To link to this article: DOI:10.1121/1.424376
URL: http://dx.doi.org/10.1121/1.424376

To cite this version: Doutaut, Vincent and Matignon, Denis and Chaigne, Antoine
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of America (JASA), vol. 104 (n° 3). pp. 1633-1648. ISSN 0001-4966
Numerical simulations of xylophones. II. Time-domain modeling of the resonator and of the radiated sound pressure

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(Received 26 November 1997; revised 22 May 1998; accepted 2 June 1998)

This paper presents a time-domain modeling for the sound pressure radiated by a xylophone and, more generally, by mallet percussion instruments such as the marimba and vibraphone, using finite difference methods. The time-domain model used for the one-dimensional (1-D) flexural vibrations of a nonuniform bar has been described in a previous paper by Chaigne and Doutaut [J. Acoust. Soc. Am. 101, 539–557 (1997)] and is now extended to the modeling of the sound-pressure field radiated by the bar coupled with a 1-D tubular resonator. The bar is viewed as a linear array of equivalent oscillating spheres. A fraction of the bar field excites the tubular resonator which, in turn, radiates sound with a certain delay. In the present model, the open end of the resonator is represented by an equivalent pulsating sphere. The total sound field is obtained by summing the respective contributions of the bar and tube. Particular care is given for defining a valid approximation of the radiation impedance, both in continuous and discrete time domain, on the basis of Kreiss’s theory. The model is successful in reproducing the main features of real instruments: sharp attack, tuning of the bar, directivity, tone color, and aftersound due to the bar-resonator coupling. © 1998 Acoustical Society of America. [S0001-4966(98)03409-2]

PACS numbers: 43.75.Kk [WJS]

INTRODUCTION

A discrete time-domain formulation of the system of equations that govern the transverse bending motion of a xylophone bar excited by the blow of a mallet has been obtained in the past with the use of finite differences. This model yields both the time history and spatial distribution of the bar velocity, among other results.1

This paper is now dealing with the time-domain calculation of the sound-pressure field radiated by mallet percussion instruments, in direct continuity with the previous work. The contribution of the bar to the pressure field is computed from its velocity, by summing together, at each time step, the pressure radiated by each discrete segment of the bar. Following Junger, it is assumed that the bar can be viewed as a linear array of dipoles.2 This radiation model accounts for the use of the instrument without resonators. However, in most musical situations, a tubular resonator is placed under the bar. In the time domain, under the assumption of free space, the physical phenomena can be then summarized as follows: In the half-space above the bar, the radiated sound pressure propagates without any modification. At the same time, a pressure wavefront propagates below the bar and reaches the open end of the tube shortly after the blow of the mallet. Thus a fraction of the bar acoustic field is transformed into stationary waves inside the tube. A part of the internal energy of the tube is then reemitted to the free space through its open end, due to the radiation impedance. The total sound field is the sum of both contributions of the bar and tube.

In this paper, it will be assumed that the lower end of the tube is closed and perfectly rigid. As a consequence, the spectrum of the sound radiated by the tube is made of frequencies nearly equal to the odd harmonics of the fundamental frequency, the wavelength corresponding to this fundamental being closely equal to four times the length \( l_t \) of the tube. It is also assumed that the open end of the tubular resonator behaves like a monopole, which is in accordance with observations made by other authors, and modeled here as an equivalent pulsating sphere.3 The continuous model of the instrument is presented in Sec. I.

In Sec. II, the numerical formulation of the problem, based on finite differences, is presented. Emphasis is put on the resonator with special considerations on the time-domain modeling of the radiation impedance. Based on Kreiss’s theory,4 it is shown that the approximation used for this impedance must fulfill specific criteria in order to ensure the stability of the resonator model. Detailed mathematical derivations of these criteria can be found in Appendixes A and B.

The results of the model are presented in Sec. III. First, the numerical scheme of the tube is validated by a comparison between analytical and numerical solutions. In a second step, it is shown to what extent the model is efficient in reproducing the main musical qualities of real instruments. Comparisons between measured and simulated xylophone sounds contribute to illustrate the capabilities and the limits of the method.

I. PHYSICAL MODELING

A. Radiation of the bar

The transverse motion of a free-free bar with a variable section is described by the classical one-dimensional Euler–
Bernoulli partial differential equation. This equation has been slightly modified in order to account for the losses and for the restoring force of the suspending cord. The action of the mallet against the bar is described by Hertz’s law. The model also includes a differential equation that governs the motion of the mallet. This vibratory model has been extensively described in a previous paper and will not be discussed further here.\(^1\) The system of equations is solved in the time domain by means of finite-difference methods. The results can be alternatively expressed in terms of displacement, velocity, or acceleration of the bar.

In this section, the model used for calculating the sound pressure resulting from the bar displacement is presented. The basic assumption is that the reaction of the radiated field on the vibrating bar is negligible, even in the case where a tubular resonator is situated close to it. This assumption is justified by a series of experiments which have been conducted in order to investigate the influence of the resonator on the vibration of the bars. Figure 1 shows, for example, a comparison between the acceleration spectrum at a given point of a F\(^4\) bar without a resonator [Fig. 1(a)], and with a resonator placed under the bar [Fig. 1(b)]. No significant differences can be detectable between these two spectra. As a consequence, the influence of the pressure radiated by the tube on the motion of the bar will be neglected.

According to Junger, the far field radiated by the flexural vibrations of an unconfined beam can be approximated, in the low-frequency range, by modeling the bar as a distribution of dipole sources. This assumption is valid if the thickness of the bar \(h(x)\) is small compared to the acoustic wavelength.\(^2\) In mallet percussion instruments the bar thickness is generally less than 2 cm, which corresponds to acoustic wavelengths for frequencies above 17 kHz. The Junger approximation is thus justified.

In order to apply this result to the sound radiated by a xylophone bar, the acoustic source is viewed as a linear array of equivalent oscillating spheres, where the volume \(\Delta V(x)\) of each sphere is equal to the volume of one spatial element of the beam of length \(\Delta x\). This approach is similar to the one adopted by Akay et al.\(^3\) for beams and can also be related to the work by Ochmann where vibrating structures are represented by equivalent distributions of multipoles.\(^4,5\) The following mathematical derivations are restricted to the plane of symmetry \(\text{xOz}\) of the system composed by the xylophone bar and the resonator and the radiation problem is expressed in polar coordinates (see Fig. 2).

Each element \(\Delta x\) of the bar is viewed as an oscillating sphere of equivalent volume:

\[
\Delta V(x) = \frac{1}{2} \pi a^3(x) = bh(x)\Delta x, \tag{1}
\]

where \(a(x)\) is the radius of the sphere. This approximation is valid in the low-frequency range, which means that we must fulfill the condition

\[
\forall x \in [0;L], \quad ka(x) \ll 1, \tag{2}
\]

where \(L\) is the length of the bar and \(k\) is the acoustic wave number.

Under the additional geometrical assumption of far field \([a(x) \ll r]\), the contribution of each oscillating sphere at po-
sition $x_i=i\Delta x$ on the bar to the sound pressure at a point of observation $M(r,\theta)$ in free space, in the time domain, is given by

$$\Delta p_{\theta}(r,\theta,t)=\frac{3}{8\pi}\rho_0\Delta V(x_i)\cos(\theta x_i)\left(\frac{1}{r_{x_i}}\frac{\partial^2w}{\partial t^2}(x_i,t)\right),$$

$$-r_{x_i}/c_0 + \frac{1}{r_{x_i}c_0}\frac{\partial^3w}{\partial t^2}(x_i,t-r_{x_i}/c_0),$$

which $w(x_i)$ represents the transverse displacement of the bar, $\rho_0$ is the density of air, and $c_0$ is the speed of sound in air. By applying the principle of superposition, the sound pressure field radiated by the discrete bar is written

$$p_g(r,\theta,t)=\sum_{i=1}^{N} \Delta p_{\theta}(r,\theta,t),$$

where $N=L/\Delta x$ is the number of discrete elements of the bar. If the resonator is removed, Eq. (4) is sufficient for calculating the sound radiated by the instrument. However, it is necessary to take the contribution of the resonator into account in order to obtain a more general model. Therefore in the next paragraph, the resonator is assumed to be excited by the bar pressure field $p_g(d,\pi,t)$. At this point it must be said that the geometrical far-field assumption may be not fully justified in this case since the distance $d$ corresponding to the position of the open end of the resonator below the bar is usually equal to a few centimeters only, which is not significantly larger than the dimensions of the bar elements. This assumption may explain some of the discrepancies between predicted and measured pressure.

### B. Resonator model

In order to account for the radiation of the resonator, the model must include a time-domain formulation for both the wave propagation in the tube and the radiation impedance at its open end. This section deals with the continuous formulation of the problem, which has to be mathematically well-posed in order to be physically relevant in the space–time domain and to allow the stability of the inferred numerical schemes.

Most resonators of bar percussion instruments are cylindrical, with a radius $a_T$ significantly smaller than the length $l_T$. The sound wave inside the tube can be considered to consist of one-dimensional plane waves below the cutoff frequency given by

$$f_c=\frac{1.8411}{2\pi} \frac{c_0}{a_T}=0.2930 \frac{c_0}{a_T}.$$  (5)

At this frequency, the wavelength is $\approx 1.7$ times the diameter of the opening. With $c_0=340$ m/s and a radius of 2.0 cm (which is a typical order of magnitude for a xylophone), Eq. (5) yields a lower limit of $f_c$ equal to 4980 Hz. In comparison, the fundamental frequency of the highest note ($C_9$) in xylophones is nearly equal to 4284 Hz. As a consequence, it is justified to describe the propagation in the pipe by the plane-wave equation

$$\frac{\partial^2p}{\partial t^2}(z,t)-c_0^2\frac{\partial^2p}{\partial z^2}(z,t)=0,$$  (6)

where $p(z,t)$ is the acoustic pressure in the resonator. However, Eq. (6) may be invalid for some marimba or vibraphone resonators having a relatively larger cross section. The resonators of bar-mallet percussion instruments are metallic with perfect rigid walls. The losses are essentially localized near the walls due to viscous thermal effects. Thus the speed of sound in the pipe and the damping factor of the wave become frequency dependent:

$$c(\omega)=c_0\left(1-\frac{\epsilon_v}{a_T\omega^2} \frac{\sqrt{\epsilon_0}}{\omega}\right), \quad \alpha(\omega)=\frac{\epsilon_v}{a_T\omega^2} \frac{\sqrt{\omega}}{\sqrt{\epsilon_0}}$$  (7)

with $\omega$ the radian frequency, $\epsilon_v=\sqrt{\epsilon_0}+(\gamma-1)\sqrt{\epsilon_1}$ and $l_T\approx 4 \times 10^{-8}$ m and $l_k\approx 5.6 \times 10^{-8}$ m are the characteristic lengths of viscous and heat propagation effects, respectively, and $\gamma=C_p/C_v$.

Equation (7) can be transposed to the time domain, but cannot be easily solved because of the dependence of both terms with the square root of frequency. However, the problem can be greatly simplified if one considers that the fundamental frequency $f_1=\omega_1/2\pi$ of the resonator is generally tuned to the fundamental of the bar and that the upper partials of the bar will generally not match one of the upper eigenfrequencies of the tube. As it is currently observed in practice, the spectrum of the wave reemitted by the tube is thus very similar to the one of a pure tone. Therefore the losses in the tube can be taken into account by adding a fluid-damping term independent of frequency in Eq. (6) which yields

$$\frac{\partial^2p}{\partial t^2}(z,t)+\gamma_T \frac{\partial p}{\partial t}(z,t)-c_0^2\frac{\partial^2p}{\partial z^2}(z,t)=0$$  (8)

with

$$c=c_0\left(1-\frac{\epsilon_v}{a_T\omega^2} \frac{\sqrt{\epsilon_0}}{\omega}\right), \quad \gamma_T=\frac{\epsilon_v}{a_T} \frac{c}{\sqrt{\omega_1} \sqrt{\epsilon_0}}.$$  (9)

Equation (8) has been used for the modeling of the wave propagation inside the tube. This propagation equation needs now to be complemented by boundary conditions, i.e., by a time-domain modeling of the radiation impedance.

The general idea behind this step of the modeling is to find a suitable time-domain approximation of the radiation impedance, starting from its continuous expression in the frequency domain. Therefore some of the most significant and well-known results, with regard to the radiation impedances of tubes, are first briefly reviewed.

Neglecting the edge effects, let us first assume that the open end of the tube acts like a baffled planar piston radiating in free space. Following Rayleigh, the radiation impedance is then given by


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TABLE I. Values of the coefficients $\alpha_i$ and $\beta_i$ for the approximate radiation impedance.

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rayleigh (first order)</td>
<td>0.0000</td>
<td>0.8488</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.8488</td>
<td>0.0000</td>
</tr>
<tr>
<td>Rayleigh (second order)</td>
<td>0.0000</td>
<td>0.8488</td>
<td>0.4000</td>
<td>1.0000</td>
<td>1.0186</td>
<td>0.4000</td>
</tr>
<tr>
<td>Levine–Schwinger (second order)</td>
<td>0.0000</td>
<td>0.6133</td>
<td>0.2100</td>
<td>1.0000</td>
<td>0.7000</td>
<td>0.2100</td>
</tr>
</tbody>
</table>

\[
Z_r(ka_T) = Z_0 \left(1 - \frac{J_1(2ka_T)}{ka_T} + j \frac{S_1(2ka_T)}{ka_T} \right)
\]

with $Z_0 = \frac{\rho_0 c_0}{S_T}$ and $k = \frac{\omega}{c_0}$, \hspace{1cm} (10)

where $Z_0$ is the characteristic impedance of the tube with section $S_T$, and where $J_1$ and $S_1$ are the Bessel function and the Struve function of first order. In the low-frequency range $(ka_T \ll 1)$, Eq. (10) reduces to

\[
Z_r^{LF}(ka_T) \approx Z_0 \left(\frac{1}{2} (ka_T)^2 + j \frac{8}{3\pi} ka_T \right) = Z_0 \left(\frac{1}{2} (ka_T)^2 + j 0.8488 ka_T \right),
\]

The Rayleigh impedance has been used as a benchmark for comparing numerical and analytical solutions for the isolated tube (see Sec. III A). However, it turns out that an unbaffled model yields simulations closer to experimental results for mallet percussion instruments (see Sec. III B). Therefore, the calculation of the radiation impedance carried out by Levine and Schwinger is more appropriate.\hspace{1cm} (11)

Due to the usual frequency range of the instruments, the following approximations were used:\hspace{1cm} (12)

For $ka_T < 1.5$:

\[
Z_r(ka_T) = \frac{Z_0}{Z_0} = \frac{(ka_T)^2}{4} + 0.0127(ka_T)^4
\]

+ $0.082(ka_T)^4 \ln ka_T - 0.023(ka_T)^6$

+ $j0.6133ka_T - 0.036(ka_T)^3$

+ $0.034(ka_T)^3 \ln ka_T - 0.0187(ka_T)^5$.

For $1.5 \leq ka_T < 3.5$:

\[
Z_r(ka_T) = j \tan \left(k \Delta l + \frac{1}{2} \ln R \right),
\]

where

\[
R = e^{-ka_T \sqrt{\pi ka_T}} \left[1 + (3/32)(1/\ell (ka_T)^2) \right]
\]

and

\[
\Delta l = 0.634 - 0.1102ka_T + 0.0018(ka_T)^2
\]

\[-0.000005(ka_T)^4\]

In the low-frequency range $(ka_T \ll 1)$, Eq. (12) reduces to

\[
Z_r^{LF}(ka_T) = Z_0 \left(\frac{1}{2} (ka_T)^2 + j 0.6133ka_T \right).
\]

In order to allow transposition to the time domain, the radiation impedance is approximated by a fraction of second-order polynomials:

\[
\tilde{\xi}_r(jka_T) = \frac{Z_r(jka_T) - Z_0}{Z_0} = \frac{B_2(jka_T)}{A_2(jka_T)}
\]

\[
= \frac{\beta_0 + \beta_1 jka_T + \beta_2(jka_T)^2}{\alpha_0 + \alpha_1 jka_T + \alpha_2 (jka_T)^2},
\]

where $\tilde{\xi}_r(jka_T)$ denotes the normalized approximated radiation impedance.

The selection of the six coefficients $(\alpha_i, \beta_i)$ is based on the minimization of the least-squares error between the Rayleigh (respectively, Levine–Schwinger) expression and Eq. (14), with the constraints of convergence to Eq. (11) [respectively, Eq. (13)] as the frequency tends to zero, and to $Z_0$ as the frequency tends to infinity. As a consequence of these last conditions we get:

\[
\alpha_0 = 1, \hspace{0.5cm} \beta_0 = 0, \hspace{0.5cm} \alpha_2 = \beta_2
\]

\[
\beta_1 = 0.8488 \hspace{0.5cm} \text{(Rayleigh)} \hspace{0.5cm} \text{or}
\]

\[
\beta_1 = 0.6133 \hspace{0.5cm} \text{(Levine–Schwinger)}
\]

In addition, the transposition to the time domain (i.e., the replacement of $j\omega$ by the time-derivative operator $\partial/\partial t$) requires that $\tilde{\xi}_r$ must fulfill the Kreiss’s stability criterion which states that\hspace{1cm} (14)

\[
\forall s \in \mathbb{C} \quad \text{with} \hspace{0.3cm} \Re e(s) > 0, \hspace{0.3cm} \Re e(\tilde{\xi}_r(s)) > 0,
\]

where $s$ is the Laplace transform variable.

It is interesting to point out here that the condition (16) corresponds exactly to the definition of positive real functions which have been widely investigated in the past in the context of electrical network synthesis, as mentioned by Smith.\hspace{1cm} (15)

This criterion yields the following sufficient conditions (see Appendix A):

\[
\alpha_1, \alpha_2 > 0 \hspace{0.3cm} \text{and} \hspace{0.3cm} \alpha_1 \beta_1 > \alpha_2.
\]

Table I gives the values obtained for the coefficients $\alpha_i$ and $\beta_i$. This second-order approximation yields a mean error of nearly 1% for the radiation impedance in the complete register of the instrument ($f \leq 4.5$ kHz, i.e., $ka_T \ll 1.5$ for $a_T = 1.8$ cm) (see Figs. 3 and 4). A higher-order approximation does not seem to be required in view of this obtained degree of accuracy.

It is possible to derive another stable approximation of the impedance $\tilde{\xi}_r$ using a fraction of first-order polynomials, instead of second-order polynomials. The results, for the Rayleigh impedance only, are shown in Table I. As expected, the first-order approximation is less accurate than the second-order one because of the reduced number of ‘degrees of freedom.’ It can be seen in Fig. 3 that the first-order approximation leads to a systematic overestimation of the
imaginary part of the impedance for $ka_T \gg 1$ and to a systematic overestimation of the real part of the impedance for $ka_T \ll 1$.

Finally, it is important to remark that the direct transposition of the low-frequency approximation of the radiation impedance to the time-domain shown in Eqs. (11) and (13) leads to an ill-posed problem since, in this case, one has $\beta_2 = -0.5$ which is not compatible with the conditions expressed in Eq. (17). In practice, such an approximation does not ensure the stability of the continuous system of equations which means that some solutions may increase exponentially with time.

The time-domain formulation of the boundary condition at the open end of the tube is now derived from Eq. (14). It is assumed that the resonator is excited at its open end by the total pressure $p_R(d, \pi, t)$ radiated by the bar [see Eq. (4)]. For the sake of convenience, the input pressure will now be denoted $p_R(t)$ and the open end of the tube, located at a distance $d$ below the bar, will be taken as the new origin of the $z$ axis (see Fig. 2).

Through application of the superposition theorem, the total sound pressure at $z = 0$ is obtained by summing the pressure radiated by the resonator and the bar pressure $p_R(t)$, which yields in the frequency domain

\[
p(0, j\omega) = -Z_r(j\omega)S_R(0, j\omega) + p_R(j\omega),
\]

(18)

where $\mu(z, t)$ is the acoustic velocity. By combining Euler’s equation with the approximate expression of $Z_r(j\omega)$, one obtains

\[
j\omega p(0, j\omega) - c_0 \xi_r \begin{pmatrix} j\omega & \frac{a_T}{c_0} \end{pmatrix} \begin{pmatrix} \frac{\partial p}{\partial z} \end{pmatrix} (0, j\omega) = j\omega p_R(j\omega).
\]

(19)

In the time domain, Eq. (19) becomes

\[
\mathbf{A} \frac{a_T}{c_0} d_i \frac{dp}{dt} (t) = \mathbf{A} \frac{a_T}{c_0} \frac{\partial}{\partial t} (0, t) - c_0 \mathbf{B} \frac{a_T}{c_0} \frac{\partial}{\partial z} (0, t),
\]

(20)

where the differential operators $\mathbf{A}$ and $\mathbf{B}$ are given by

\[
\mathbf{A} \frac{a_T}{c_0} \frac{\partial}{\partial t} - \alpha_0 + \alpha_1 \frac{\partial}{\partial t} + \alpha_2 \frac{a_T}{c_0} \frac{\partial^2}{\partial t^2}.
\]
At the closed end of the tube \((z = l_T)\), the acoustic velocity is equal to zero (the condition for a perfect rigid wall), and the boundary condition can be written
\[
\frac{\partial p}{\partial z} (l_T,t) = 0.
\]

(21)

In summary, Eqs. (8), (9), (20), and (21) form the continuous time-domain model used in this study for the tubular resonator of mallet percussion instrument. The numerical formulation of these equations will be now examined together with the numerical model used for the calculation of the total sound pressure radiated by the instrument.

II. NUMERICAL FORMULATION

A. A finite difference formulation for the resonator

A uniform grid of \(N_T\) segments is considered for the discrete resonator. Let \(\Delta z = l_T/N_T\) and \(\Delta t = 1/f_s\) be the spatial step and the time step, respectively, the sampling frequency being \(f_s\). Approximating Eq. (8) with central difference derivatives of second order in time and space, one obtains the following explicit formulation for the inner mesh points:

\[
\forall i \in [1;N_T-1],
\]

\[
(1 + \gamma_T \Delta t/2)p_i^{n+1} = 2p_i^n - (1 - \gamma_T \Delta t/2)p_i^{n-1} + c^2 \frac{\Delta t^2}{\Delta z^2} (p_{i+1}^n - 2p_i^n + p_{i-1}^n),
\]

(22)

where \(i\) and \(n\) are the spatial and times indices, respectively, \(p_i^n\) denotes the calculated value of the acoustic pressure at position \(z_i = i\Delta z\), and time \(t_n = n\Delta t\). In the undamped case (\(\gamma_T = 0\)), the scheme remains stable under the condition

\[
r_{CFL} = \frac{c}{2c_0} \Delta z \leq 1.
\]

(23)

where \(r_{CFL}\) represents the so-called Courant–Friedrichs–Levy number. It can be shown that the introduction of the fluid damping term \(\gamma_T\) tends to stabilize the scheme.\textsuperscript{14,15}Another remarkable and well-known property of this 2-2 explicit scheme for the one-dimensional wave equation is that no numerical dispersion occurs if \(r_{CFL} = 1\). Under this condition, the eigenfrequencies of the discretized simulation are equal to those of the continuous boundary value problem. Therefore the time and spatial steps were selected in accordance with the equality \(c\Delta t = \Delta z\) for the tube model.

If no loss of energy occurs at the boundaries, one can use the method of images, which guarantees both the stability and second-order precision of the entire scheme, in order to obtain an appropriate numerical formulation of the boundary condition.\textsuperscript{14,15}This is applied here to the closed end of the pipe, which yields

\[
\frac{\partial p}{\partial z} (l_T,t) = 0 = p_{N_{r}+1}^n - p_{N_{r}+1}^{n-1}.
\]

Thus for \(i = N_T\) and with \(r_{CFL} = 1\), the difference equation becomes

\[
p_{N_{r}+1}^n = 2p_{N_{r}+1}^n - p_{N_{r}+1}^{n-1}.
\]

(24)

If an absorbing boundary condition is introduced, as it is the case for the radiation impedance, the stability is not guaranteed unless a thorough analysis of the numerical scheme is conducted on the basis of the Kreiss criterion.\textsuperscript{8}This criterion leads here to the following sufficient condition in terms of time step (see Appendix B):

\[
\Delta t < \frac{a_T}{2c_0}\left(\frac{\alpha_1 - \beta_1}{2\alpha_2}\right).
\]

(26)

With the second-order approximation of the radiation impedance (see Table I), \(a_T = 2\) cm and \(c_0 = 340\) m/s, Eq. (26) yields \(\Delta t_{\text{max}} = 2.5 \times 10^{-5}\) s (Rayleigh), or \(\Delta t_{\text{max}} = 1.2 \times 10^{-5}\) s (Levine–Schwinger), which means that the sampling frequency must be greater than 40 (respectively, 84) kHz. In practice a sampling frequency of 192 kHz has been selected in order to guarantee the fine tuning of the bar (see Ref. 1), which fulfills the condition expressed in (26) for both the baffled and un baffled radiation impedance. Thus under the assumption that both the continuous and discrete stability condition are fulfilled, these conditions are being expressed in terms of the coefficients of the approximate impedance \((\alpha_i \text{ and } \beta_i)\) and of the time step \(\Delta t\), respectively, the second-order finite difference formulation of Eq. (20) for the open end of the resonator, i.e., for \(i = 0\), is written

\[
(1 + a_1 + b_1 + 2a_2)p_0^{n+1} = (a_1 + 3a_2)p_0^n - a_2p_0^{n-1} + (b_1 + a_2)p_1^n + a_2p_1^{n-1} + a_2p_1^{n+1},
\]

\[
(a_1 + a_2)p_1^{n+1} = (a_1 + 3a_2)p_1^n - a_2p_1^{n-1} + (b_1 + a_2)p_2^n + a_2p_2^{n-1},
\]

\[
\text{where}
\]

\[
a_1 = \frac{a_T}{\Delta t c_0}, \quad b_1 = \frac{\beta_1 a_T}{\Delta t c_0}, \quad \text{and} \quad a_2 = \frac{a_T}{\Delta t c_0} \left(\frac{\alpha_1 - \beta_1}{2\alpha_2}\right).
\]

(27)

Equation (27) yields explicitly the radiated sound pressure \(p_0\) at the open end of the tube as a function of both the internal pressure \(p_1\) at position \(z_1 = \Delta z\) and pressure \(p_B\) radiated by the bar.

B. Total sound pressure radiated by the instrument

The numerical formulation of the acoustic pressure due to the bar derives from the continuous model presented in Sec. I. At each time step, the finite difference scheme used in Ref. 1 for the vibrating bar yields the velocity

\[
u_i(n\Delta t) = (w_i^{n+1} - w_i^{n-1})/(2\Delta t).
\]

(29)

Thus Eqs. (3) and (4) become
By combining Eq. (32) with Euler’s equation, one obtains

\[ \Delta p_T(r, \theta, n \Delta t) = \frac{S_T}{4\pi r_T} \frac{\partial p}{\partial r} (0, n \Delta t - r_T/c_0). \]  

which, in numerical form, becomes

\[ p_T(r, \theta, n \Delta t) = \frac{S_T}{4\pi r_T} \frac{p(1, n \Delta t - r_T/c_0) - p(0, n \Delta t - r_T/c_0)}{\Delta z}. \]  

This simplified radiation model is valid in the low-frequency range \((ka_T \ll 1)\) and yields an acceptable model for the radiation of bar-mallet percussion instruments. Finally, the total pressure field of the bar-resonator system is obtained by adding the two contributions:

\[ p(r, \theta, n \Delta t) = p_T(0, \theta, n \Delta t) + p_T(r_T, n \Delta t). \]  

Equation (35) illustrates one interesting feature of the time-domain modeling, which allows independent control of the bar and tube contributions in the total sound radiated by the instrument. The model can be easily transposed to the case of a baffled tube. In this case, one should represent the open end by a half-pulsating sphere, rather than by a com-

![Figure 5](image-url)  

FIG. 5. Comparison between analytical and numerical impulse response of a baffled tube of length \(l_T = 19.3\) cm and radius \(a_T = 1.85\) cm closed at one end. (a) Pressure waveform at the open end; (b) pressure spectrum. Solid line: analytical solution with Rayleigh radiation impedance at the open end. Dashed line: second-order approximation with sampling rate \(f_s = 192\) kHz. (c) Pressure waveform at the open end; (d) pressure spectrum. Solid line: analytical solution with Rayleigh radiation impedance at the open end. Dash-dotted line: second-order approximation with sampling rate \(f_s = 44.1\) kHz.
complete sphere, and the right-hand side of Eqs. (32)–(34) should be consequently multiplied by a factor of 2.

III. EXPERIMENTS AND SIMULATIONS

From a musical point of view, a xylophone model should be able to reproduce the following main relevant qualities of a real instrument: initial sharp attack, tuning, directivity, tone color, and aftersound due to the tubular resonator. The control of the tuning and of the initial sharp attack are mainly dependent on exciter and bar properties. These two points have been widely investigated in a previous paper and will not be discussed further here.

The reproduction of the directivity pattern of the instrument is directly related to the above-presented linear array of dipoles model used for the bar and to the pulsating sphere model used for the tube. The ability of the model to account for the directivity measured on a real instrument is presented in Fig. 8. In addition, it is shown to what extent variations of the tube diameter in the model can change the balance between “bar” sound and “tube” sound (see Fig. 10).

During the initial transient, the tone color of the instrument is essentially due to the interaction between bar and mallet. However, after a short delay, the magnitude of the tube contribution becomes predominant and the spectrum of the tone is clearly altered. Thus it is essential to examine whether the model is adequate for controlling the delay, magnitude, and spectrum of this aftersound. The capability of the model for reproducing the characteristic beats observed when the bar and the tube are detuned will be presented in Fig. 11.

A. Comparison between numerical and analytical solution for the isolated tube

The resonator model to be validated is composed of two parts: the acoustic wave propagation in the tube and the radiation impedance. The 2-2 explicit scheme used for the internal wave is standard and gives no difficulty. It yields no dispersion, in particular, since it is used with the condition $r_{CHL} = 1$ [see Eq. (23)]. As a consequence, the validation test for the resonator model has been essentially conducted in order to check the efficiency of the radiation impedance approximation.

The reference solution has been obtained using the analytical formulation of the Rayleigh radiation impedance shown in Eq. (10) for a baffled tube. From this equation, the transfer function $H_T(j\omega)$ between the resulting sound pressure at the open end $p_0$ and the incoming bar pressure $p_B$ at
this point has been calculated (see Appendix A). The corresponding impulse response of the tube $h_T(t)$ is obtained by inverse Fourier transform. The pressure $p_0$ radiated by the tube is calculated by convolving $h_T(t)$ with a Gaussian pulse:

$$p_0(t) = \exp \left( -\frac{(t-t_0)^2}{2\Delta^2} \right).$$

The duration of the Gaussian pulse $2\Delta \tau$ is taken equal to 10 $\mu$s in order to be significantly lower than the propagation time in the tube. The reference solution is compared with the radiated pressure obtained by means of the numerical model presented in Sec. II.

Figure 5(a) shows the comparison between analytical and numerical time responses for an ideal lossless ($\gamma_T=0$) $A_4$ resonator of length $l_T=19.3$ cm with fundamental $f_1 = 440$ Hz, closed at one end. The radiation at the open end is modeled by the second-order approximation of the Rayleigh impedance presented in the previous section. The calculations have been made at a sampling frequency $f_s = 192$ kHz, i.e., $\Delta z = c_0 \Delta t = 1.8$ mm, for a radius $a_T = 1.85$ cm. The relative error with the analytical solution is equal to 0.03% for the first 100 ms of the signal. Figure 5(b) shows the frequency responses of these two waveforms, obtained by means of a FFT analysis. The agreement is excellent below 1 kHz and is equal to a few dB between 1 and 4 kHz with a slight deviation of the maxima.

In comparison, Fig. 5(c) and (d) show how the model performs at $f_s=44.1$ kHz, which is the most commonly used audio sampling rate. In this case, the simulated waveform shows significant artifacts and the agreement between theoretical and simulated spectra is restricted to frequencies smaller than 1.5 kHz. Therefore for audio applications where a high degree of accuracy for the radiation impedance would be requested, the simulation should be made first at nearly four times the audio sampling rate and followed by a decimation with a factor of 4 before digital-to-analog conversion.

B. Radiation of the bar: Comparison between simulation and experiments

In a first step, only the bar radiation is investigated. Figure 6 shows the comparison between measurements and
simulations both in the time and frequency domains. The recording position for the comparisons between measured and simulated sound pressure has been taken at a point corresponding roughly to the location of the player’s ear ($r = 0.8$ m, $\theta = \pi/6$). The sound pressure has been measured with a microphone Schöps CMC3-D and the output signals were recorded on a Digital Audio Tape Sony TCD10-Pro. The comparisons were made for the G4 note ($f_1 = 396$ Hz) of a xylophone Concorde X 4001, played with a boxwood mallet at the third of the bar length. A good agreement can be observed in the results. The general shape of the pressure waveforms are similar, except during the first few milliseconds. This discrepancy is mainly a consequence of the phase shift due to dispersion in the rapidly damped high-frequency range ($f > 5$ kHz). The spectra of the measured and simulated sound pressure show a large degree of similarity, at least for the first three partials where the observed discrepancies are less than 3 dB. For higher partials, the differences can be explained by the model used for the bending vibration of the bar, which takes neither torsional waves nor shear and rotary inertia effects into account.¹

C. Simulation of total sound field: Function of the resonator

Figure 7 illustrates the capability of the model to account for the directivity of the instrument. In these numerical experiments, the resonator is tuned to a frequency close to the fundamental frequency of the bar. The waveform envelope is very sensitive to small variations of the tuning. Temperature and humidity changes during the experiments may alter these waveforms significantly. These differences are more clearly seen in the time domain than in the frequency domain. Figure 7(a) and (c) corresponds to the case $\theta = 0$ whereas Fig. 7(b) and (d) corresponds to the case $\theta = 80^\circ$. For $\theta = 0$ the bar contribution is relatively significant whereas this contribution is largely reduced for $\theta = 80^\circ$. During the first 10 ms of the sound, the radiativity is of dipole type. The pressure signal reaches its maximum 40–60 ms after the impact. Systematic simulations show that this delay primarily depends on the bar-tube tuning and, more generally, on the coupling parameters: bar-tube distance, tube radius, and tube length. Notice that this delay is much larger than the time needed for the sound wave below the bar to reach the resonator before being reemitted, which would

![Figure 8: Directivity (spectral plots). Comparison between measurements and simulations of sound pressure radiated by a xylophone A₄ bar (with resonator) struck by a rubber mallet near the center. (a) Measured pressure with $r=0.8$ m, $\theta=0$; (b) $r=0.8$ m, $\theta=80^\circ$. (c) Simulated pressure $r=0.8$ m, $\theta=0$; (d) $r=0.8$ m, $\theta=80^\circ$.](image)
yield a delay equal to \(2(d+\tau)/c_0 = 1.2\) ms, a confusion which is frequently encountered. Between typically 10 and 50 ms, the directivity pattern of the instrument changes gradually from dipole to monopole type. These results confirm previous experiments made by other authors on a vibraphone. The model is able to reproduce the main features of the measured sounds although some differences can be seen for \(\theta = 80^\circ\) where the magnitude of the measured waveform is about twice the simulated waveform during the attack.

Figure 8 shows the spectral plots corresponding to the waveforms displayed in Fig. 7. These plots were obtained from short-time Fourier transform using a window length of 40 ms and a step size of 5 ms. It is confirmed from these plots that the tube contribution is prominent and that the bar contributes to the sound essentially during the first 50 ms. The model is able to reproduce the time-envelope of the main spectral peaks.

Figure 9 shows variations of simulated sound pressure with tube radius. The recording point here is such that \(\theta = 45^\circ\) and \(r_\tau = 60\) cm. On the left-hand side of the figure, the tube radius \(a_\tau = 18.5\) mm whereas \(a_\tau = 11\) mm on the right-hand side. As expected, the magnitude of the tube field becomes larger as the radius increases. In the experiments, the length of the tube has been slightly readjusted in order to compensate the consecutive small variations of the fundamental frequency. Both measurements and simulations clearly show the dramatic changes in bar-tube balance and sharpness of the attack due to modification of the tube radius.

The spectral plots corresponding to these sounds are shown in Fig. 10. It can be seen that, with the smaller radius, the initial magnitude of the bar components are of the same order of magnitude than the fundamental of the tube whereas, with the larger radius, these components are nearly 20 dB lower than the main tube component. Here again, the simulated time-envelopes of the main peaks agree well with the experiments.

Finally, Fig. 11 shows the simulated pressure waveforms obtained when the resonator is not tuned to the fundamental frequency of the bar. In order to illustrate this point, a \(B_4\) resonator of length \(l_\tau = 159\) mm is placed under a \(A_4\) bar. Here the main effect of the tube is to produce the character-
istic beats which can be clearly heard on real instruments.

It has been also observed experimentally that the fundamental frequency of the tube decreases slightly with the bar-tube distance $d$. This phenomenon becomes significant if $d$ is typically smaller than 3 cm and is due to the fact that the bar is an inertial obstacle to the sound field emitted by the open end which, in turn, modifies the radiation impedance. In our model, this detuning can be compensated by adjusting the imaginary part of the simulated radiation impedance.

**IV. CONCLUSION**

A theoretical model has been developed which accounts for the sound field radiated by mallet percussion instruments. This time-domain modeling is based on the one-dimensional Euler–Bernoulli equation for the flexural vibrations of the bar coupled with the one-dimensional wave equation in the resonator under appropriate boundary conditions. Particular attention has been paid to the time-domain modeling of the radiation impedance of the tube, in order to guarantee that the problem is well-posed. Simulation of real instruments tones are obtained as a result of the numerical formulation of the problem. These simulations are based on finite difference approximations of the complete system of equations. Here again, a thorough numerical analysis is conducted in order to ensure stability and sufficient precision of the numerical algorithms in the audio range. From a practical point of view, the computing time is not significantly longer for a complete instrument than for the bar equation only (typically 10³ s for 1 s of sound on a Sun-Sparc10 workstation). The most time-consuming part of the model is due to the necessary fine tuning of the bar (see Ref. 1).

Various types of measurements and simulations have been conducted. First, it has been checked experimentally on wooden xylophone bars that the tube sound field has no appreciable influence on both eigenfrequencies and decay times of the bar vibrations, which confirms one of the basic assumptions of the model. This negligible influence of the tube is due first to the fact that, for wooden xylophone bars, the coupling with the tube is relatively weak and, second, to the relatively high internal losses compared to the radiation losses. However, this assumption is not justified for marimbas and vibraphones in the low-frequency range since these instruments exhibit a stronger bar-tube coupling and a lower internal damping in the bars. As a consequence, differences in the decay times, with and without the tube, are clearly seen. In order to take this coupling into account, the present

![Simulation plots](image-url)
model should be modified in future versions by adding a tube pressure term in the vibrating bar equation.1

The tube model has been validated by comparisons between the numerical results and an analytical solution based on Rayleigh radiation impedance. In a third series of experiments, it has been shown to what extent the numerical model is able to reproduce the main features of real instruments, with regard to the balance between bar and tube contributions and to the directivity pattern of the instrument. The influence of tube length and bar-tube distance were also investigated. With the present one-dimensional model of the resonator, it is not yet possible to investigate with great detail the influence of the geometry of the tube on the sound field. However, the general structure of the model has proven to be valid and a number of problems of interest for makers could be solved in the future through the generalization of the acoustic wave equation in Eq. (6) to higher dimensions.

This time-domain approach yields a better understanding of the physics of mallet percussion instruments which complements previous works conducted in the frequency domain.3 The results are currently of use for the design of real instruments and for psychoacoustical studies. Sequences of simulated tones sound very realistic and can be used in the context of musical sound synthesis.

APPENDIX A: STABILITY OF THE CONTINUOUS RESONATOR MODEL

The following derivations are based on Kreiss’s theory, which has been developed in order to analyze whether a boundary value problem is well-posed or not. In practice, the boundary conditions must fulfill a number of conditions in order to ensure that no waves of increasing magnitude with time can be the solution of the problem.4 This theorem applies to problems where the boundary conditions are complex impedances. In the Laplace domain, the specific impedances are then written in the form of a ratio of two polynomials:

$$\xi(s) = \frac{B(s)}{A(s)}, \quad (A1)$$

where $s$ is the Laplace transform variable. It can be shown that the problem is well-posed in the sense of Kreiss theory, providing that

$$\forall s \in \mathbb{C} \quad \text{with} \quad \Re\xi(s) > 0, \quad \Re\xi(s) = \Re\left(\frac{B(s)}{A(s)}\right) > 0. \quad (A2)$$

One can equivalently write the condition (A2) in terms of the reflection coefficient $R(s)$ as follows:

$$\forall s \in \mathbb{C} \quad \text{with} \quad \Re\xi(s) > 0, \quad \left|R(s)\right| < 1. \quad (A3)$$

From a physical point of view, Eq. (A3) implies energy losses at the boundary.

The resonator model

The previous criterion is applied to a quarter-wavelength lossless tube, excited at its open end ($z = 0$) by the pressure signal $p_B(t)$ generated by the bar. The total pressure at this end is written $p(0,t)$. In the Laplace domain, the transfer function $\mathcal{H}_\tau(s)$ of the tube is written

$$\mathcal{H}_\tau(s) = \frac{p(0,s)}{p_B(s)} = \frac{1-R_0(s)}{2} \frac{1+e^{-2\Gamma(s)}}{1-R_0(s)e^{-2\Gamma(s)}}. \quad (A4)$$

where $\Gamma(s) = s/c_0$, and $R_0(s) = R(0,s)$ is the reflection coefficient at the open end:

$$R_0(s) = \frac{\xi(s)-1}{\xi(s)+1} \Rightarrow \xi = \frac{1+R_0(s)}{1-R_0(s)}. \quad (A5)$$

A sufficient condition for ensuring the stability of the system represented by the transfer function $\mathcal{H}_\tau(s)$ is given by $\left|R_0(s)\right| < 1$. A direct consequence of this condition is that $\xi$ must have no poles with a non-negative real part. In our case, the specific radiation impedance is given by the second-order approximation:

$$\xi(s) = \frac{a_T}{c_0} \frac{\beta_1 s + \alpha_2 (a_T/c_0)s^2}{1+\alpha_1(a_T/c_0)s+\alpha_2(a_T/c_0)^2 s^2}. \quad (A6)$$

FIG. 11. Effect of tube length. Sound pressure radiated by a xylophone A4 bar (with resonator) struck by a rubber mallet near the center: $r = 0.8 \text{ m}$, $\theta = 0$, $l_\tau = 159 \text{ mm}$ (B4 tube). (a) Measured; (b) simulated.
The condition of no poles with a non-negative real part for \( \bar{\xi} \) implies first:

\[
\alpha_1 > 0 \quad \text{and} \quad \alpha_2 > 0.
\]  

(A7)

Second, it turns out that the condition (A2) needs to be checked on the imaginary axis \( \overline{s} = i \omega \) only, through application of the maximum modulus principle to the function \( R_0(s) \) which must be analytic in the right-half complex plane. It is easy to establish in our case that

\[
\Re(\bar{\xi}_n(j\omega)) = \left( \frac{\alpha_2^2}{c_0^2} \right) \left[ (-\alpha_2 \omega^2)(1-\alpha_2 \omega^2) + \omega^2 \alpha_1 \beta_1 \right] 
\]

\[
> 0 \iff \alpha_1 \beta_1 > \alpha_2.
\]  

(A8)

In summary, Eqs. (A7) and (A8) yield the stability conditions for the continuous system:

\[
\alpha_1, \alpha_2 > 0 \quad \text{and} \quad \alpha_1 \beta_1 > \alpha_2.
\]  

(A9)

APPENDIX B: STABILITY OF THE DISCRETE RESONATOR MODEL

For the numerical formulation of the problem, the approach for investigating the stability conditions is similar to the one presented in Appendix A, apart from the fact that the partial derivatives are now approximated by finite differences. It has been shown in another context that a good strategy consists in approximating the partial differential operators by the following expressions:

\[
\frac{\partial p}{\partial t} \approx (\delta_t p)_i^{n+1} = \frac{p_i^{n+1} - p_i^n}{\Delta t},
\]

\[
\frac{\partial^2 p}{\partial t^2} \approx (\delta_t^2 p)_i^n = \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{\Delta t^2},
\]

\[
\frac{\partial p}{\partial z} \approx (\delta_z p)_i^n = \frac{p_i^{n+1} - p_i^{n-1}}{\Delta z}.
\]

With these notations, the discrete wave equations in the isolated tube are written

\[
(\delta_t^2 p)_i^n - c_0^2 (\delta_z^2 p)_i^n = 0 \quad \text{for} \quad z \neq 0
\]  

(B2)

and

\[
A_k(\delta_k^-)(\delta_k^-)p_i^{n+1} - c_0 B_k(\delta_k^-)(\delta_k^+)^n = 0
\]

\[
\text{for} \quad z = 0.
\]  

(B3)

Solutions to the problem of the form

\[
p_i^n = \xi^n \kappa^n \quad \text{with} \quad |\xi| > 1, |\kappa| > 1,
\]  

(B4)

where the case \( \xi = \kappa = 1 \) corresponds to the particular case of harmonic waves, are not allowed to propagate in the system since their magnitudes are increasing with time. With (B4), Eqs. (B2) and (B3) become

\[
(\xi^2 - 2 + \xi^{-1}) - r_{\text{CFM}}^2 (\kappa^2 - 2 + \kappa^{-1}) = 0,
\]  

(B5)

\[
(1 - \xi^{-1})A_k\left(1-\xi^{-1}\right) - r_{\text{CFM}} (\kappa-1)B_k\left(1-\kappa^{-1}\right) = 0.
\]

With the second-order radiation impedance Eq. (A6) and the finite difference approximations (B1), Eq. (B3) becomes

\[
1 + \frac{\alpha_1}{\Delta t} \left( \frac{\alpha_T}{c_0} \right) (1-\xi^{-1}) + \frac{\alpha_2}{\Delta t^2} \left( \frac{\alpha_T}{c_0} \right)^2 (1-\xi^{-1})^2
\]

\[
- \frac{\alpha_T}{\Delta t} (\kappa-1) \left[ \beta_1 + \frac{\alpha_T}{\Delta t} \left( \frac{\alpha_T}{c_0} \right) (1-\xi^{-1}) \right] = 0.
\]  

(B6)

The stability theory in the discretized case is essentially the same as in the continuous case. It implies here that \( |\kappa| > 1 \) in Eq. (B6), which yields

\[
2\alpha_2\alpha_1 \sigma + \alpha_2^2 - 2\alpha_2 (1+\beta_1 \sigma) > 0
\]

(B7)

with \( \sigma = \alpha_T / c_0 \Delta t \). Taking further the stability condition of the continuous problem (A9) into account, Eq. (B7) is written finally in terms of the time step:

\[
\Delta t < \frac{\alpha_1^2}{c_0^2} \frac{2 \beta_1 (\alpha_1 - 2 \beta_1)}{2 \beta_1^2 - \alpha_1^2}.
\]  

(B8)

Notice that, in order to ensure the positivity of \( \Delta t \), we must have in addition

\[
\beta_1 < \alpha_1 < 2 \beta_1.
\]  

(B9)

17. V. Doutaut, A. Chaigne, and G. Bedrane, “Time-domain simulation of the
