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A New Implementation of the Extended Helmholtz Resonator Acoustic Liner Impedance Model in Time Domain CAA

L. Pascal*, E. Piot and G. Casalis
Onera – The French Aerospace Lab
F-31055 Toulouse, France
{lucas.pascal@onera.fr

The application of wall acoustic lining is a major factor in the reduction of aircraft engine noise. The extended Helmholtz Resonator (EHR) impedance model is widely used since it is representative of the behavior of realistic liners over a wide range of frequencies. Its application in time domain CAA methods by means of z-transform has been the subject of several papers. In contrast to standard liner modeling in time domain CAA, which consists in imposing a boundary condition modeling both the cavities and the perforated sheet of the liner, an alternative approach involves adding the cavities to the computational domain and imposing a condition between these cavities and the duct domain to model the resistive sheet. However, the original method may not be used for broadband acoustics since it implements an impedance condition with frequency independent resistance. This paper describes an extension of this method to implement the EHR impedance model in a time domain CAA method.

Keywords: Duct aeroacoustics; acoustic lining; time domain impedance modeling; CAA.

1. Introduction

Acoustic liners are widely used to reduce sound. A liner is generally made of an array of honeycomb cavities on a rigid backing sheet and under a resistive sheet which may either be a perforated or wire-mesh sheet. In order to circumvent the calculation of all the physical phenomena taking place in the liner, an impedance boundary condition is usually imposed at the lined wall on the acoustic perturbation. The acoustic impedance at a lined wall, written $\tilde{Z}$, is defined as being the ratio between pressure and normal velocity of the acoustic perturbation.

Among the different existing impedance models, the extended Helmholtz Resonator (EHR) model is widely used. Richter et al. have shown that this model “is shown to approximate or even reproduce the frequency response for a wide variety of other models” such as the standard Helmholtz resonator model (or model of Ko). Moreover, although defined in frequency domain, this model may be applied for time domain simulations through z-transform. Such simulations have been presented in several papers, see for instance Refs. 2.
and 4. EHR model may be effectively used for broadband acoustic problems, as shown by Richter et al.\textsuperscript{5} Time domain computations present the advantage of being suitable for nonlinear and multi-tones noise simulations.

More than 10 years ago, Sbardella et al.\textsuperscript{6} proposed an alternative method to implement the standard Helmholtz resonator model. This method consists in modeling the resistive sheet by an impedance condition but now the cavities belong to the computational domain. This method features, for instance, simple time domain implementation and allows to write the temporal eigenproblem under a linear form although the impedance model is nonlinear with respect to the frequency. This might be of great benefit when performing temporal stability analysis. However, this method has not received much attention\textsuperscript{7,8} probably because it is not able to model realistic liners. This is due to the fact that this method assumes a frequency invariant resistance of the impedance model. This paper is intended to extend the method of Sbardella et al.\textsuperscript{6} by giving it the capacity of implementing the EHR impedance model. We believe that this method might be an interesting alternative to the standard implementation of EHR impedance model in time domain and might in some cases be easier to integrate in a CAA or CFD code.

Sections 2 and 3 review respectively the EHR impedance model and the method of Sbardella et al.\textsuperscript{6} Section 4 is devoted to the base-flow boundary layer developing on a liner. In Sec. 5, shown how to extend the method of Sbardella et al.\textsuperscript{6} to implement the EHR impedance model. Finally, an example of numerical implementation followed by a validation case and a discussion on numerical implementation and cost is given in Sec. 6.

In the following, variables are made dimensionless thanks to reference velocity $a_0$ (sound celerity), length $H$ (duct height), pressure $\rho_0 a_0^2$ ($\rho_0$ is the base-flow density) and time $H/a_0$. The hat symbol $\hat{\cdot}$ denotes the Fourier transform. The symbol $\ast$ denotes dimensional parameters.

2. Extended Helmholtz Resonator

The EHR model\textsuperscript{1} is defined by the following impedance frequency evolution\textsuperscript{a}:

$$\hat{Z}(\omega) = R - i m \omega + i \phi \cotan \left( \omega h + \frac{i \varepsilon}{2} \right),$$  \hspace{1cm} (1)

where $R$ is the face-sheet resistance, $\omega m$ is the face-sheet reactance, $h$ is the cavity depth, $\varepsilon$ corresponds to the damping in the cavity’s fluid and $\phi$ is related to the porosity. Parameters $R$, $m$, $h$, $\varepsilon$ and $\phi$ are frequency independent. See Refs. 1, 2, 5 and 9 for more details on how to set those parameters to match the behavior of realistic liners.

Acoustic normal velocity and pressure are related in frequency domain by the impedance through the following equation:

$$\hat{p}(\omega) = \hat{Z}(\omega) \hat{\mathbf{u}} \cdot \mathbf{n},$$  \hspace{1cm} (2)

where $\mathbf{n}$ is the unitary normal vector pointing into the lined wall.

\textsuperscript{a}In this paper, the harmonic ansatz $e^{(-i \omega t)}$ has been chosen.
By writing the cotangent term under exponential form, $\hat{Z}(\omega)$ can be expressed as (see Sec. 6.A in Ref. 1):

$$\hat{Z}(\omega) = \frac{(e^{2i\omega h}e^{-\varepsilon} - 1)(R - i\omega m) - \phi(e^{2i\omega h}e^{-\varepsilon} + 1)}{e^{2i\omega h}e^{-\varepsilon} - 1}$$

(3a)

or (see Sec. III.D in Ref. 5):

$$\hat{Z}(\omega) = R - i\omega m + \phi + 2\phi\sum_{n=1}^{\infty} e^{2in\omega h}e^{-n\varepsilon}$$

(3b)

Equation (3b) is derived by using the formula: $\sum_{n=0}^{N-1} ar^n = (1 - r^N)/(1 - r)$ (sum of the $N$ first terms of a geometric series), with $N \to \infty$.

Replacing $\hat{Z}(\omega)$ by Eq. (3a) or Eq. (3b) in Eq. (2) and performing an inverse Fourier transform yields a temporal-domain boundary condition:

$$p(t) = e^{-\varepsilon}p(t - 2h) + (R + \phi)u(t) \cdot n + (R - \phi)e^{-\varepsilon}u(t - 2h) \cdot n$$

$$+ m\partial_t u(t) \cdot n - me^{-\varepsilon}\partial_t u(t - 2h) \cdot n$$

(4a)

or

$$p(t) = (R + \phi)u(t) \cdot n + m\partial_t u(t) \cdot n + 2\phi\sum_{n=1}^{\lfloor t/(2h) \rfloor} e^{-\varepsilon n}u(t - 2nh) \cdot n.$$  

(4b)

Equation (4a) is used in Refs. 2 and 5 while Eq. (4b) has been chosen by Chevaugon et al. In both cases, the method requires to store values at the discretization points which are on the lined wall. It may be chosen either to define $2h$ as a multiple of the time-step or to interpolate.

3. Method of Sbardella et al.

The method proposed by Sbardella et al. consists in “treating the duct and the liner backing air cavity as two different domains which interact with each other through novel boundary conditions simulating the presence of the liner porous sheet”. Thus, for each discretization point, one-dimensional (1D) elements modeling a cavity are added, in which 1D linearized Euler equations are solved. The resistive face-sheet between these 1D elements and the duct domain is modeled through a Darcy boundary condition (see Fig. 1). The latter relates the acoustic pressure jump across the resistive sheet to the velocity through the sheet $u_l$ by the resistance $R$:

$$Ru_l = p^{\text{duct}} - p^{\text{cavity}}.$$  

(5)

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\[\cot(X) = \cos(X)/\sin(X) = \text{i}(\exp(\text{i}X) + \exp(-\text{i}X))/(\exp(\text{i}X) - \exp(-\text{i}X)).\]

This paper is restricted to linear acoustics. As done in Ref. 6, it is as well possible to solve nonlinear Euler equations in the 1D elements.
If the base-flow satisfies a no-slip condition, \( u_l \) equals the acoustic normal velocity \( u \cdot n \) (where \( n \) is the unitary vector pointing into the cavities) in the duct. The case of a slipping base-flow is addressed in Sec. 4. For both cases, \( u_l \) equals the acoustic velocity in 1D elements modeling the cavities.

It might be shown that because of the hard-wall boundary condition imposed at the reflective backing sheet, acoustic pressure and velocity at the facing sheet in the cavity satisfy the following relationship in frequency domain:

\[
\hat{p}_{\text{cavity}} = i \cotan(\omega h).
\]  

It follows that this method implements the basic resonator Helmholtz model (or model of Ko\(^3\)) for a purely resistive face-sheet:

\[
\hat{Z}(\omega) = R + i \cotan(\omega h).
\]

4. Fully Resolved Boundary-Layer or Uniform Flow Assumption

Instead of considering a sheared flow satisfying no-slip condition at the lined wall, it is common to assume the flow as uniform. In that case, it is necessary to modify the boundary condition in order to take into account the boundary layer. This latter is usually considered infinitely thin and then reduces to a vortex-sheet. The boundary condition is then derived by invoking continuity of acoustic particle displacement and Eq. (2) is then replaced by the well-known Ingard–Myers\(^{10,11}\) boundary condition:

\[
-i \omega \hat{Z}(\omega) \hat{u} \cdot n = (-i \omega + U_0 \cdot \nabla - (n \cdot \nabla U_0) \cdot n) \hat{p},
\]

where \( U_0 \) is the base-flow velocity vector. This boundary condition associated to Eq. (1) might be written in time domain, see for instance Eqs. (57)–(58) in Ref. 1.

As far as the method of Sbardella et al.\(^6\) is concerned, continuity of particle displacement is invoked to construct a relationship between \( u_l \) and \( u \cdot n \). For a two-dimensional (2D) straight duct of axis \( x \) carrying a uniform flow of Mach number \( M \), the following relationship

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![Diagram](image)
Implementation of EHR Impedance Model in Time Domain

is obtained:

\[ \partial_t \mathbf{u} \cdot \mathbf{n} = \partial_t u_l + M \partial_x u_l. \]  

(9)

However, Brambley\textsuperscript{12} has shown that this boundary condition is ill-posed and leads in some cases to numerical instabilities. These instabilities are in practice suppressed by filtering.\textsuperscript{4,5} A well-posed boundary condition may be derived by considering a thin but finite-thickness boundary layer.\textsuperscript{13,14}

On the contrary, if the boundary-layer is fully resolved, these numerical treatments are unnecessary since the base-flow is zero at the wall. For instance, Richter \textit{et al.}\textsuperscript{2} have performed computations on a base-flow obtained by RANS simulation while Burak \textit{et al.}\textsuperscript{15} have integrated time domain impedance condition in a Navier–Stokes code.

5. Improvement of the Method of Sbardella \textit{et al.}\textsuperscript{6}

The method proposed by Sbardella \textit{et al.}\textsuperscript{6} implements the basic Helmholtz resonator impedance model which has a frequency invariant real part (resistance). Therefore, it is not able to be representative of the behavior of a realistic liner over a wide range of frequencies. We propose in this section some improvements to implement the EHR model (Eq. (1)).

Compared to the EHR model (Eq. (1)), three parameters are missing in Eq. (7): \( m, \varepsilon \) and \( \phi \). In the following sections is shown how to extend the method of Sbardella \textit{et al.}\textsuperscript{6} to account for these variables.

5.1. Mass-reactance \( m \) and porosity related term \( \phi \)

The mass-reactance \( m \) and the porosity related term \( \phi \) may be incorporated into the model by modifying the Darcy boundary condition Eq. (5) into:

\[ Ru_l + m \partial_t u_l = p^{\text{duct}} - \phi p^{\text{cavity}}. \]  

(10)

For \( \phi = 1 \), Eq. (10) is similar to Eq. (8) in Ref. 16. How \( \phi \) is introduced into the Darcy boundary condition (5) is not based on a physical reasoning but on an heuristic approach to achieve the required form of impedance behavior.

5.2. Cavity’s fluid damping \( \varepsilon \)

Let \( y \) be the coordinate in a cavity. The linearized Euler equations in a cavity are:

\[ \partial_t \varphi + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y \varphi = 0, \]  

(11)

where \( \varphi = (v, p)^T \) is the unknown vector.
It is found by heuristic search that modifying Eq. (11) into
\[ \partial_t \varphi + A_y \partial_y \varphi + \frac{\varepsilon}{2h} \varphi = 0 \] (12)
transforms Eq. (6) into:
\[ \frac{p^{\text{cavity}}}{u_l} = \text{icotan} \left( \omega h + i \frac{\varepsilon}{2} \right), \] (13)
which is the desired relationship between \( p^{\text{cavity}} \) and \( u_l \) in frequency domain in order to implement the EHR impedance model Eq. (1). Indeed, combining Eq. (10) written in frequency domain and Eq. (6) yields \( \frac{p^{\text{duct}}}{u_l} = \hat{Z}(\omega) \). The term \( \varepsilon \) in Eq. (12) corresponds mathematically to a damping term.

### 6. Example of Numerical Implementation and Simulation

In this section is shown how to implement the model proposed in a 2D discontinuous Galerkin solver. A validation computation is then performed and the numerical implementation and cost of the model are addressed. The validation case is the well-known NASA Grazing Incidence Tube (GIT) for which experimental data are available in Ref. 17. The configuration is depicted Fig. 2 (the duct is upside down compared to Ref. 17, where the acoustic treatment was on the upper wall).

The main flow (denoted with subscript 0) is assumed to be subsonic, stationary and homentropic. Moreover, the main density \( \rho_0 = 1.23 \text{ kg.m}^{-3} \) and the sound speed \( a_0 = 340 \text{ m.s}^{-1} \) are taken as constant. The sheared flow is supposed to be parallel: \( U_0 = U_0(y)e_x \) and to satisfy no-slip boundary condition at the walls.

#### 6.1. Numerical method

**6.1.1. Duct domain**

In the duct domain are solved the 2D linearized Euler equations. The linearized Euler, written under a matrix form, reads (the Einstein summation is used on \( x \) and \( y \)):
\[ \partial_t \varphi + A_j \partial_j \varphi + B \varphi = 0, \] (14)
where:
\[ A_x = \begin{pmatrix} U_0 & 0 & 1 \\ 0 & U_0 & 0 \\ 1 & 0 & U_0 \end{pmatrix}, \quad A_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \partial_y U_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
\( \varphi(x, y, t) \) correspond to the acoustic perturbation and is composed of the perturbation velocity vector \( \mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y \) and by the perturbation pressure \( p: \varphi = (u, p) \).

The computational domain \( \Omega_h = \Omega_d + \Omega_c \), bordered by \( \partial \Omega_h \), is meshed by \( N_t \) triangles \((T_l)_{l \in [1, N_t]}\) on \( \Omega_d \) (duct domain) and \( N_l \) 1D element \((L_i)_{i \in [1, N_l]}\) on \( \Omega_c \) (cavities). The discontinuous Galerkin formulation is obtained by imposing an orthogonality condition between the governing equations (Eq. (14)) and test-functions \( \psi^l_m \) belonging to \( P^2(T_l) \), the space of second-order polynomial defined on \( T_l \). The solution is as well decomposed on \( P^2(T_l) \). The solution being locally defined on a triangle, numerical fluxes are imposed between elements and to impose boundary conditions. Numerical fluxes are generally defined as a function of the solution in the triangle \( \varphi^- \) and in the neighboring triangle \( \varphi^+ \) and of the outward pointing vector normal to the triangle edge \( \mathbf{n} = (n_x, n_y) \): \( \Pi(\varphi^-, \varphi^+, \mathbf{n}) \).

By following Ref. 18, the formulation on the \( l \)th triangle is:

\[
\forall \psi^l_m \in P^2(T_l) \quad \langle \partial_t \varphi + A_j \partial_j \varphi + B \varphi; \psi^l_m \rangle_{T_l} + \langle \Pi(\varphi^-, \varphi^+, \mathbf{n}) \cdot \psi^l_m \rangle_{\partial T_l} = 0, \quad (15)
\]

where the inner product \( \langle \cdot, \cdot \rangle \) reads: \( \langle \varphi_1; \varphi_2 \rangle = \int_\Omega \varphi_1 \cdot \varphi_2 \, d\Omega \).

- If \( \partial T_l \cap \partial \Omega_h = \emptyset \) (between two triangles): upwind flux is imposed by: \( \Pi(\varphi^-, \varphi^+, \mathbf{n}) = [A_j n_j]^- (\varphi^+ - \varphi^-) \) where \( \otimes \) denotes tensor product:

\[
[A_j n_j]^- = \min(0; U_0 \cdot n) \begin{pmatrix}
   n_y^2 & -n_x n_y & 0 \\
   -n_x n_y & n_x^2 & 0 \\
   0 & 0 & 0
\end{pmatrix} 
+ \frac{U_0 \cdot n - 1}{2} \begin{pmatrix}
   n \otimes n & -n \\
   -n^T & 1
\end{pmatrix}.
\]

- If \( \partial T_l \cap \partial \Omega_h \neq \emptyset \) at outflow/inflow: characteristics boundary condition is imposed through \( \Pi(\varphi^-, \mathbf{n}) = -[A_j n_j]^- \varphi^- \).

- If \( \partial T_l \cap \partial \Omega_h \neq \emptyset \) on duct walls: hard wall boundary condition is imposed by: \( \Pi(\varphi^-, \mathbf{n}) = M(1) \varphi^- \) where:

\[
M(1) = \begin{pmatrix}
   n \otimes n & 0 \\
   -n^T & 0
\end{pmatrix}.
\]

- If \( \partial T_l \cap \partial \Omega_h \neq \emptyset \) on the liner: details are given in Sec. 6.1.3.

6.1.2. Cavities

The derivation for 1D elements is similar. The formulation reads on the \( l \)th element \( y \in [y^l_b, y^l_a] \):

\[
\forall \psi^l_m \in P^2(L_l) \quad \langle \partial_t \varphi + A_y \partial_y \varphi; \psi^l_m \rangle_{L_l} + \langle \Pi(\varphi^-, \varphi^+, \mathbf{n}) \cdot \psi^l_m \rangle_{y=y^l_b} = 0. \quad (18)
\]
The perturbation vector is \( \mathbf{\varphi} = (v, p) \) and the numerical flux \( \Pi(\varphi^-, \varphi^+, n) \) is defined as:

- \( [A_y n]^- (\varphi^+ - \varphi^-) \) between elements. \( [A_y n]^- \) corresponds to the restriction of \( [A_j n_j]^- \) (see Eq. (14)) to 1D case without flow:
  \[
  [A_y n]^- = \max(0, n_y/2) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \min(0, n_y/2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
  \] (19)

- \( \tilde{M}(1) \varphi^- \) on the rigid backing sheet to ensure hard-wall boundary condition. It is obtained by restricting \( M(1) \) to the 1D case:
  \[
  \tilde{M}(1) = \begin{pmatrix} 1 & 0 \\ -n_y & 0 \end{pmatrix}.
  \] (20)

The numerical flux ensuring the coupling with the duct is given in the next section.

6.1.3. Coupling between duct elements and cavities elements

The coupling between the duct and the cavities is modeled by Eq. (10). The velocity through the perforations \( u_l \) equals \( u \cdot n \) (since the mean flow satisfies no-slip boundary condition) and the acoustic velocity in the cavities (see Sec. 3). On the element \( E \), the numerical flux is based on a centered flux:

- If \( E \in \Omega_d \) :
  \[
  \Pi(\varphi^-, \varphi^+, n) = A_{jn_j} (\varphi^D - \varphi^-),
  \] (21a)

- If \( E \in \Omega_c \) :
  \[
  \Pi(\varphi^-, \varphi^+, n) = A_{ny} (\varphi^D - \varphi^-).
  \] (21b)

\( \varphi^D \) is a fictitious exterior trace relating \( \varphi^- \) and \( \varphi^+ \) through Eq. (10). It is defined as:

- If \( E \in \Omega_d \) :
  \[
  p^d = R \frac{u^- \cdot n + u^+ \cdot n}{2} + m \partial_t u^- \cdot n + \phi p^+.
  \] (22a)

- If \( E \in \Omega_c \) :
  \[
  p^d = R \frac{u^- \cdot n + u^+ \cdot n}{2\phi} + m \frac{\partial_t u^- \cdot n + p^+}{\phi}.
  \] (22b)

Finally, the flux reads:

- If \( E \in \Omega_d \) :
  \[
  \Pi(\varphi^-, \varphi^+, n) = \frac{R}{2} \begin{pmatrix} n \otimes n & 0 \\ 0^T & 0 \end{pmatrix} \{\varphi\} - \frac{1}{2} \begin{pmatrix} 0 & n \\ n^T & 0 \end{pmatrix} \varphi^- + \frac{1}{2} \begin{pmatrix} 0 & \phi n \\ n^T & 0 \end{pmatrix} \varphi^+ + \frac{m}{2} \begin{pmatrix} n \otimes n & 0 \\ 0^T & 0 \end{pmatrix} \partial_t \varphi^-.
  \] (23a)
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\[
\text{if } E \in \Omega_c: \Pi(\varphi^-, \varphi^+, n) = \frac{R}{2\phi}\begin{pmatrix} n \otimes n & 0 \\ 0 & 0 \end{pmatrix} \{\{\varphi}\} - \frac{1}{2}\begin{pmatrix} 0 & n \\ n^T & 0 \end{pmatrix} \varphi^- \\
+ \frac{1}{2}\begin{pmatrix} 0 & n/\phi \\ n^T/\phi & 0 \end{pmatrix} \varphi^+ + \frac{m}{2\phi}\begin{pmatrix} n \otimes n & 0 \\ 0 & 0 \end{pmatrix} \partial_t \varphi^- , \quad (23b)
\]

where \(\{\{\varphi}\}\) = \((\varphi^- + \varphi^+)/2\).

Note that the numerical flux (Eq. (23)) is given in generic form and may be used for 1D, 2D or three-dimensional (3D) configurations.

6.2. 2D validation

6.2.1. Numerical set-up

The incident plane wave is generated at \(f_0 = 2000\) Hz by a buffer domain following Ref. 19. Reflection free termination is achieved by adding a PML downstream (see Fig. 2). PML parameters are set-up following Ref. 20.

As chosen by Özyörük and Long,\(^{21}\) Poiseuille (i.e. parabolic) mean flow is imposed. Its section-averaged Mach number is 0.335. Following Ref. 17, CT57 liner is chosen. Its parameters have been educed by Richter et al.\(^5\):

\[
R = 0.000926, \quad 1/m = 2136.5, \quad \phi = 1.925, \quad h^* = 82.7\, \text{mm}, \quad \epsilon = 0.7367. \quad (24)
\]

Figure 3 compared the impedance obtained by combining Eqs. (1) and (24) with the impedance educed by Jones et al.\(^{17}\) for a section-averaged Mach number of 0.335.

In order to validate the model proposed, the computation performed in time domain is compared against harmonic computation (i.e. where \(\partial_t\) is replaced by \(-i\omega\) and \(\varphi\) is complex). In the latter, the liner model proposed in this paper is replaced by a standard impedance

\[
\text{Fig. 3. Liner impedance } \hat{Z} \text{ defined by Eq. (1) with the parameters Eq. (24).}
\]
boundary condition with \( \hat{Z} = 5.21 - 1.07i \) which is found by evaluating Eq. (1) at \( \omega = 2\pi f_0 \).

The impedance boundary condition is imposed by the following numerical flux:

\[
\Pi(\varphi^-, \varphi^+, n) = M(\beta)\varphi^- = \frac{1}{2} \left( (\beta + 1)n \otimes n \begin{pmatrix} \beta & -1 \\ -1 & \beta \end{pmatrix} n^T (1 - \beta) \right) \varphi^-,
\]

with \( \beta = (\hat{Z} - 1)/(\hat{Z} + 1) \), see Ref. 18 for more details. Hard wall boundary condition \( (\hat{Z} \to \infty) \) corresponds to \( \beta = 1 \), which explains the notation \( M(1) \) previously introduced.

6.2.2. Results

Fourth-order low storage explicit Runge–Kutta (LSERK) method\(^{22}\) is chosen for time-integration. In Fig. 4, is shown the pressure recorded at the middle position on the liner. After a transient regime, a permanent regime is established. The SPL and the phase on the wall of the duct opposite the liner (i.e. the upper wall) are then computed and compared to the results of the harmonic computation and to the measurements of Jones et al.,\(^{17}\) see Fig. 5. Time domain simulation agrees very well with results of harmonic computation (the two curves are almost superimposed), which shows that the impedance boundary condition is correctly implemented. A rather good agreement is obtained with the experimental results (the error is on the same order of magnitude as in Ref. 23) given the fact that in

![Fig. 4. Pressure signal measured at the middle position on the liner.](image)

![Fig. 5. SPL (a) and phase (b). Comparison between time domain computation (solid line), harmonic computation (dashed line) and experiments of Jones et al.\(^{17}\) (symbols).](image)
the experiments of Jones et al., the impedance at $f = 2000$ Hz is $\hat{Z} = 4.93 + 1.95i$, the flow profile is different (with the same section-averaged Mach number) as well as the exit impedance.

An additional 1D validation case is presented in Appendix A.

6.3. Discussion on numerical implementation and cost

The initial method (i.e. with $\varepsilon = m = 0$ and $\phi = 1$) derived in Ref. 6 has been successfully integrated in a 2D finite-volume code with zero base-flow and with nonzero slipping base-flow in Ref. 6 and in 3D CFD finite-volume code with slipping base-flow in Refs. 7 and 8. The numerical flux Eq. (23), ensuring the coupling between the duct and the cavities, is readily applicable to 3D configurations.

The 1D elements modeling cavities are placed under each control point of a triangle neighbor to the liner, see Fig. 1. For each triangle neighboring the liner, $n_t^c \times n_t \times n_v \times h/\Delta h$ degrees of freedom are added to the problem: $n_t^c$ is the number of control points per triangle edge (here $n_t^c = 3$), $n_t$ is the number of points per 1D element (here $n_t = 3$), $n_v$ is the size of the solution vector $\varphi$ in the cavities ($n_v = 2$) and $\Delta h$ is the spatial step size in the cavities. $\Delta h$ is chosen as a function of the frequency of the phenomenon of interest, for instance $f_0$ in Sec. 6.2.2. The associated wavelength in the cavities is $\lambda_0 = a_0 f_0$. Analyzing the dispersion and dissipation properties of the 1D scheme (Eq. (18)) following Ref. 24 shows that having at least 9.09 discretization points per wavelength ensures that both the dispersion and dissipations errors are below 1%. For the validation case presented in Sec. 6.2.2, $\Delta h = h/5$ is enough to satisfy this requirement. It results in 1800 1D elements added to model the cavities which represents 3.5% of the total number of degrees of freedom.

The implementation of Richter et al. requires to store 10 previous time-steps, to perform interpolation and to compute the time derivative $\partial_t u \cdot n$.

As the applications envisaged by the authors involve fully resolved boundary-layers only, it has not been chosen to discuss the implementation of Myers impedance boundary condition. Nevertheless, the proposed method does not preclude the use of Myers boundary condition which has been originally considered by Sbardella et al. and presented in Sec. 4.

7. Conclusion

In this paper, was presented an alternative method to implement the EHR impedance model based on the work of Sbardella et al. The implementation of the method in a 2D discontinuous Galerkin solver has been presented and validated by comparing the results of harmonic computation and experiments.

Extending the method of Sbardella et al. only requires to slightly modify the equations in the cavities and the equation modeling the resistive sheet. Contrary to the initial method the extended method is able to model realistic liners. While the presented numerical implementation concerns 2D nonslipping base-flow, it is readily applicable to 3D configuration: The numerical flux ensuring the coupling between the duct and the cavities has been given into a generic multi-dimensional form and the 1D scheme in the cavities remains the same.
Slipping base-flows are not considered in this paper, but Myers-type boundary condition is compatible with the method and has been originally derived by Sbardella et al.\textsuperscript{6}

As far as time domain simulations are concerned, this method might be easier to implement than the application of z-transform depending on the considered code. On one hand the extended method of Sbardella et al.\textsuperscript{6} requires to introduce additional 1D elements in the mesh where Eq. (12) is imposed, which increases the computational cost. On the other hand, application of z-transform requires to store data from previous times and either to assume the cavity depth to be a multiple of the time-step\textsuperscript{1,4} or to perform interpolations,\textsuperscript{2,5}

We believe that this method represents an interesting solution to integrate liner modeling in a standard CAA or CFD code.

Moreover, an appealing feature of this method appears when writing the governing equations as a temporal eigenproblem (i.e. where $\omega$ is the eigenvalue), as done for instance in the context of stability analysis. As a matter of fact, the resulting eigenproblem is linear with respect to $\omega$ although the EHR impedance model is nonlinear with respect to $\omega$. The authors have used this method in Ref. 25 (in French) to perform stability analysis in a lined duct with nonzero base-flow. Finally, this method might easily be extended to multiple degrees of freedom liner.

Appendix A: 1D Validation

In this section, the numerical test case proposed in Sec. 7 of Ref. 1 is considered as an example and extended in order to account to the full EHR model. It consists in computing the 1D reflection of an incident wave on a liner. The acoustic field is initialized at $t = 0$ by a Gaussian pulse of half-width $b = 1/10$ centered around $y = 1/2$: The initial condition is $v(y,t = 0) = 0$ and $p(y,t = 0) = e^{-\alpha(y-1/2)^2}$ with $\alpha = \ln 2/b^2$. At $y = 0$, a nonreflecting boundary condition is imposed and at $y = 1$ the liner of impedance Eq. (1) is placed.

Following Sec. 7 in Refs. 1 and 26, $v(1,t)$ and $p(1,t)$ are given by:

\[
\begin{align*}
  v(1,t) &= f(t) - g(t) \\
  p(1,t) &= f(t) + g(t)
\end{align*}
\]

where:

\[
\begin{align*}
  f(t) &= e^{-\alpha(1/2-t)^2} / 2 \quad \text{for } t \geq 0 \quad \text{and} \quad f(t) = 0 \quad \text{for } t < 0, \\
  g(t) &= f(t) - \frac{2}{m} \int_0^t f(t-\tau) \sum_{n=0}^{\lfloor \tau/(2h) \rfloor} e^{-\frac{R+\phi+1}{m}(\tau-2nh)-\eta L_n(-1)} \\
  &\quad \times (2\phi(\tau-2nh)/m)d\tau \quad \text{and} \quad g(t) = 0 \quad \text{for } t < 0,
\end{align*}
\]

where $L_n(-1)$ is a generalized Laguerre polynomial (see Ref. 26 for more details).

A.1. Numerical method

Equation (18) is used as 1D discontinuous Galerkin scheme, but now test and interpolation functions are chosen in $P^6(D_l)$, the space of sixth-order polynomials defined on $L_l$. The computational domain is $\Omega = \{y \in [0,1] \cup [1,1+h]\}$ where $[1,1+h]$ represents a cavity.
Implementation of EHR Impedance Model in Time Domain

Fig. A.1. Pressure (solid line) and velocity (dotted line) time-evolution at the lining. The circle and square symbols correspond respectively to the velocity and the pressure obtained analytically.

The 1D characteristics outflow condition at $y = 0$ is enforced by the flux $\Pi(\varphi^-, n) = -[Ay^\eta]^-\varphi^-$. 

A.2. Results

The liner parameters are $R = 1.5$, $h = 1/3$, $m = 10^{-4}$, $\varepsilon = 0.5$ and $\phi = 1.7$. In Fig. A.1 are shown the time-evolutions of the pressure and velocity at the lining obtained with the method developed in Sec. 5 and analytically. Since there is no acoustic source, only the transient regime is of interest (in the permanent regime, the solution equals zero). The analytical solution is computed by discretizing the integrand of Eq. (A.1) with the same time-step as for the numerical simulation $\Delta t \approx 3 \times 10^{-3}$. The mesh in $[0, 1]$ and in the cavity $[1, 1+h]$ is uniform with $\Delta y = 1/15$.

The results between both methods are identical to the numerical precision.

References