrix}; \quad K_{Q} = \begin{bmatrix}
\dot{K}_{Q} & 0 \\
0 & \dot{K}_{Q}
\end{bmatrix}
\tag{27}
\]

Equation (22) with Eqs. (23)–(27) form the double-port model, \( G_{F,Q}(s) \), of the flexible substructure \( A \) in chainlike assembly, called TITOP model. This model allows to interconnect different flexible substructures in chainlike assembly taking into account...
flexible motions. A simplified scheme of the TITOP model is shown in Fig. 5. In the 6DOF case, \( G^A_{P,Q}(s) \) of the flexible sub-structure, \( A \) is a 12 × 12 TM (that is, \( r = 6 \) and \( c = 6 \)).

The physical interpretation of Eq. (22) is similar to the one connection point case. In this case, rigid-body displacements of the appendage \( A \) are transmitted to the connection point \( P \) through the whole of the appendage, and this excites the fixed-boundary natural modes (the modes obtained when clamping the appendage at point \( P \) and \( Q \)) through the modal participation matrices, \( [M_x]_i \) and \( [M_y]_i \), and thus the constraint point \( Q \). These natural modes produce a load transmitted to substructure \( P \) modifying the load that appendage \( A \) will induce to \( P \), which depends on the load received at point \( Q \). For the two-port model, \( \{ \ddot{u}_P \} \), and the natural modes. It can be observed that the rigid-body substructure of flexible \( A \), \( [M_x]_i \), influences the transfer as well.

Another advantage of this kind of approach is its versatility. By setting inputs to 0, \( G^{A_{P,Q}}(s) \) represents the free (at \( P \))-clamped (at \( Q \)) model of \( A \). In the same way, \( G^{A_{P,Q}}(s) \) represents the free (at \( P \))-clamped (at \( Q \)) mode of \( A \). Both “channels” are invertible, and thus, the models

\[
\begin{align*}
[G^A_{P,Q}(s)]^{-1} \begin{bmatrix}
F_{Q,A} \\
F_{A,P} \\
\end{bmatrix}_A & = [G^A_{P,Q}(s)]^{-1} \begin{bmatrix}
\ddot{u}_Q \\
\ddot{u}_P \\
\end{bmatrix}_A \quad (28) \\
[G^A_{P,Q}(s)]^{-1} \begin{bmatrix}
\ddot{u}_Q \\
\ddot{u}_P \\
\end{bmatrix}_P & = [G^A_{P,Q}(s)]^{-1} \begin{bmatrix}
F_{Q,A} \\
F_{A,P} \\
\end{bmatrix}_P \quad (29)
\end{align*}
\]

can be used to take into account boundary conditions at \( P \) or \( Q \). Indexes “a” and “i” are used to describe the “upper” channel and “lower” channel, respectively. It should be noted that removing connection point \( Q \), the same model as for the one connection point case is found. Thus, Eqs. (28) and (29) correspond to the clamped-clamped and free-free models, respectively.

### 3.3 Revolute Joint

The double-port approach allows taking into account constraints at the level of the connection points by simply restricting or releasing DOF. This study shows that a revolute joint at the connection point \( P \) between the bodies \( A \) and \( P \), as depicted in Fig. 6, can be modeled as well for the two connection points.

Augmenting the double-port model \( [G^A_{P,Q}(s)]_R \) of the body \( A \) projected in the frame \( R \) : let \( c = \{ x_a \ y_a \ z_a \} \) be a unit vector along the revolute joint axis, then

\[
\begin{bmatrix}
\ddot{u}_Q \\
F_{A,P} \\
\end{bmatrix}_{R} \quad t_{r,j,p} \quad \begin{bmatrix}
\ddot{u}_P \\
F_{Q,A} \\
\end{bmatrix}_{R} = \begin{bmatrix}
\ddot{u}_Q \\
F_{A,P} \\
\end{bmatrix}_{R} \quad t_{r,j,p}
\]

(30)

with the selection matrix

\[
E_a = -[0_{1 	imes 9} \quad x_a \quad y_a \quad z_a] \quad (31)
\]

where \( [H^A_{P,Q}(s)]_R \) is the double-port model augmented with a 13th input: \( \ddot{\alpha} \), the angular acceleration inside the revolute joint.

\[
\begin{align*}
\ddot{u}_P & = \begin{bmatrix}
\ddot{u}_P \\
\ddot{\alpha} \\
\end{bmatrix} \\
\ddot{u}_Q & = \begin{bmatrix}
\ddot{u}_Q \\
\ddot{\alpha} \\
\end{bmatrix} \\
\end{align*}
\]

Figs. 6 and 7 show the two-channel scheme of the TITOP model.

### 3.4 Modeling of FMSs

The state-space realizations found for FE models transformed with CMS decomposition and double-port approach serve as elemental bricks for building FMS with small deflections. Indeed, the one-connection-point TITOP model can be used to model flexible systems in starlike structures or to end chainlike structures. The two-connection-point TITOP model can be used to connect every type of chainlike structures between them, taking into account its flexibility.

For instance, the FMS shown in Fig. 8 can be modeled as different TITOP models interacting among them as depicted in Fig. 9. The flexible multibody spacecraft is composed of a rigid main body or hub in which other appendages are attached, such as an antenna, masts, and solar panels. For control purposes, it is useful to choose as inputs the loads applied to the hub, \( F_G \), and as outputs the induced accelerations at the hub, \( \ddot{u}_G \). These accelerations are transported to the connection point \( P \) (\( P_T \) for the antenna connection point and \( P_z \) for one of the masts) through the kinematic model \( T_{P,G} \) [20], where they are transmitted to the TITOP models of the flexible appendages. Eventually, rotation matrices can be included in the diagram in order to change from the hub’s frame.
to the appendage’s frame. These models transmit what can be called “disturbance” loads at the level of the hub, thus taken into account their flexibility. A more illustrative example of FMS modeling is proposed in Sec. 4.

Therefore, the TITOP model allows synthetic, simple, and intuitive modeling schemes for control purposes. Given its simplicity and the easy access to some measurements and inputs, such as external forces and accelerations in different parts of the FMS, this kind of modeling approach has been used by authors, such as Alazard et al. [30,31] for integrated control/structure design.

3.5 Parameterization. This section underlines another attribute of the TITOP model. The model can be parameterized as a function of structural configuration parameters. For a structure with varying configuration or varying mass and stiffness properties, like most space structures, the TITOP modeling technique may be especially efficient since it can take into account such variations (see Ref. [4] as an illustrative example). It can also be used for structural/control integrated design allowing structural sizing parameters to be simultaneously optimized with the attitude control gains [32].

Physical parameters are accessible in the TITOP model through the rigid-body matrix, denoted as $[M_{rr}]$ or $[J_A]_P$ in Eq. (16). Indeed, total system mass or geometric parameters can be parameterized by accessing to this matrix. Natural modes can be parameterized by accessing to matrix $[K_{nn}]$ or $[I_A]_P$ in the one connection point case. Matrix $[\phi_{ij}]$ reflects the geometric properties of the appendage, since it transports the kinematics from point $P$ to $Q$. Its modifications can also be taken into account within the model, affecting its dynamics. As an illustrative example of modeling of varying parameters, consider a beamlike substructure that is modeled as a TITOP state-space representation. If length $L$ has to be varied to analyze its influence in the FMS, its variation $d$ can be modeled in the skew symmetric matrix $\tilde{\tau}_{PQ}$ associated with the vector $\bar{PQ}$ (see Fig. 4) in the following way:

$$\tilde{\tau}_{PQ}(d) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -(L + d) \end{bmatrix},$$

where $F_l(N, \Delta)$ is the lower linear fractional transformation of $N$ by $\Delta$, and $\Delta = \delta L$ [33]. Such a linear fractional representation (LFR) of $(\tilde{\tau}_{PQ}(d))$ can be propagated in $[\phi_{ij}(\Delta)] = \tau_{PQ}(\Delta)$ and

![Fig. 8 Example of FMS](image)

![Fig. 9 FMS modeling with the TITOP model (mast II is not represented for simplicity)](image)
in the whole state-space model using standard functions of robust control theory [34]. The final LFR model contains the nominal model, \( G_{FQ}(s) \), and the \( \Delta \) block, as depicted in Fig. 10. Such a formalism of parametric variations is commonly used in control system theory for sensitivity analysis [33,35]. It can be noticed that not all the parameters are easily accessible in the TITOP formulation, as, for example, the cross-sectional area or the Young’s modulus. For elementary substructures, like a mast or a boom, Murali et al. [32] proposed the analytic TITOP formalism of parametric variations inside the block \( \Delta \).

4 Robust Modeling and Parameterization

To demonstrate the less sensitivity of the TITOP modeling method to changes in boundary conditions and parameterization possibilities, a maneuvering flexible spacecraft is considered. The results are compared with another flexible spacecraft model based on AMM developed in Junkins [27].

4.1 System Description. The system is composed of a rigid main hub with four identical cantilevered flexible appendages and tip masses as shown in Fig. 11. The configuration parameters are provided in Table 3. Under normal operation, the spacecraft undergoes planar rotational maneuvers about the inertially fixed axis \( z \). The spacecraft body frame is attached to the mass center of the rigid hub, and it is denoted by a right-handed triad \( x, y, \) and \( z \). The rotation about the axis \( z \) is denoted by the angle \( \theta \) and the translational deformation of each tip by \( w_{tip} \), with superscript \( i \) denoting the appendage number.

The system is actuated by three different torques. The main torque, \( t_{hub} \), is provided by the main hub about the axis \( z \). Two additional input torques, \( t_{tip,1} \) and \( t_{tip,2} \), are applied at the tip masses 1–3 and 2–4, respectively. These torques can be applied purposely for control reasons or can be the result of environment disturbances.

The purpose is to model this particular FMS using the TITOP method and compare it with the AMM approach and superelement method.

Fig. 11 Maneuverable flexible spacecraft configuration

\[
J_{\text{Hub}} = \begin{bmatrix} m_h & 0 & 0 \\ 0 & m_h & 0 \\ 0 & 0 & J_b \end{bmatrix} \quad J_{\text{Tip}} = \begin{bmatrix} m_t & 0 & 0 \\ 0 & m_t & 0 \\ 0 & 0 & J_t \end{bmatrix}
\] (34)

The kinematic models [20] between points \( G \) and \( P_i \), being \( i \) the appendage number \( i \), are in the planar case

\[
\tau_{P,G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \end{bmatrix} \tau_{P,G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{bmatrix}
\] (35)

and the rotation matrices can be written as follows:

\[
R_i = \begin{bmatrix} \cos \beta_i & -\sin \beta_i & 0 \\ \sin \beta_i & \cos \beta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\text{App} \rightarrow \text{Hub}}
\] (36)

where \( \beta_i \) is the angle of the ith appendage \( i \) with \( x \). Beam’s FE model is obtained with classical FE discretization in five elements, and mass and stiffness matrices are transformed as explained in Sec. 3.2 to get the TITOP model of the beam. As the tip mass is considered as rigid, there is no need of applying CMS to this substructure, Eq. (34) is used for such a purpose.

The assembly for each appendage is the one shown in Fig. 12. Accelerations at the hub are transmitted to the attachment point \( P_i \), through the kinematic model \( \tau_{P,G} \), and then changed to the appendage frame through \( R_i^T \). The acceleration of the hub, together with the load exerted by the tip mass at the opposite end, is the inputs of the beam TITOP model, which delivers the acceleration transmitted to the tip mass and the load transmitted to the hub, which has to be transported to the hub and change its frame.

TITOP being a generic 6DOF approach, it is restricted to the three planar DOF. Thus, the acceleration and loads vectors used in Figs. 12 and 13 have three components corresponding to the two translations in the plane \( \pi(x, y) \) and one rotation around \( z \).

The same process is performed to the four appendages, obtaining the final assembly shown in Fig. 13. It can be observed that...
where the resulting system, \((J_{st}^G)^{-1}\), has the applied torques as inputs when the following inputs are assigned the following values: \(\{F_{ext} = \{0 \ 0 \ \hat{\theta}_{hub}\}\}\), and the hub accelerations as outputs (\(\hat{\theta}_{G} = \theta\)). Tip acceleration can be observed through the signal transmitted from the beam to the tip and tip torques \(F_{tip,1}\) and \(F_{tip,1}\), can be added to \(F_{tip,beam,G}\) loads with the help of a sum block.

4.2.2 AMM Approach. For the comparison objective, the classical assumed modes (AM) solution is exploited. Although AMM can be applied in many different ways, the most general case is deriving the hub-beam-tip equations.

The AMM assumes a decoupled spatial and time deformation approximated by the series

\[
\mathbf{w}(x,t) = \sum_{i=1}^{asm} \phi_i(x) u_i(t) \tag{37}
\]

where \(\phi_i(x)\) denotes the assumed mode shape, \(u_i(t)\) denotes the \(i\)th generalized coordinate, \(asm\) denotes the number of terms retained in the approximation, and \(x\) the distance from the considered point in the beam to the attachment point.

Then, the kinetic and potential energy of the spacecraft, containing space and time partial derivatives of \(w(x,t)\), are derived using the approximation in Eq. (37) and performing the integration with respect to \(x\), writing the kinetic energy and potential energy in the quadratic forms

\[
T(t) = \frac{1}{2} \sum_{i=1}^{asm} \sum_{j=1}^{asm} M_{ij} \dot{u}_i(t) \dot{u}_j(t) = \frac{1}{2} \{\dot{u}(t)\}^T \{M\} \{\dot{u}(t)\} \tag{38}
\]

\[
V(t) = \frac{1}{2} \sum_{i=1}^{asm} \sum_{j=1}^{asm} K_{ij} u_i(t) u_j(t) = \frac{1}{2} \{u(t)\}^T \{K\} \{u(t)\} \tag{39}
\]

where \(M_{ij}\) denotes the \((i,j)\)th element of the symmetric mass matrix \([M]\) (respectively, for the stiffness matrix \([K]\)). The equations of motion follow on introducing \(T\) and \(V\) into Lagrange’s equations:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{u}_r} \right) - \frac{\partial T}{\partial u_r} + \frac{\partial V}{\partial u_r} = Q_r, \quad r = 1, \ldots, asm \tag{40}
\]

where \(Q_r\) denotes the generalized nonconservative forces, the applied torques. The following equations of motion are obtained:

\[
\sum_{j=1}^{asm} M_{ij} \ddot{u}_j(t) + \sum_{j=1}^{asm} K_{ij} u_j(t) = Q_r, \quad r = 1, \ldots, asm \tag{41}
\]

which written in matrix compact form gives

\[
[M]\{\ddot{u}(t)\} + [K]\{u(t)\} = \{Q(t)\} \tag{42}
\]

The analytic formulation of mass and stiffness submatrices in Eq. (42) for the rotatory spacecraft can be found in Junkins [27], which have been developed with the following admissible functions satisfying the boundary conditions for clamped–free appendages:

\[
\phi_i(x) = 1 - \cos \left( \frac{j \pi x}{L} \right) + \frac{1}{2} \left( \frac{1}{2} \right)^{j/2} \left( \frac{j \pi x}{L} \right)^2 \tag{43}
\]

The admissible functions in Eq. (43) are known to produce very accurate results and have been adopted widely [36]. Equation (42) provides thus the desired equations of motion in which the time-varying amplitudes are generalized coordinates. Given the instantaneous vector \(\{u(t)\}\), the instantaneous deformation of the structure is approximated by the assumed modes expansion.

4.3 Comparison of the Modeling Methods. A comparison between the TITOP modeling technique and the numerical AMM is presented. The TITOP model uses FE models of five elements for each beam, and the AMM uses 13 modes per beam. Both methods are compared with the reference FE model (FEM) of the whole structure which can be found in Ref. [27], considered to be the most accurate.

4.3.1 Accuracy. First, the accuracy of the proposed TITOP modeling technique is verified. A comparison of all the methods (AMM, TITOP, and FEM) among the first six flexible modes is shown in Table 1. The computed frequencies converge accurately for the 8DOF per appendage TITOP solution, whereas the AMM solutions are not accurate for modes 4–6. The relative mean square error of these values is shown in Fig. 14, showing that for the same number of DOF the TITOP modeling technique is slightly more accurate. Therefore, the TITOP modeling is able to provide accurate models which have less DOF and achieves more accurate results than the AMM.

4.3.2 Robustness to Variations in Boundary Conditions. Figure 15 presents the effect of flexible appendages on the main hub motion, \(\theta_{hub}\). Figure 16 shows the dynamic response of the tip accelerations to the hub torque, \(\theta_{hub}\). Both figures are in perfect agreement with the frequency response of the reference model, FEM, until the fourth flexible mode, located at around 50 Hz. At that point, the response \(\theta_{hub} \rightarrow w_{tip}\) of the AMM presents a significant error in the first antiresonance frequency.

<table>
<thead>
<tr>
<th>No.</th>
<th>AMM</th>
<th>TITOP</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.3722</td>
<td>4.3722</td>
<td>4.3722</td>
</tr>
<tr>
<td>2</td>
<td>7.9066</td>
<td>7.9066</td>
<td>7.9066</td>
</tr>
<tr>
<td>3</td>
<td>51.7234</td>
<td>51.4518</td>
<td>51.4259</td>
</tr>
<tr>
<td>4</td>
<td>53.0829</td>
<td>53.0829</td>
<td>53.0829</td>
</tr>
<tr>
<td>5</td>
<td>160.2661</td>
<td>157.5382</td>
<td>157.5382</td>
</tr>
<tr>
<td>6</td>
<td>161.0962</td>
<td>158.3683</td>
<td>158.3683</td>
</tr>
</tbody>
</table>

\(N\) denotes the number of DOF per appendage. The reference frequency \((\omega_{ref})\) is obtained with an FE model of 100DOF per appendage.
As it can be evaluated, the AMM is no longer accurate for frequencies beyond the third flexible mode for this spacecraft configuration. The TITOP method, however, is accurate for the considered frequency range. This difference is due to the AMM assumption of clamped–free mode shapes, whereas real mode shapes of a hub-beam-mass system are different from a clamped–free system, as theoretically demonstrated in Elgohary et al. [36]. The TITOP modeling technique does not make any approximation of mode shapes, they naturally arise when the whole system is assembled. This demonstrates that the TITOP method is less sensitive to the imposed boundary conditions, while AMM remains more sensitive due to the choice of the mode shapes. As a result, the TITOP method is valid for every type of configuration, whereas AMM is limited by the selected mode shapes.

Such sensitivity to boundary conditions is highlighted by studying the influence of a mass at the tip of each beam. Figure 17 shows the frequency response $t_{hub} \rightarrow w_{tip}$ when there is no mass at the tip of the beam, $M_t = 0$ kg, whereas Fig. 18 shows the same frequency response with a heavy mass at the tip, $M_t = 114.5$ kg. When there is no tip mass, the AMM is perfectly valid and coincides with the reference model, since its admissible functions fully respect the boundary conditions clamped–free. However, when a heavy mass is located at the tip, the clamped–free boundary conditions no longer apply, which makes AMM fail in predicting the frequency response. On the other hand, the TITOP method perfectly fits the frequency response of the reference model in all the cases.

### 4.4 System Parameterization

The TITOP modeling technique allows taking into account the variations of certain structural parameters inside the model, since they can be easily found inside the state-space representation of the substructures. In this section, parametric variations are performed to the rotatory spacecraft, including variations on beams’ lengths and tip masses.

Considering the appendage as a beam, its length variations are introduced through the superelement model explained in Murali et al. [32]. Tip mass variations are introduced through the rigid-body matrix of the tip mass, Eq. (34), as follows:

$$J^{tip}_0(\delta_m) = \begin{bmatrix} m_t + \delta_m & 0 & 0 \\ 0 & m_t + \delta_m & 0 \\ 0 & 0 & J_t(\delta_m) \end{bmatrix}$$  

(44)

After assembly, the system appears like a model as the one shown in Fig. 19, with variations in tip mass $\Delta_m$ and beam’s length $\Delta_l$, included in the $\Delta$-block.
Using this approach, dynamic behavior sensitivity analysis can be performed. As it can be seen in Fig. 20, the first natural frequency of the system decreases by either increasing length or increasing tip mass in all the appendages in the same manner.

Nevertheless, the most interesting remark can be done when only one appendage varies its beam's length and tip's mass. As it is shown in Fig. 21, a new frequency mode appears, not present in the case where all the appendages varied their length and tip mass simultaneously. Indeed, if all the appendages are identical, the symmetric modes of the appendages are uncontrollable from the hub's torque and they do not appear in the frequency response. In the case of only one appendage variation, however, the asymmetry due to the variation of one single appendage makes these modes now controllable and can be found in the frequency response.

As it has been demonstrated, parameterization can be easily taken into account with the TITOP method. For this problem, AMM approach of Junkins and Kim [27] considers several simplifications for the model, such as symmetric displacements between appendages. If such kind of variations was done with the AMM approach, the whole model would have been changed, reinitializing the modeling process. On the contrary, this step is avoided with the TITOP modeling technique. The TITOP model does not require reformulating the problem since all the variations are considered from the beginning of the modeling process and individually for each appendage. The TITOP model represents simultaneously the spacecraft nominal configuration with all the possible parameter variations, whereas the AMM approach needs to compute a new model for each parameter variation.

5 Comparison With Nonlinear Modeling

In this section, the TITOP modeling technique is compared to a nonlinear modeling technique for the case of a planar two-link flexible arm, a flexible multichain example where the kinematic nonlinearities can be large. The objective is to evaluate the...
The accuracy of the TITOP linear model for a control application and to determine if the nonlinear terms could restrict its usage. In addition, the modeling process is explained for taking into account the revolute joint’s actuator with mass and inertia at the connection points.

The nonlinear model of the planar two-link flexible arm can be found in its explicit closed-form in Luca and Siciliano [28]. It consists of two flexible arms with a payload at the end of the second arm. This system is illustrated in Fig. 22, and its corresponding parameters are described in Table 4 of Appendix C.

5.1 TITOP Modeling of the Planar Two-Link Flexible Arm. The TITOP modeling of a n-link flexible arm composed of a chain of n link flexible segments and joints starts with the individual assembly of each flexible segment to the joint. The rigid-body matrix of the joint can be derived straightforward for the planar case as stated in Sec. 4.2.1

\[ J_{\text{joint}} = \begin{bmatrix} m_h & 0 & 0 \\ 0 & m_h & 0 \\ 0 & 0 & J_h \end{bmatrix} \] (45)

The inverse dynamics TITOP model of the flexible link i is obtained as depicted in Fig. 23. The connections follow the same principles as explained in Sec. 3.3, where the angular acceleration induced by the revolute joint to the system is added to the angular acceleration of the hub (the joint’s rigid-body matrix). The total acceleration (the one received by the joint and the one imposed by the joint’s rotation) is transmitted to the channel which corresponds to the acceleration at point P of the ith flexible arm’s TITOP model. The load transmitted by the ith flexible arm to the joint at point P is subtracted from the inertial load obtained at the joint to get the resulting load to the ith flexible arm, \( F_{i-1, P_i} \). It is assumed that the flexible arm is perfectly connected to the center of the joint, i.e., no kinematic transport matrix is needed since \( G_i \equiv P_i \). Finally, the applied joint’s torque, \( t_{j, P_i} \), is obtained as the third component of the transmitted loads for the planar case. The second channel, the one which exchanges acceleration–load at the other end of the segment (point \( Q_i \)) remains unchanged for future connection to the next flexible link.

**Fig. 21**  Bode system comparison when varying length and tip mass for one appendage only

**Fig. 22**  The planar two-link flexible arm

**Fig. 23**  TITOP assembly of a single flexible link i
5.2 Modeling and Simulation Results. In order to test the system’s TITOP model, the planar two-link flexible arm with the physical parameters described in Table 4 is compared with the nonlinear closed-form model found in Luca and Siciliano [28]. The TITOP model uses a two-element FE model for each flexible segment. The FE model takes into account translations along the $x$ and $y$ axis of the segment, and rotations around the $z$ axis. The model of Ref. [28] only considers translations along the $y$ axis and the rotation around $z$, and it uses two assumed modes for each segment.

First, the natural frequencies of the system are compared for different nominal configurations. The comparison is shown in Table 2. It can be noticed that the error is not larger than 1.0% of the nonlinear value for the first frequency mode, 0.01% for the second frequency mode, and 0.2% for the third frequency mode. The fourth frequency shows a discrepancy of 2.8%. However, nothing can be concluded regarding the accuracy of the frequency modes since the nonlinear model uses four assumed modes computed for the nominal configuration $\omega_2(0) = 0$ deg. This could be an error source for the highest frequency modes. On the other hand, the TITOP model considers translations along the $x$ axis, which might explain the differences, and it is more robust to changes in the nominal configuration (since it is equivalent to changes in the boundary conditions).

Once the models have been proved to have the same dynamics, a numerical simulation is performed to validate the direct dynamics TITOP model with the nonlinear direct dynamics model in controlled evolution. The joints’ rotations of the flexible arm are controlled through proportional–derivative controllers, taking the angular position and rate as inputs as shown in Eq. (48) and Fig. 7:

\[
\begin{align*}
t_{11,P_1} &= k_{p1}(\omega_{ref1} - \omega_1) - k_{d1}\dot{\omega}_1 \\
t_{22,P_2} &= k_{p2}(\omega_{ref2} - \omega_2) - k_{d2}\dot{\omega}_2
\end{align*}
\] (48)

A numerical simulation has been performed for a $\omega_{ref1} = 60$ deg step command given to the first joint when the arm is fully extended ($\omega_2(0) = 0$ deg). The controller’s gains are $k_{p1} = 160$ N·m, $k_{d1} = 11$ N·m/s, $k_{p2} = 60$ N·m, and $k_{d2} = 1.1$ N·m/s. The nonlinear equations of motion have been integrated via a fourth-order adaptive Runge–Kutta (Dormand–Prince) method. Figure 25 shows the dynamic response of the joints for the first 4 s. The nonlinear model and the TITOP model are in perfect agreement, even when the nonlinearities are expected to be large ($z \gg 1$ rad/s). In the subplot corresponding to $\omega_2$, an additional frequency can be observed at the peaks over the first cycle, which it is not present in the nonlinear model.

Therefore, the linear model provided by the TITOP modeling technique can be used as an approximation even when the nonlinear terms can be large, as in the case of a two-link flexible arm. Furthermore, for the same level of modeling complexity, the TITOP model is able to provide additional frequency modes which have a more significant impact than the nonlinear terms in the system’s response.

Table 2 Table showing the natural frequencies ($\omega$, rad/s) corresponding to the first four flexible modes for each modeling method of the two-link flexible arm

<table>
<thead>
<tr>
<th>Mode</th>
<th>Nonlinear</th>
<th>TITOP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega_2(0) = 0$ deg</td>
<td>$\omega_2(0) = 30$ deg</td>
</tr>
<tr>
<td>No.</td>
<td>$\omega_2(0)$</td>
<td>$\omega_2(0)$</td>
</tr>
<tr>
<td>1</td>
<td>8.8335</td>
<td>8.8348</td>
</tr>
<tr>
<td>3</td>
<td>101.2585</td>
<td>100.9695</td>
</tr>
<tr>
<td>4</td>
<td>144.3649</td>
<td>144.3386</td>
</tr>
</tbody>
</table>

Three initial configurations are considered: $\omega_2(0) = 0$ deg, $\omega_2(0) = 30$ deg, and $\omega_2(0) = 90$ deg.
6 Conclusions

This study proposes a new FMS linear modeling approach for control purposes, called the TITOP model. The TITOP method is based on the formulation of FE models in component-mode synthesis. Connections among elastic substructures are established using the double-port approach, which uses the exchange of acceleration–load at the connection points to express the dynamic overlapping among the components.

Chainlike and/or star-structures can be managed by this technique using an intuitive assembly process. This eases the access to certain measurements needed for control purposes, such as accelerations or applied loads at the connection points. The TITOP model takes into account arbitrary boundary conditions without the need of recomputing the model. This makes the TITOP model less sensitive than other approaches (AMM) to changes in boundary conditions. In addition, several parametric variations, such as changes in mass or geometry, can be taken into account for an effective integrated control/structure design in preliminary design phases. Furthermore, the TITOP model can be used as an accurate modeling technique for systems where kinematics nonlinearities can be large.

The modeling and design variations of a rotatory flexible spacecraft and a two-link flexible arm demonstrate the feasibility, accuracy, and effectiveness of the approach compared with other accepted methods. Dynamic responses, in the frequency domain, are in complete agreement with AMM and with nonlinear models. When boundary conditions change, the TITOP model is more accurate than the AMM formulation since it is less sensitive to changes in boundary conditions. For the two-link flexible arm, it appears to be more determinant to add more frequency modes with the TITOP technique rather than considering nonlinear terms. Continued research in extending the proposed method will be focusing on examining alternative structure/control configurations, performing integrated structure/control design with structured $H_{\infty}$ techniques and developing control approaches for complex space structures.

Acknowledgment

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Nomenclature

Generalized displacements in equations and figures are expressed as follows:
\[ \{ u \} = \text{column matrix of generalized displacements} \]
\[ \{ \dot{u} \} = \text{column matrix of generalized velocities} \]
\[ \{ \ddot{u} \} = \text{column matrix of generalized accelerations} \]

The vector $\{ q \}$ of generalized displacements is often decomposed as follows (identically for generalized velocities and generalized accelerations):
\[ \{ q \} = \text{generalized translations} \]
\[ \{ \theta \} = \text{generalized rotations} \]

Generalized coordinates can be projected in the following directions:
\[ \text{x} = \text{unit vector along x axis} \]
\[ \text{y} = \text{unit vector along y axis} \]
\[ \text{z} = \text{unit vector along z axis} \]

The equations of motion and figures may include one of the following notations:
\[ [D] = \text{square matrix of system generalized damping coefficients} \]
\[ \{ F \} = \text{column matrix of generalized loads} \]
\[ \vec{F}_c = \text{vector of externally applied forces at the constraint degrees-of-freedom} \]
\[ \vec{F}_c = \text{vector of loads acting on a substructure as a result of its connection to adjacent substructure at the constraint degrees-of-freedom} \]
\[ \vec{F}_t = \text{vector of externally applied forces at the rigid-body degrees-of-freedom} \]
\[ \vec{F}_t = \text{vector of loads acting on a substructure rigid-body degrees-of-freedom as a result of its connection to adjacent substructure} \]
Appendix A: Component Modes Obtention

In component-mode synthesis, the substructure’s physical displacements can be expressed in terms of substructure generalized coordinates \( \eta \) by the Rayleigh–Ritz coordinate transformation

\[
\{ u \} = \phi \{ \eta \} \tag{A1}
\]

where the component-mode matrix \( \phi \) is a matrix of preselected component modes including: fixed-constraint modes, constraint modes, and rigid-body modes. Then, the matrix \( \phi \) is obtained as follows:

- Using a set of \( N_n \) substructure fixed-constraint normal modes, \( \phi^N \), obtained from the solution of the eigenproblem:

\[
[K_m - \omega^2 M_m] \{ \phi_m \} = \{ 0 \}, \quad j = 1, 2, \ldots, N_n \tag{A2}
\]

\[
\phi^N_{N \times N_n} = \begin{bmatrix}
\phi_{1n} & \cdots & \phi_{N_n}
\end{bmatrix}
\tag{A3}
\]

- Using a set of redundant constraint modes, defined relative to the redundant boundary coordinate set:

\[
\phi^C_{N \times N_r} = \begin{bmatrix}
\phi_{1r} & \cdots & \phi_{N_r}
\end{bmatrix}
\tag{A4}
\]

- Using a set of rigid-body modes, obtained by solving the equation resulting from restraining the rigid-body motion of the substructure:

\[
\phi^R_{N \times N_r} = \begin{bmatrix}
\phi_{1r} & \cdots & \phi_{N_r}
\end{bmatrix}
\tag{A5}
\]

The set of \( N_r \) fixed-interface normal modes can be reduced to a smaller set of kept normal modes, denoted as \( \phi_i \). The combined set \( \{ \phi_i \} \) spans the static response of the substructure to interface loading and allows for arbitrary interface displacements \( \mathbf{u}_i \). These interface displacements can be accompanied by the displacements of the interior of the substructure as shown in Fig. 1.

Appendix B: Maneuverable Spacecraft Parameters

The values for the maneuverable spacecraft modelization in Sec. 4 are given in Table 3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hub radius</td>
<td>( r )</td>
<td>0.305 m</td>
</tr>
<tr>
<td>Hub mass</td>
<td>( m_s )</td>
<td>233.502 kg</td>
</tr>
<tr>
<td>Hub rotatory inertia</td>
<td>( J_b )</td>
<td>10.847 kgm^2</td>
</tr>
<tr>
<td>Mass density of beams</td>
<td>( \rho )</td>
<td>1.302 kg/m</td>
</tr>
<tr>
<td>Elastic modulus of beams</td>
<td>( E )</td>
<td>75,842 GPa</td>
</tr>
<tr>
<td>Beam length</td>
<td>( L )</td>
<td>1.219 m</td>
</tr>
<tr>
<td>Beam thickness</td>
<td>( t )</td>
<td>3.175 mm</td>
</tr>
<tr>
<td>Beam height</td>
<td>( h )</td>
<td>0.152 m</td>
</tr>
<tr>
<td>Tip mass</td>
<td>( m_t )</td>
<td>2.290 kg</td>
</tr>
<tr>
<td>Tip mass rotatory inertia</td>
<td>( J_t )</td>
<td>2.440 kgm^2</td>
</tr>
<tr>
<td>Nodes for beam FE model</td>
<td>nom</td>
<td>11</td>
</tr>
<tr>
<td>Number of AM</td>
<td>asm</td>
<td>13</td>
</tr>
</tbody>
</table>

Appendix C: Two-Link Flexible Arm Parameters

The values for the two-link flexible arm modelization of Sec. 5 are given in Table 4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link 1 inertia</td>
<td>( J_{a1} )</td>
<td>0.1 kg/m^2</td>
</tr>
<tr>
<td>Link 2 inertia</td>
<td>( J_{a2} )</td>
<td>0.1 kg/m^2</td>
</tr>
<tr>
<td>Link 2 mass</td>
<td>( m_{a2} )</td>
<td>1.0 kg</td>
</tr>
<tr>
<td>Linear density of beam 1</td>
<td>( \rho_1 )</td>
<td>0.2 kg/m</td>
</tr>
<tr>
<td>Linear density of beam 2</td>
<td>( \rho_2 )</td>
<td>0.2 kg/m</td>
</tr>
<tr>
<td>Cross section properties beam 1</td>
<td>( E_{11} )</td>
<td>1 Nm^2</td>
</tr>
<tr>
<td>Cross section properties beam 2</td>
<td>( E_{12} )</td>
<td>1 Nm^2</td>
</tr>
<tr>
<td>Beam 1 length</td>
<td>( l_1 )</td>
<td>0.5 m</td>
</tr>
<tr>
<td>Beam 2 length</td>
<td>( l_2 )</td>
<td>0.5 m</td>
</tr>
<tr>
<td>Payload mass</td>
<td>( m_p )</td>
<td>0.1 kg</td>
</tr>
<tr>
<td>Payload inertia</td>
<td>( J_p )</td>
<td>0.5 g/m^2</td>
</tr>
</tbody>
</table>

References


