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Finite-difference numerical modelling of gravitoacoustic wave propagation in a windy and attenuating atmosphere

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SUMMARY
Acoustic and gravity waves propagating in planetary atmospheres have been studied intensively as markers of specific phenomena such as tectonic events or explosions or as contributors to atmosphere dynamics. To get a better understanding of the physics behind these dynamic processes, both acoustic and gravity waves propagation should be modelled in a 3-D attenuating and windy atmosphere extending from the ground to the upper thermosphere. Thus, in order to provide an efficient numerical tool at the regional or global scale, we introduce a finite difference in the time domain (FDTD) approach that relies on the linearized compressible Navier–Stokes equations with a background flow (wind). One significant benefit of such a method is its versatility because it handles both acoustic and gravity waves in the same simulation, which enables one to observe interactions between them. Simulations can be performed for 2-D or 3-D realistic cases such as tsunamis in a full MSISE-00 atmosphere or gravity-wave generation by atmospheric explosions. We validate the computations by comparing them to analytical solutions based on dispersion relations in specific benchmark cases: an atmospheric explosion, and a ground displacement forcing.

Key words: Numerical solutions; Acoustic-gravity waves; Tsunamis; Earthquake ground motions; Computational seismology; Wave propagation.

1 INTRODUCTION
Propagation of acoustic and gravity waves in the atmosphere of planets has a wide range of scientific interests, from the interplay between these waves and atmosphere dynamics to the detection of tectonic events. Historically, this research topic was initially supported by ground-based observations of atmospheric infrasounds (see Le Pichon et al. (2010) for a review) and observations of thermospheric gravity waves through air-glow measurements, or electron content variations in the ionosphere (Hines 1960). Over the past twenty years the development of new observation tools allowing to recover electron density variations in the ionosphere (such as GNSS receivers located on the ground or in satellites, ionosondes, over-the-horizon and incoherent scattering radars...) has enabled the study of additional phenomena such as the emission of infrasounds by seismic surface waves or volcanic eruptions, as well as the emission of gravity waves by tsunamis or by large-scale atmospheric disturbances. Understanding these physical processes required the development of new tools capable of modelling wave propagation from the ground to the upper thermosphere (Lognonné et al. 1998; Occhipinti et al. 2006), and coupling with the ionosphere (Kherani et al. 2009). Recently, new types of observations based on air-glow emissions (Garcia et al. 2009 Makela et al. 2011;) or in-situ measurements of air density in very low Earth-orbit satellites (Garcia et al. 2013, 2014) have provided respectively an increase of space/time coverage and resolution. Making optimal use of such improved precision and resolution in observations requires more sophisticated and accurate modelling tools. Thus, the propagation of both acoustic and gravity waves should be studied in a windy 3-D atmosphere model, including the thermosphere.

In order to provide realistic modelling at the regional or the global scale, physical simulations should include effects of attenuation, heterogeneous and realistic atmosphere models and strong wind perturbations. In this paper, we present a first step towards this complex goal through the modelling of acoustic and gravity wave propagation in a planetary atmosphere based on a finite-difference numerical technique.

In a fluid two main approaches can be used, one based on a linearization of the full Navier–Stokes equations (Nappo 2002) and another one based on a decomposition of the gravitoacoustic equations in terms of potentials (Chaljub 2000). A third one, the full Navier–Stokes equations embedding nonlinearities is also sometimes used (e.g. for shock capturing or the study of turbulence) for atmospheric applications. In the context of nonlinearities, Lecoanet
et al. (2015) studied gravity wave generated by interface or Reynolds stress forcing in a coupled ocean/atmosphere model. Taking into account nonlinearities they solved the 2-D incompressible Navier–Stokes equations in a Fourier domain along x and over a Chebyshev grid along z. Wilson et al. (2004) studied 3-D acoustic inviscid wave propagation based on finite differences and included turbulence and wind in their modelling. They provided a tool to study scattering phenomena affecting atmospheric remote-sensing systems. Finally, Snively & Pasko (2008) solved 2-D Navier–Stokes equations for gravity waves with both wind and viscosity based on a finite volume method and focused on ducted gravity waves in the lower thermosphere. Another approach called the General Circulation Model, based on the compressible Navier–Stokes equations taking into account the Coriolis force but without gravity, gives interesting results about gravity wave propagation in a windy atmosphere (Miyoshi et al. 2014).

In the potential formulation, one makes the time evolution of the perturbations derive from a displacement potential and a gravity potential. In the presence of bulk attenuation only, such a decomposition into potentials can easily be applied (Chaljub 2000). However, in the presence of deviatoric stress and/or of wind, this field representation is not valid anymore because the potentials will not fully describe the solution of the Navier–Stokes equations (Valette 1987).

In this work, we thus use the acoustic and advection parts of the compressible and viscous linearized version of the Navier–Stokes equations. As we will see below this system of equations allows one to couple gravity, wind velocity effects and acoustic wave propagation in the same unified numerical framework. Accuracy and limitations of the linear approximation were studied by Dörnbrack & Nappo (1997) by comparing the results of a linear model with a nonlinear, time-dependent, hydrodynamic numerical model. They pointed out that similar results are obtained from linear and nonlinear models for wave stress, wave breaking height and wave dissipation through the critical level (Nappo 2002).

Linearization of the Navier–Stokes equations has been proposed by different authors: de Groot-Hedlin et al. (2011) resorted to a 2-D finite-difference discretization but focused only on acoustic waves for realistic atmosphere models with wind and sound speed gradients. Ostashev et al. (2005) used the same discretization and considered 2-D gravity waves but without atmospheric viscosity. Both articles considered atmospheric sources only. Mikhailenko & Mikhailov (2014) relied on 2-D Laguerre/Fourier discretization to study low-altitude inviscid gravity waves in simple atmosphere models. Finally, Wei et al. (2015) focused on the tropopause and inviscid gravitoacoustic waves in the low atmosphere by means of a spectral/Laplace method. That study used a ground forcing technique in order to model tsunami-induced gravity waves: however to our knowledge acoustic-gravity wave propagation with stratified profiles of wind and strongly varying density, sound speed and viscosity has never been implemented in 3-D.

Atmospheric attenuation is crucial for realistic simulations. Landau & Lifshitz (1959), Coulouvrat (2012) and Godin (2014) have established a formulation of the dynamic and volume viscosities and also developed analytical solutions for the evolution of pressure in the frequency domain in the presence of bulk and/or shear viscosity. In our simulations, we will take into account both processes and their fluctuations through altitude because attenuation parameters vary strongly owing to the drastic density decrease when altitude increases (Godin 2014).

In terms of numerical implementation, for spatial discretization we will use a classical staggered grid (Yee 1966; Madariaga 1976) because it provides an efficient and stable way of reaching high order for the discretization. This grid is widely used for wave propagation in solid and fluid media (Graves 1996; Chaljub et al. 2007) but to our knowledge the fourth-order implementation has not been used before for atmospheric studies. Another version of a staggered grid for the atmosphere has been used in Ostashev et al. (2005) and de Groot-Hedlin et al. (2011) in particular to treat advection terms. Contrary to these articles, here we perform the implementation of advection terms through upwind (non-centred backward/forward) schemes (Ferziger & Peric 2012) to take into account wind velocities of different signs and to avoid possible stability issues, mainly at outgoing boundaries. We will validate our numerical technique by making comparisons with analytical solutions derived for benchmark cases for the different physical features involved.

In this paper, we first recall the governing equations, including their linearization and decomposition in terms of wind advective components and propagative perturbation components (acoustic and gravity waves). We then describe the wave attenuation parameters and link them to the parameters usually used in the acoustic and geophysics communities. We also introduce the finite-difference numerical implementation and validate the 2-D code by performing comparisons to analytical solutions in simplified atmosphere models. We present examples of 2-D applications for atmosphere bottom forcing by tsunamis and by seismic waves, and then for atmospheric explosions in realistic atmosphere models. We finally validate the 3-D code by performing comparisons to analytical solutions in simplified atmosphere models.

2 LINEAR GRAVITO-ACOUSTIC PROPAGATION IN A WINDY, ABSORPTIVE MEDIUM STABLY STRATIFIED

2.1 Governing equations

In this section, we recall how the Eulerian form of the equations of motion is derived from the Eulerian momentum, mass conservation and state equations. One starts from the conservation of energy (Vallis 2006)

$$D_t I = D_t Q - P D_t (V/\rho_T),$$

(1)

where $D_t = \partial_t + V \cdot \nabla$ denotes the Lagrangian derivative, $I$ is the internal energy, $Q$ is the heat input to the body, $P$ the pressure, and $\rho_T$ the atmospheric density. From the Eulerian formulation of the momentum equation (Landau & Lifshitz 1959),

$$\rho_T D_t V - \nabla \cdot \Sigma_T = \mathcal{F}_{ex} = \rho_T G,$$

(2)

in which $V$ is the velocity, $\Sigma_T$ the Eulerian stress tensor and $\mathcal{F}_{ex}$ an external volumic force, equal to gravity forces in our case, where $G$ is the gravitational acceleration, and from the mass conservation equation

$$D_t \rho_T = -\rho_T V \cdot V$$

(3)

the following assumptions are then made:

(i) The atmosphere is considered as a Newtonian fluid. Thus in the Eulerian description, the stress tensor reads

$$\Sigma_T_{ij} = -P \delta_{ij} + (\Sigma_T^v)_{ij},$$

(4)

where $P$ is the pressure, $\Sigma_T^v$ is the viscous stress tensor and $\delta$ the Kronecker symbol.

(ii) The atmosphere is considered as an ideal gas

$$d I = C_d dT$$

and $P = \rho_T R T$, 

(5)
where $P$ is the pressure, $R = C_p - C_v$ the gas constant, $C_v$ the heat capacity at constant volume, $C_p$ the heat capacity at constant pressure, $T$ the temperature and $\rho T$ the atmospheric density.

(iii) State variables can be split into a stationary component (subscript 0) and a small space/time variable component (subscript 1):

$$P = P_0 + P_1; \quad \rho T = \rho_0 + \rho_1; \quad \mathbf{G} = \mathbf{G}_0 + \mathbf{G}_1;$$

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1; \quad \mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1; \quad \mathbf{\Sigma}_T = \mathbf{\Sigma}_0 + \mathbf{\Sigma}_1$$

(6)

where $P, \rho T, \mathbf{G}, \mathbf{U}, \mathbf{V}$ are respectively the pressure, atmospheric density, gravitational acceleration, displacement and velocity.

(iv) The atmosphere is stratified and thus physical parameters $\rho_0, \mathbf{G}_0, \eta, \mu, \mathbf{V}_0$ (respectively the atmospheric density, volume viscosity, dynamic viscosity and wind velocity) only vary along $z$.

(v) The background velocity $\mathbf{V}_0$ is a stationary stratified horizontal wind, that is, $\mathbf{V}_0(\mathbf{x}) = \mathbf{V}_{0,x}(z) \cdot \mathbf{e}_x + \mathbf{V}_{0,y}(z) \cdot \mathbf{e}_y$, where $\mathbf{x} = (x, y, z)$. This assumption will lead to a divergence-free wind ($\nabla \cdot \mathbf{V}_0 = 0$) and remove the influence of background wind on the hydrostatic equilibrium specified in assumption (vi).

(vi) The hydrostatic equilibrium is considered as a reference state

$$\mathbf{\Sigma}_0 = -P_0 \mathbf{I}_d$$

(7)

with $\mathbf{\Sigma}_0$ the reference state tensor, $P_0$ the background pressure and $\mathbf{I}_d$ the identity tensor in $\mathbb{R}^3$. By assuming that the initial atmosphere is stratified and at hydrostatic equilibrium (2) one can formulate an equation describing this initial state as

$$\nabla \cdot \mathbf{\Sigma}_0 + \rho_0 \mathbf{G}_0 = 0.$$  (8)

By injecting eq. (7) into eq. (8) one obtains

$$\rho_0 \mathbf{G}_0 = -\nabla P_0.$$  (9)

(vii) We will make a linear assumption, that is, we will neglect second-order terms by removing the $O(u^2)$ terms.

(viii) The wave perturbations are considered close to the adiabatic condition: $D_t Q = 0$.

(ix) One makes the Cowling approximation (Cowling 1941) for the gravitational field. It consists in ignoring perturbations in the gravitational field, such that

$$\rho_1 \mathbf{G} = \rho_0 \mathbf{G}_0 + \rho_1 \mathbf{G}_0.$$  (10)

(x) We consider a regional scale domain and neglect the Coriolis force.

Hypothesis (ii) can be recast in a more convenient form

$$D_t I = C_v D_t T \quad \text{and} \quad T = \frac{P}{\rho T} \Rightarrow D_t I = \frac{C_v}{R} D_t (P/\rho T).$$  (11)

 Injecting it into the energy conservation eq. (1) and taking into account the adiabatic condition (viii) with $P_0$ being the adiabatic pressure, this yields

$$\frac{C_v}{R} D_t \left( \frac{P}{\rho T} \right) = -P D_t (1/\rho T)$$

$$\frac{C_v}{R} \left( \frac{1}{\rho T} \right) D_t P_0 + \frac{P_0 D_t (1/\rho T)}{\rho T} \right) = -P_0 D_t (1/\rho T)$$

$$\frac{C_v}{R} \left( \frac{1}{\rho T} \right) D_t P_0 - \frac{P_0 D_t (1/\rho T)}{\rho T} \right) = P_0 D_t (1/\rho T)$$

$$D_t P_u = P_u \frac{D_t \rho T}{\rho T} \left( \frac{R}{C_v} + 1 \right).$$  (12)

Since we used the adiabatic assumption to get eq. (12), we will only work with the adiabatic pressure $P_a$. Thenceforth, we will use the notation $P = P_a$, where $P$ will refer to the adiabatic pressure.

Combining this with the mass conservation eq. (3) yields

$$D_t P = -P \gamma \nabla \cdot \mathbf{V}$$

(13)

where $\gamma = \frac{C_p}{C_v}$ is the specific heat ratio.

One then gets a coupled system of equations: the pressure evolution eq. (13) and the Eulerian form of the momentum eq. (2) (details can be found in Vallis (2006) and Chaljub (2000)):

$$D_t P = -P \gamma \nabla \cdot \mathbf{V}$$

(14)

$$\rho T D_t \mathbf{V} - \nabla \cdot \mathbf{\Sigma}_T = \rho T \mathbf{G}.$$  (15)

Unknowns are then split into ambient and perturbation values (iii), and from the linear hypothesis (vii), the reference state considered (vi) and the Cowling approximation (ix), eq. (14)-1 then reads (see e.g. Chaljub (2000))

$$\partial_t P_1 + (\mathbf{V}_0 + \mathbf{V}_1) \nabla (P_0 + P_1) = -(P_0 + P_1) \gamma \nabla \cdot \mathbf{V}_1$$

$$\partial_t P_1 = -\nabla \cdot \nabla \mathbf{P}_1 - \rho_0 c^2 \nabla \cdot \mathbf{V}_1$$

(16)

$\rho_0$ is the atmospheric density, volume viscosity, dynamic viscosity and wind velocity.

Considering the divergence-free background wind (v) and the linear assumption (vii) one then has

$$\partial_t P_1 = \nabla \cdot (\mathbf{V}_0 \rho_1 + \mathbf{V}_1 \rho_0)$$

(17)

$$\partial_t \rho_1 = -\nabla \cdot (\mathbf{V}_0 \rho_1 + \mathbf{V}_1 \rho_0)$$

(18)

Now turning to the momentum equation, considering the divergence-free background wind (v) and the linear assumption (vii), (14)-2 reads

$$\rho_0 \partial_t \mathbf{V}_1 = -\rho_0 (\mathbf{V}_1 \cdot \nabla) \mathbf{V}_0 + (\mathbf{V}_0 \cdot \nabla) \mathbf{V}_1 + \nabla \cdot \mathbf{\Sigma}_1 + \mathbf{G}_0 \rho_1.$$  (19)

Combined with the static equilibrium eq. (8) this yields

$$\rho_0 \partial_t \mathbf{V}_1 = -\rho_0 (\mathbf{V}_1 \cdot \nabla) \mathbf{V}_0 + (\mathbf{V}_0 \cdot \nabla) \mathbf{V}_1 + \nabla \cdot \mathbf{\Sigma}_1 + \mathbf{G}_0 \rho_1.$$  (20)

Using eqs (15), (18) and (20), the whole system (14) then reduces to:

$$\partial_t P_1 = -\nabla \cdot \nabla P_1 - \rho_0 c^2 \nabla \cdot \mathbf{V}_1 - \rho_0 \mathbf{V}_1 \mathbf{G}_0$$

$$\partial_t \rho_1 = -\nabla \cdot \nabla \rho_1 - \nabla \cdot (\rho_0 \mathbf{V}_1)$$

$$\rho_0 \partial_t \mathbf{V}_1 = -\rho_0 (\mathbf{V}_1 \cdot \nabla) \mathbf{V}_0 + (\mathbf{V}_0 \cdot \nabla) \mathbf{V}_1 + \nabla \cdot \mathbf{\Sigma}_1 + \mathbf{G}_0 \rho_1.$$  (21)
where the stress tensor $\Sigma_1$, under assumption (i), reads, $\forall (i, j) \in [1, 3] \times [1, 3]$,
\[
(\Sigma_1)_{ij} = -P_i \delta_{ij} + \mu \left( \partial_i V_j + \partial_j V_i - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{V} \right) + \eta V_i \delta_{ij} \nabla \cdot \mathbf{V},
\]
(22)
where $\delta$ is the Kronecker symbol.

To simplify the writing in what follows we will drop subscripts and write
\[
\rho_0 = \rho; \ P_0 = \rho c^2 \ V_0 = 1; \ U_1 = 1; \ \Sigma_1 = \Sigma.
\]
(23)
Eq. (21) then reads
\[
\partial_t p = -\mathbf{w} \cdot \nabla p - \rho c^2 \nabla \cdot \mathbf{v} - \rho \mathbf{v} \mathbf{g} - \rho \mathbf{v} \mathbf{g} - \rho \mathbf{v} \mathbf{g}.
\]
(24)
With these notations (23) the stress tensor $\Sigma$ then reads, $\forall (i, j) \in [1, 3] \times [1, 3]$,
\[
(\Sigma)_{ij} = -P_i \delta_{ij} + \mu \left( \partial_i V_j + \partial_j V_i - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{V} \right) + \eta V_i \delta_{ij} \nabla \cdot \mathbf{V}.
\]
(25)
The system of eq. (24) describes simultaneously the propagation of both acoustic and gravity waves in a viscous fluid subject to wind. Note that in order to establish eq. (24), we did not use the stratified atmosphere assumption (iv) for density, adiabatic sound speed, viscosity nor gravity but only for wind profiles. It means that any 3-D varying profile of atmosphere can be considered for background parameters besides background wind. However, in simulations presented later on in this paper, for validations or applications, we only considered stratified media since it enabled us to get simple analytical solutions. Expansion of eq. (24) in component form can be found in Appendix A. In the remainder of this paper, we will refer by ‘wind-convective’ terms to the following terms in eq. (A1):

(a) $\mathbf{w} \cdot \nabla p$, $\mathbf{w} \cdot \nabla \rho$

(b) $(\mathbf{w} \cdot \nabla) \mathbf{w}$

(c) $(\mathbf{v} \cdot \nabla) \mathbf{w}$.

2.2 Atmospheric viscosity and acoustic attenuation

Atmospheric absorption can occur by two main mechanisms (Bass et al. 1984): classical losses due to dissipation of mechanical energy and relaxation losses due to the conduction of heat energy. The dynamic viscosity $\mu$, due to shear stress applied to a fluid, reads (Zuckerwar & Ash 2006; Bass & Chambers 2001)
\[
\mu = \frac{2}{3} L p c \sqrt{2/\pi} \gamma,
\]
(26)
where $L$ is the mean free path, $c$ the adiabatic sound speed (16) and $\gamma = C_p / C_v$, the specific heat ratio. The volume viscosity $\eta V$ due to the relaxation of dilatational disturbances (i.e. heat conduction and molecular relaxations) reads
\[
\eta V = \frac{4}{3} \mu + \frac{(\gamma - 1) \kappa}{\gamma C_v},
\]
(27)
where $\mu$ is the dynamic viscosity, $\gamma = C_p / C_v$, the specific heat ratio, $C_v$ the molar low-frequency specific heat at constant volume and $\kappa$ the thermal conductivity. The acoustic absorption coefficient $\alpha$ (in $\text{m}^{-1}$) describes the frequency dependence of the attenuation process. This coefficient is the imaginary part of the wavenumber $k = Re(k) - i\alpha$ (Landau & Lifshitz 1959); From Bass & Chambers (2001), it writes
\[
\alpha(f) = \frac{2(\pi f)^2}{\rho c^3 \eta V}.
\]
(28)
When acoustic or seismic waves are modelled, Zener, Maxwell or Kelvin–Voigt are commonly used to introduce attenuation effects in the time domain (Moczo & Kristek (2005) show that several of these models are equivalent). Viscoelasticity in solids, modelled using the Zener model in the time domain, is introduced in the discretized equations through memory variables (Carcione 2014). Doing so avoids having to explicitly handle a convolution process with the whole past of the viscoelastic material, which is a complicated process from a numerical point of view (Carcione et al. 1988; Moczo 1989; Robertsson et al. 1994). However, in the Earth atmosphere, volume and dynamic viscosities tend to act as a Kelvin–Voigt viscoelastic mechanism. For a Kelvin–Voigt solid, one can represent the absorption coefficient, which is proportional to the inverse of its quality factor, as a function of frequency. Using this formulation, we will show in a simple case that this choice of viscoelastic mechanism is reasonable by comparing its absorption coefficient to the theoretical one in eq. (28).

We consider a simple homogeneous (i.e. with constant density and sound velocity) atmosphere model in which the volume viscosity $\eta V$ is constant and the shear viscosity is not taken into account (for $i \neq j$, $\Sigma_{ij} = 0$). We also neglect background velocity ($\mathbf{w} = 0$) and gravitation ($\mathbf{g} = 0$). Eq. (24) then yields
\[
\partial_t p = -\rho c^2 \nabla \cdot \mathbf{v}
\]
(29)
\[
\partial_t \rho = -\nabla \cdot (\rho \mathbf{v})
\]
(30)
\[
\rho \partial_t \mathbf{v} = \nabla \cdot \Sigma,
\]
where
\[
\Sigma = (-p + \eta V \nabla \cdot \mathbf{v}) \mathbf{I}_d,
\]
(31)
while $\mathbf{I}_d$ being the identity tensor in $\mathbb{R}^3$. After replacing the pressure term in eq. (29)-3 with the primitive of eq. (29)-1, one obtains the formulation of the stress–strain relationship for a Kelvin–Voigt solid, as described for instance by Carcione (2014), eq. (2.159):
\[
(\Sigma)_{ij,1,\ldots,3,\ldots,3} = (M \mathbf{e} + \eta \mathbf{e}) \delta_{ij},
\]
(32)
where $\Sigma$ is the stress tensor, $M \mathbf{e}$ the bulk modulus, $\eta \mathbf{e}$ the bulk viscosity and $\mathbf{e}$ the strain, defined from the displacement $\mathbf{u}$ by $\mathbf{e} = \nabla \cdot \mathbf{u}$. Unknowns are assumed to be Fourier functions in time and space such that
\[
\mathbf{u}(x, t) = u_0 e^{i(k x - \omega t)}.
\]
(33)
From eq. (29), the dispersion relation in this case reads
\[
k^2 \left( 1 - \frac{\omega^2}{2D} \right) - \left( \frac{\omega}{D} \right)^2 = 0,
\]
(34)
where $D = \frac{\rho c^2}{\eta V}$, which yields
\[
k^2 = \left( \frac{\omega}{D} \right)^2 \left( 1 + \frac{\omega^2}{2D} \right).
\]
(35)
The quality factor then reads (Carcione et al. 1988)
\[
Q = \frac{Re(k^2)}{Im(k^2)} = D = \frac{\rho c^2}{2\pi f \eta V}.
\]
(36)
One then gets the final dispersion relation by taking the square root of eq. (34):

\[
k \approx w \sqrt{2c} \left\{ \frac{1}{1 + (1/Q)^2} + \frac{1}{[1 + (1/Q)^2]^{1/2}} \right\}^{1/2} + i \frac{w}{\sqrt{2c}} \left\{ \frac{1}{1 + (1/Q)^2} - \frac{1}{[1 + (1/Q)^2]^{1/2}} \right\}^{1/2}. \tag{36}
\]

This expression enables one to compute both the phase velocity and the absorption coefficient. From Carcione (2014), the phase velocity reads

\[
v_\phi = \frac{\omega}{Re(k)}, \tag{37}
\]

where \(Re(k)\) is the real part of the wavenumber \(k\). Eq. (36) then yields

\[
v_\phi = \sqrt{2c} \left( \frac{1}{1 + (1/Q)^2} + \frac{1}{[1 + (1/Q)^2]^{1/2}} \right)^{-1/2}. \tag{38}
\]

Considering altitudes below typically 400 km and low-frequency signals smaller than typically 1 Hz, it is reasonable to make the assumption \(Q \gg 1\) and then to develop this expression to the second order in \(1/Q\). Doing so the phase velocity (38) can then be written as

\[
v_\phi \approx \frac{c}{1 - \frac{3}{8} \left( \frac{1}{Q} \right)^2}. \tag{39}
\]

This expression is the one given by Blackstock (2000), page 306. Consequently, acoustic wave propagation in an attenuated medium with bulk viscosity follows the Kelvin–Voigt relation and is dispersive. However, the \((1/Q)^2\) term is usually ignored in acoustics because it is a second-order term in \(1/Q\). The wavenumber reads

\[
k = Re(k) - i\alpha, \quad \alpha = \text{the absorption coefficient}
\]

\[
\alpha^2 = -\left( \frac{\omega}{c} \right)^2 \frac{1}{2(1 + (1/Q)^2)} \left( 1 - \sqrt{1 + (1/Q)^2} \right). \tag{40}
\]

By a Taylor expansion when \(Q \gg 1\), one then gets

\[
\alpha \approx \left( \frac{\omega}{c} \right) \frac{1}{2Q}, \tag{41}
\]

\[
\alpha \approx \frac{2(\pi f)^2}{\rho c^3} \eta_f. \tag{42}
\]

The Kelvin–Voigt absorption coefficients \(\alpha\) and \(Q\) can thus be defined in terms of volume viscosity and frequency according to formulae (35) and (42). This result has been extended by Godin (2014) to the full absorption process, choosing into account both shear and volume viscosities. Eq. (9) in Godin (2014) shows that the traditional choice of picking constant coefficients leads to substantial quantitative errors, and in the infrasound limit Eq. (12) in that article gives a similar result as eq. (28). Finally, the background flow that causes the Doppler effect will shift the wave frequency and will thus impact its absorption. Variations of viscosity coefficients (\(\mu\) and \(\eta_f\)) with altitude and background flows will be taken into account in our numerical simulations.

### 3 ATMOSPHERE MODELS

We will simulate wave propagation in several atmosphere models. We will first use simplified models for validation of our numerical technique with respect to analytical solutions. In these first models, all atmospheric parameters will be set to constant values. We will then design an isothermal atmosphere model to test the stability of the calculations relative to the exponential density decrease in the atmosphere. We will finally build a more realistic atmosphere model from empirical atmosphere models with only vertical variations of the parameters.

#### 3.1 Isothermal model

In order to infer the validity of our computations relative to the exponential density decrease in the atmosphere, we first create an isothermal model. Air density and pressure decrease exponentially with a constant scale height, all other parameters being constant. The values of these constant parameters are representative of those observed in the thermosphere and are summarized in Table 1. The model is built assuming an ideal gas in hydrostatic equilibrium at constant temperature (Nappo 2002).

#### 3.2 Realistic atmosphere model

In order to verify the stability of our calculations relative to realistic vertical variations of atmospheric parameters, we create a model that exhibits only vertical variations of the atmospheric parameters that are extracted from the MSISE-00 atmosphere model (Picone et al. 2002), and from the HWM93 atmospheric wind model (Hedin 1991) when atmospheric winds are included. The thermodynamic properties of the atmospheric compounds are extracted from the NIST web-book database (http://webbook.nist.gov/chemistry/). We extracted a vertical profile of these atmospheric properties for conditions corresponding to a surface point at latitude 36.5°, longitude 158.7°, at 7:47:40 UTC on 2011 March 11. This space and time location corresponds to the coordinates of the crossing between the post-seismic infrasonic waves generated by the Tohoku earthquake in Japan and the GOCE satellite (Garcia et al. 2013). This vertical profile is extended in 2-D and 3-D by invariant prolongation of the whole set of physical properties in the direction orthogonal to the 2-D plane. Density, adiabatic sound speed and Brunt–Väisälä frequency versus altitude are presented in Fig. 1. Other charts can be found in Appendix C.

### 4 NUMERICAL DISCRETIZATION

Time discretization is carried out based on a fourth-order Runge–Kutta scheme and spatial discretization is based on a fourth-order staggered scheme. We have performed comparisons, not shown here, that demonstrate that in our case there is no significant benefit of using a more sophisticated scheme such as a low-dissipation and low-dispersion fourth-order Runge–Kutta algorithm (LDDRK; Berland et al. 2006). For spatial discretization, we use the following stencil (Fig. 2):

For a scalar unknown \(u\) computed at time step \(m\) and at grid point \((i, j, k)\),

\[
u_m^{i,j,k} = u(i \Delta x, j \Delta y, k \Delta z, m \Delta t), \tag{43}
\]
within the domain $\Omega$ the finite-difference operators read

$$
(\partial_x u)^{i,j,k}_m = \frac{27 u^{i+1,j,k}_m - 27 u^{i,j,k}_m - u^{i+2,j,k}_m + u^{i-1,j,k}_m}{24\Delta x},
$$

$$
(\partial_y u)^{i,j,k}_m = \frac{27 u^{i,j+1,k}_m - 27 u^{i,j-1,k}_m - u^{i,j+2,k}_m + u^{i,j-2,k}_m}{24\Delta y},
$$

$$
(\partial_z u)^{i,j,k}_m = \frac{27 u^{i,j,k+1}_m - 27 u^{i,j,k-1}_m - u^{i,j,k+2}_m + u^{i,j,k-2}_m}{24\Delta z}.
$$

Note that in eq. (24) the pressure and density perturbation evolution equations require the calculation of $\nabla p$ and $\nabla \rho$ at the same spatial location as, respectively, $p$ and $\rho$, in term (a). We select a non-staggered upwind or downwind scheme depending on the sign of $w_u$ in order to properly treat the advective terms (Ferziger & Peric 2012). The instability that can otherwise arise from a centred scheme comes from the fact that the flow goes from upstream to downstream and thus the derivative computed at any point should not take into account information downstream since it has no physical meaning. Similar instabilities appear when using the staggered grid described in eq. (44). This upwind/downwind scheme writes:

if $w_u < 0$ \[ (\partial_x p)^{i,j,k}_m = \frac{1}{6\Delta x} \{ 2(p^{i+1,j,k}_m - p^{i-1,j,k}_m) \]

\[ + 6(p^{i+1,j,k}_m - p^{i-1,j,k}_m) - (p^{i+2,j,k}_m - p^{i-2,j,k}_m) \} \]

if $w_u > 0$ \[ (\partial_x p)^{i,j,k}_m = \frac{1}{6\Delta x} \{ 2(p^{i+1,j,k}_m - p^{i-1,j,k}_m) \]

\[ + 6(p^{i-1,j,k}_m - p^{i+1,j,k}_m) - (p^{i-2,j,k}_m - p^{i+2,j,k}_m) \}. \]
scheme). It exhibited instabilities when performing various tests on atmospheric backgrounds with a strong wind (of about 100 m s\(^{-1}\)) and a ‘high-frequency’ wave generated by a point source (of about 5 s dominant time period). We thus choose to use the discretization (44)-(45) instead.

Regarding boundary conditions, we perform simulations in a simple Cartesian mesh in which \(\Delta x = \Delta y = \Delta z\). On the left and right boundaries of the domain, we implement periodic boundary conditions. This implies that the atmosphere model should be continuous between the right and left boundaries, which is the case since our models only vary along \(z\). On the top edge of the grid, referred to as \(\Gamma_D\), we enforce a homogeneous Dirichlet boundary condition that consists in imposing, at any time \(t\), for \(x \in \Gamma_D\),

\[
u(x, t) = 0. \tag{46}
\]

This choice has no real physical meaning but is implemented here for simplicity. It will lead to reflection when waves hit the boundary, but for large-enough meshes this choice has no measurable impact on signals observed. When simulating an atmospheric explosion, that is, when the source is located inside the grid, we apply Dirichlet boundary conditions (46) on the bottom boundary as well.

In the other cases, that is, when the seismic source is located outside the grid, we apply a forcing boundary condition along the bottom edge of the grid to simulate incoming seismic waves impinging from the bottom: at any time \(t\), for \(x \in \Gamma_F\), for \(f \in \mathbb{R}\)

\[
u(x, t) = f(x, t), \tag{47}
\]

where \(f\) is the forcing function.

At the edges of the computational grid, the discretization (44) requires the computation of unknown terms at position \(j = 0, -1\). We compute these terms using a mirror condition, meaning that for all spatial wavelengths \(\lambda\),

\[
p_{m, j}^{(i, 0, k)} = 2p_{m, j}^{(i, 1, k)} - p_{m, j}^{(i, 2, k)} \tag{48}
\]

which is the linear interpolation of \(p_{m, j}^{(i, 1, k)}\) from neighbouring values \(p_{m, j-1}^{(i, 1, k)}\), \(p_{m, j+1}^{(i, 1, k)}\) expressed in terms of \(p_{m, j}^{(i, 0, k)}\). The same holds for \(p_{m, j}^{(i, -1, k)}\), such that

\[
p_{m, j}^{(i, -1, k)} = 2p_{m, j}^{(i, 0, k)} - p_{m, j}^{(i, 1, k)} \tag{49}
\]

Since we will often perform simulations over large domains we resort to parallel computing implemented using the Message-Passing Interface libraries (Gropp et al. 1994), decomposing the mesh into regular slices cut along the \(x\)-coordinate axis.

5 2-D ACOUSTIC WAVE VALIDATION

5.1 Construction of the analytical solutions

We compute the analytical solution in the time domain for validation purposes for each test case using the following process:

(i) Calculation of the forcing signal for the whole time domain along the forcing boundary or at the point source,

(ii) Calculation of the 3-D (or 2-D) Fourier transform (spatial and time transformations) of that function,

(iii) Calculation of wavenumbers \(k_x = 2\pi / \lambda_x\) and \(k_y = 2\pi / \lambda_y\), for all spatial wavelengths \(\lambda_{x,y}\),

(iv) Calculation of \(k\), from dispersion relations for all wavenumbers \(k_x, k_y\), and time frequencies (see Appendix B),

(v) Multiplication, in the Fourier domain, of the forcing function with a complex filter based on the representation of the solution in the case of an harmonic source or forcing term (see Appendix B for more details),

(vi) Calculation of the inverse Fourier transform of the resulting at the recording stations to obtain the solution in the time domain.

5.2 Bottom ‘high-frequency’ forcing in a windless atmosphere with exponentially decaying density and without attenuation

The first validation step concerns acoustic waves and the underlying physical processes of dispersion and amplitude growth with altitude.

We consider the following forcing function, \(\forall x \in \Gamma_F:\)

\[
f(x, t) = e^{-\frac{t - t_0}{P} - \frac{P}{\rho v_p}} - e^{-\frac{t - t_0}{P} + \frac{P}{\rho v_p}}, \tag{50}
\]

where \(P\) is the time period of the forcing signal and \(t_0\) the starting forcing time. We set \(P = 60\) s and \(t_0 = 55\) s.

The atmosphere is considered isothermal and described in Table 2.

Two particular features for atmospheric waves associated with density variations can be noticed in Fig. 3: first, the amplification of vertical velocity/displacement amplitude due to the decrease of atmospheric density, since kinetic energy \(E_k \propto \rho \| \mathbf{v} \|^2\) is conserved; second, the dispersion effect on the waveform. This latter point is due to the frequency dependence of phase velocity \(v_p = v_p(\omega, c, H)\) (Landau & Lifshitz 1959).

5.3 Bottom ‘high-frequency’ forcing in an attenuating, windless atmosphere with exponentially decaying density

Here let us study and validate the effect of viscosity on acoustic waves. To do so in eq. (50), we set \(P = 15\) s and \(t_0 = 30\) s. These parameters are slightly different from the previous case because the absorption coefficient \(\alpha\) in (28) is frequency dependent and thus in order to clearly see its effect we need to select a frequency larger than in the previous case. The atmosphere is considered isothermal and described in Table 3.

Fig. 4 shows a good fit between the analytical and numerical signals in terms of both wave amplitude and traveltime, the error being less than 5 per cent in maximum amplitude over time. Several physical phenomena can be observed: first, the decay in amplitude due to the atmospheric viscosity. In this case, only the volumic viscosity impacts the propagation because acoustic (pressure) waves are not sensitive to shear stress. The other phenomenon is the apparent frequency dispersion, coming from the fact that the absorption coefficient (28) is frequency dependent and thus high frequencies are more attenuated than lower ones, which leads to a larger apparent period for the attenuated signal than for the non-attenuated one.

Table 2. Simulation parameters for the isothermal model 3.1 without considering attenuation, used in Simulation 5.2, that is, the case of a Bottom ‘high-frequency’ forcing in a windless atmosphere with exponentially decaying density and without attenuation. In this table, we express parameters with the following dimensions: \(\rho\) (kg m\(^{-3}\)), \(c\) (m s\(^{-1}\)), \(\eta_v\) (kg m\(^{-1}\) s\(^{-1}\)), \(\mu\) (kg m\(^{-1}\) s\(^{-1}\)), \(\rho g_z\) (m s\(^{-2}\)) and \(u_w\) (m s\(^{-1}\)).

<table>
<thead>
<tr>
<th>(L_x \times L_y) (km)</th>
<th>(\Delta x) (m)</th>
<th>(\Delta t) (s)</th>
<th>(\rho)</th>
<th>(c)</th>
<th>(\eta_v)</th>
<th>(\mu)</th>
<th>(\rho g_z)</th>
<th>(u_w)</th>
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</thead>
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<td>10^{-2}</td>
<td>varying</td>
<td>652.82</td>
<td>0</td>
<td>0</td>
<td>-9.831</td>
<td>0</td>
</tr>
</tbody>
</table>

Downloaded from http://gji.oxfordjournals.org/ at Bibliothéque des Planètes on June 10, 2016
5.4 Atmospheric explosion in a windy homogeneous atmosphere

Here let us study and validate the effect of wind on acoustic waves. In this case, we consider an atmospheric explosion, that is, a sudden increase in volume inserted into the pressure equation, such that

\[
Q = \frac{-2\pi}{P}(t - t_0)e^{-\frac{2}{P}(t - t_0)},
\]

\[
\frac{\partial p}{\partial t} = -w \cdot \nabla p - \rho c^2 (\nabla \cdot v + Q),
\]

where \( P = 100 \) s is the dominant time period of the explosion and \( t_0 = 75 \) s is the starting time. The source is located at \( x_S = 400 \) km and \( z_S = 400 \) km. The atmosphere is again considered isothermal and described in Table 4.

Fig. 5 shows that the waveform and the traveltime are both computed accurately by the numerical simulation. We have scaled the source amplitude \( A \) in eq. (B2) to that recorded at the first station in the far field because the analytical solution (B3) is valid only in the far-field domain. Several aspects of spherical acoustic waves propagation in a moving medium can be noticed. As expected the amplitude decreases as \( 1/\sqrt{r} \) in terms of geometrical spreading, compared to \( 1/r \) in a 3-D medium, because this simulation is 2-D.
Table 5. Simulation parameters corresponding to the isothermal model 3.1 for Simulation 6.1, that is, the case of a bottom ‘low-frequency’ forcing in an atmosphere with exponentially decaying density. In this table, we express parameters with the following dimensions: \( \rho \) (kg m\(^{-3}\)), \( c \) (m s\(^{-1}\)), \( \eta_f \) (kg m\(^{-1}\) s\(^{-1}\)), \( \mu \) (kg m\(^{-1}\) s\(^{-1}\)), \( g_z \) (m s\(^{-2}\)) and \( \psi_w \) (m s\(^{-1}\)).

<table>
<thead>
<tr>
<th>( L_x \times L_z ) (km)</th>
<th>( \Delta x ) (m)</th>
<th>( \Delta t ) (s)</th>
<th>( \rho )</th>
<th>( c )</th>
<th>( \eta_f )</th>
<th>( \mu )</th>
<th>( g_z )</th>
<th>( \psi_w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1200 \times 400</td>
<td>500</td>
<td>10(^{-2})</td>
<td>varying</td>
<td>652.82</td>
<td>0</td>
<td>0</td>
<td>-9.831</td>
<td>0</td>
</tr>
</tbody>
</table>

The Doppler shift due to wind is noticed through the frequency shift and amplitude variations between upwind and downwind acoustic propagation. As expected (Nappo 2002), upwind waves have larger periods and amplitudes than downwind ones.

6 2-D GRAVITY WAVES

6.1 Bottom ‘low-frequency’ forcing in an atmosphere with exponentially decaying density

In order to study and validate gravitoacoustic wave propagation and the underlying physical processes, we consider the forcing function, \( \forall x \in \Gamma_{FS}: \)

\[
f(x, t) = \left( e^{-\frac{(t-t_0-P/4)}{\Delta t}} - e^{-\frac{(t-t_0+P/4)}{\Delta t}} \right) \\
\times \left( e^{-\frac{x-x_0-S/4}{\gamma}} - e^{-\frac{x-x_0+S/4}{\gamma}} \right),
\]

where \( P = 1600 \) s is the dominant time period of the forcing signal, \( S = 80 \) km is the dominant spatial period along \( x \), \( t_0 = 1400 \) s is the starting forcing time and \( x_0 = 600 \) km is the position of the bottom forcing along \( x \). The atmosphere is considered isothermal and described in Table 5.

We will now compare numerical and analytical particle displacement at several recording stations. In this case, station locations must be chosen wisely because internal gravity waves do not propagate in all directions. Indeed, the dispersion relation (B4) without wind, that is, with \( \psi_w = 0 \) yields the angle of propagation \( \beta \)

\[
\cos \beta = \frac{\omega}{N}.
\]

where \( N \) is the Brunt–Väisälä frequency and \( \beta \) is the angle between the horizontal axis \( x \) and the wave vector \( \mathbf{k} \), such that

\[
\mathbf{k} = |\mathbf{k}|(\cos \beta \mathbf{e}_x + \sin \beta \mathbf{e}_z).
\]

We thus only select stations at positions for which \( \beta_{nat} < \beta \), where \( \beta_{nat} \) is the angle between the horizontal axis \( x \) and the position vector of the station.

In Fig. 6, one can see low-frequency gravity waves propagating in the stratified atmosphere. The figure shows low amplitude errors and a good fit in terms of phase velocity.

6.2 Bottom ‘low-frequency’ forcing in a windy atmosphere with exponentially decaying density

6.2.1 Comparison with analytical solution

In order to study and validate the impact of wind on gravity wave propagation, we use the same forcing (52) as above, with parameters \( P = 1600 \) s, \( S = 80 \) km, \( t_0 = 1400 \) s and \( x_0 = 800 \) km. The atmosphere is considered isothermal and described in Table 6.

In Fig. 7, as in the acoustic case of Fig. 5, one can notice the impact of the Doppler shift on gravity wave propagation: upwind waves have a larger period and larger amplitude than downwind ones.

In Fig. 8, we present snapshots of the simulation. A typical feature of gravity wave propagation can be observed: the group (\( V_g \)) and phase velocities (\( V_p \)) are orthogonal. The pictures also illustrate an interesting aspect of our numerical modelling tool, which is that we can compute and show the propagation of gravity and acoustic waves simultaneously. One can thus notice an acoustic wave front propagating from the bottom to the top, ahead of gravity waves owing to the small positive amplitude of displacement perturbation \( \psi_w \) between the bottom forcing function (52) at \( t_i \) and \( t_i + \Delta t \) such...
Gravitoacoustic wave propagation in the atmosphere

Figure 6. Vertical displacement for the finite-difference solution ('Numerical') and the analytical solution ('Analytical') as well as the difference between the two ('A-N'). The signals are shown through time for Simulation 6.1, that is, the case of a bottom 'low-frequency' forcing in an atmosphere with exponentially decaying density, at the four recording stations located at the same altitude \( z = 80.25 \text{ km} \), their positions along \( x \) being, from top to bottom, \( x = 375, 450, 750 \) and \( 825 \text{ km} \). The atmosphere is considered isothermal (Table 5).

Table 6. Simulation parameters corresponding to the isothermal model 3.1 subject to wind for Simulation 6.2.1, that is, the case of a comparison with analytical solution. In this table, we express parameters with the following dimensions: \( \rho \) (kg m\(^{-3}\)), \( c \) (m s\(^{-1}\)), \( \eta_V \) (kg m\(^{-1}\) s\(^{-1}\)), \( \mu \) (kg m\(^{-1}\) s\(^{-1}\)), \( g_z \) (m s\(^{-2}\)) and \( w_x \) (m s\(^{-1}\)).

<table>
<thead>
<tr>
<th>( L_x \times L_z ) (km)</th>
<th>( \Delta x ) (m)</th>
<th>( \Delta t ) (s)</th>
<th>( \rho )</th>
<th>( c )</th>
<th>( \eta_V )</th>
<th>( \mu )</th>
<th>( g_z )</th>
<th>( w_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1600 \times 400</td>
<td>500</td>
<td>10(^{-2})</td>
<td>varying</td>
<td>652.82</td>
<td>0</td>
<td>0</td>
<td>-9.831</td>
<td>10</td>
</tr>
</tbody>
</table>

Figure 7. Vertical displacement for the finite-difference numerical solution ('Numerical') and the analytical solution ('Analytical') as well as the difference between the two ('A-N'). The signal is shown through time for Simulation 6.2.1, that is, the case of a comparison with analytical solution, at four recording stations located at the same altitude \( z = 100.25 \text{ km} \) and whose positions along \( x \) are, from top to bottom, \( x = 875.25, 850.25, 750.25 \) and \( 725.25 \text{ km} \). The atmosphere is considered isothermal (Table 6).

that \( f(t_1) = 0; f(t_1 + \Delta t) > 0 \). This step, similar to a Dirac, generates a high-frequency wave (an acoustic wave).

Note that in the comparisons presented in Fig. 7 the impact of this ‘high-frequency’ wave on the seismogram is not seen because its amplitude is tiny compared to that of the gravity wave.

6.2.2 Resolution analysis

In order to see the impact of resolution (spatial and time steps) on the displacement amplitude error, and since we have not performed any rigorous mathematical stability and accuracy analysis of the problem, we perform tests with various resolutions in the case of gravity waves propagating in a windy atmosphere.

We use a similar bottom forcing as in eq. (52), with parameters \( P = 800 \text{ s}, S = 25 \text{ km}, t_0 = 800 \text{ s} \) and \( x_0 = 250 \text{ km} \) and consider the atmosphere model specified in Table 7. In Fig. 9, the left and right panels show how spatial resolution impacts the amplitude error through time. First, as one decreases the spatial step one can see that the error decreases, in particular for the largest error peaks (i.e. between \( t = 2000 \text{ s} \) and \( t = 3000 \text{ s} \) in the left panel). Owing to
the cumulative nature of such numerical error with time, accuracy is more impacted for long time periods than at the beginning of the simulation, where the error is similar for all spatial resolutions considered. At time \( t = 2000 \) s in the bottom-left chart one notices that the simulation with a larger spatial step exhibits lower error, but only temporarily. The ‘averaging’ implied by a large spatial step \( \Delta x = 1000 \) m seems to, surprisingly, reproduce the phase of the analytical signal well, but as one decreases \( \Delta x \) the error gets lower than that for \( \Delta x = 1000 \) m. In the right panel, the difference in displacement amplitude between simulations with resolution \( \Delta x = 125 \) m and \( \Delta x = 250 \) m shows that for this set of parameters (Table 7) the solution seems to be converging. If one decreases the time step \( \Delta t \) for the resolution \( \Delta x = 125 \) m one notices an improvement in accuracy at the beginning of the simulation, but after \( 1600 \) s both simulations give a similar result. With this resolution \( (\Delta x = 125 \) m and \( \Delta t = 0.01 \) s) the numerical solution has converged and decreasing the time step will not decrease the error any longer.

Another source of error comes from the numerical computation of the analytical solution. Indeed, a numerical Fourier transform and then a numerical inverse Fourier transform are required to compute the solution (refer to Section 5.1), which introduces numerical approximations. In Fig. 10, we show the absolute error between numerical evaluations of the analytical solution computed with various resolutions. The resolution impacts directly the Fourier transform since it leads to a lower boundary (specified by the Nyquist frequency) for the number of points required, in order to overcome aliasing, in both spatial and time Fourier transforms. Thus, in the chart one notices that the number of spatial points has a significant impact on the analytical signal: one obtains almost a 5 per cent difference in absolute amplitude between the signal computed for \( \Delta x = 125 \) m and for \( \Delta x = 500 \) m. The very small difference (about \( 10^{-14} \) m) obtained between spatial resolutions \( \Delta x = 125 \) m and \( \Delta x = 250 \) m shows that the solution has converged and then captured low vertical wavelength values.

This illustrates the fact that small errors sometimes observed in the validation cases presented in this work can be explained by the resolution used in these simulations. If we had chosen smaller spatial steps for the numerical and analytical computations we could have decreased the error in phase and amplitude but the computation time would have thus significantly increased. As often with numerical schemes there is a trade-off to find between accuracy and numerical cost.

### 7 2-D APPLICATIONS

#### 7.1 Bottom ‘low-frequency’ forcing in an isothermal atmosphere subject to a wind duct

In this case, we set up a wind duct (a strong wind velocity gradient) to show specific gravity-wave features studied by several authors (Ding et al. 2003; Nappo 2002). We use the same type of forcing as in Simulation 6.1, with parameters \( P = 2800 \) s, \( S = 80 \) km, \( t_0 = 1600 \) s and \( x_0 = 1280 \) km. The atmosphere is considered isothermal and described in Table 8.

In this case, the wind profile is a wind duct, that is, a Gaussian bump such that

\[
w_w(z) = 10 + w_{w,0} e^{-\left(\frac{z - \bar{z}}{\sigma}\right)^2}
\]

with \( w_{w,0} = 200 \) m s\(^{-1}\), \( \bar{z} = 100 \) km and \( \sigma = 5000 \) m.

In Fig. 11, three main features of gravity-wave propagation subject to a wind duct are seen: first, waves can go beyond the wave duct but the altitude reached by the upwind waves is much higher than the downwind ones. Second, some downwind waves seem to be concen trated around the wave duct. Finally, in the bottom left of the upwind waves, one can observe a refracted wave due to reflection on the wave duct owing to the strong wind velocity gradient.
velocity is the position of the bottom forcing along $x$.

For both stations, the amplitude of the error decreases with increasing resolution, as expected. The signals are shown through time for Simulation 6.2.2, at the same recording station located at altitude $z = 20$ km, their position along $x$ being, from left to right, $x = 263$ and 316.5 km. The atmosphere is considered isothermal (Table 7). In the bottom charts we display amplitude through time of the absolute difference of vertical displacement between the analytical signal and the numerical one for various spatial steps: between signals for $x = 125, 250, 500, 1000$ m and also for $t = 0.01$ s for the spatial resolution $x = 125$ m. For both stations, the amplitude of the error decreases with increasing resolution, as expected.

7.2 Tsunami-like bottom forcing in a full MSISE-based atmosphere

In this case, let us consider the forcing function, $f(x, t)$, for all $x \in \Gamma_F$,

$$f(x, t) = \left( e^{-\left(\frac{x - \psi_0}{S\eta_1}\right)^2} \right) \left( e^{-\left(\frac{t - \psi_0}{c\eta_1}\right)^2} \right) \times \left( e^{-\left(\frac{2}{\xi_2} x + \psi_0 \eta_1^2 - \frac{\eta_1^2}{8}} \right) \right) \left( e^{-\left(\frac{2}{\xi_2} t + \psi_0 \eta_1^2 - \frac{\eta_1^2}{8}} \right) \right),$$

where $P = 800$ s is the dominant time period of the forcing signal, $S = 80$ km is the dominant spatial period along $x$ of the forcing signal, $t_0 = 800$ s is the starting forcing time and $x_0 = 266.5$ km is the position of the bottom forcing along $x$. The tsunami wave velocity is $v_t = 100$ m s$^{-1}$. We define the atmosphere according to the MSISE-00 model described in Table 10.

Table 8. Simulation parameters corresponding to the isothermal model 3.1 subject to wind for Simulation 7.1, that is, the case of a Bottom ‘low-frequency’ forcing in an isothermal atmosphere subject to a wind duct. In this table we express parameters with the following dimensions: $\rho$ (kg m$^{-3}$), $c$ (m s$^{-1}$), $\eta_1$ (kg m$^{-1}$ s$^{-1}$), $\mu$ (kg m$^{-3}$ s$^{-1}$), $g_z$ (m s$^{-2}$) and $w_z$ (m s$^{-1}$).

In Fig. 12, gravity waves propagate in a realistic atmosphere, which highlights the fact that simulations are stable in a relatively complex medium with strong gradients of the physical parameters. Waves coming from the right of the domain are due to the periodic boundary conditions implemented on the left and right boundaries. One can notice that gradients in sound and wind velocities (see Table 9) have a strong effect on the gravity wave curvature. Also,
the Doppler shift is visible as one observes that the right part of the wave front has a smaller apparent spatial period than the left part.

### 7.3 Seismic-like bottom forcing in a full MSISE-based atmosphere

In this case, we implement a large bottom \( x \)-velocity forcing \( v_x \) and a dominant time period \( P \) smaller than in eq. (56), such that, \( \forall x \in \Gamma_F \),

\[
\begin{align*}
    f(x, t) &= e^{-\left[\frac{(x-x_0-t/2)^2}{\Delta x^2}\right]}e^{-\left[\frac{(t-t_0-P/2)^2}{\Delta t^2}\right]} - e^{-\left[\frac{(x-x_0+tu_x)^2}{\Delta x^2}\right]} \quad \text{if } t < t_0 - P/2, \\
    f(x, t) &= e^{-\left[\frac{(x-x_0-tu_x)^2}{\Delta x^2}\right]} - e^{-\left[\frac{(x-x_0+tu_x)^2}{\Delta x^2}\right]} \quad \text{if } t > t_0 - P/2
\end{align*}
\]

(57)

with \( P = 60 \text{ s}, t_0 = 200 \text{ s}, S = 320 \text{ km}, x_0 = 266.5 \text{ km}, \) and \( u_x = 4000 \text{ m s}^{-1} \). We define the atmosphere according to the MSISE-00 model described in Table 10.

In Fig. 13, one can notice that the large ground forcing velocity \( v_x \) (see (57)) has a strong impact on the direction of wave propagation. We obtain almost horizontal wave fronts that can propagate in the upper atmosphere, with their trajectory impacted by wind and sound velocity gradients. The curvature of the wave front in the thermosphere is due to the sudden increase of sound velocity.

### 7.4 Atmospheric explosion in a full MSISE-based atmosphere

In this simulation, we consider the same source as in Simulation 5.4 but with parameters \( P = 20 \text{ s} \) as the dominant time period of the explosion and \( t_0 = 50 \text{ s} \) as its starting time. The source \( Q \) is located at \( x_S = 500 \text{ km} \) and \( z_S = 100 \text{ km} \). We define the atmosphere according to the MSISE-00 model described in Table 11.

In Simulation 7.4 (Fig. 14), one can notice that a point source with a small dominant time period compared to the gravity wave frequency range still generates both acoustic and gravity waves in the atmosphere, the latter propagating around the source location only, as predicted by observations and theory (Ben-Menahem & Singh 2012). Gravity waves are seen as this ‘low-frequency’ oscillating signal that follows the acoustic wave front and that has a similar shape as in the gravity-wave Simulation 6.1. Once again one can observe the impact of wind that shifts the frequency spectrum.
of the gravity wave. Finally, when wind and sound velocity gradients are present they lead to atmospheric waveguides that impose a direction of propagation for acoustic and gravity waves.

8 3-D VALIDATION

8.1 Atmospheric explosion in a 3-D homogeneous atmosphere

We perform a validation test in the 3-D case by checking the impact of geometrical attenuation due to a point source generating a spherical wave. The pressure pulse is the same as in the 2-D Simulation 5.4 but with parameters $P = 30$ s and $b_0 = 25$ s. The source is located at $x_S = 65$ km, $y_S = 65$ km and $z_S = 100$ km. The atmosphere is considered isothermal and described in Table 12.

In Fig. 15, the waveform and traveltime accurately match the analytical solution in this simple case. The maximum error over time is around 2 per cent. This could be further reduced by increasing the spatial resolution, at the expense of larger computational times. The analytical pressure solution is not the same as in Simulation 5.4 but rather described by eq. (B5). This comparison validates the geometrical spreading of acoustic waves in a 3-D case.
To validate 3-D gravity wave propagation, we first perform a test with a ground forcing uniform along \( y \) and \( z \) for numerical simulations for acoustic and gravity wave propagation in a stratified atmosphere. We have implemented a high-order moving medium. We have implemented a high-order

Table 12. Simulation parameters corresponding to the isothermal model 3.1 not subject to wind for Simulation 8.1, that is, the case of an atmospheric explosion in a 3-D homogeneous atmosphere. In this table, we express parameters with the following dimensions: \( \rho \) (kg m\(^{-3}\)), \( c \) (m s\(^{-1}\)), \( \eta_f \) (kg m\(^{-1}\) s\(^{-1}\)), \( \mu \) (kg m\(^{-1}\) s\(^{-1}\)), \( g_e \) (m s\(^{-2}\)) and \( w_e \) (m s\(^{-1}\)).

<table>
<thead>
<tr>
<th>( L_x \times L_y \times L_z ) (km)</th>
<th>( \Delta x ) (m)</th>
<th>( \Delta t ) (s)</th>
<th>( \rho )</th>
<th>( c )</th>
<th>( \eta_f )</th>
<th>( \mu )</th>
<th>( g_e )</th>
<th>( w_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>130 ( \times ) 130 ( \times ) 200</td>
<td>500</td>
<td>10(^{-2})</td>
<td>1.2</td>
<td>652.82</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 15. Pressure for the finite-difference numerical solution (‘Numerical’) and the analytical solution (‘Analytical’) as well as the difference between the two (‘A-N’). The signals are shown through time for Simulation 8.1, that is, the case of an atmospheric explosion in a 3-D homogeneous atmosphere, and for three recording stations located at altitude \( z = z_S = 65 \) km and at \( y = y_S = 100 \) km, and at distances \( x = 14.75, 29.25 \) and 43.75 km away from the source. The atmosphere is considered isothermal (Table 12).

8.2 Bottom 1-D ‘low-frequency’ forcing in a 3-D windy atmosphere with exponentially decaying density

To validate 3-D gravity wave propagation, we first perform a test with a ground forcing uniform along \( y \) and \( z \) for numerical simulations for acoustic and gravity wave propagation in a stratified atmosphere. We have implemented a high-order moving medium. We have implemented a high-order

The atmosphere is considered isothermal and described in Table 13.

As expected, in Fig. 16 we obtain a good fit in terms of amplitude and phase between the 2-D analytical signal and the 3-D numerical one. Results could be made even more accurate if one picked a smaller spatial step (identical to Simulation 6.2.1) \( \Delta x \) for numerical simulation.

8.3 Bottom 2-D ‘low-frequency’ forcing in a 3-D windy atmosphere with exponentially decaying density

In order to further validate 3-D gravity wave propagation, let us now use a similar approach than for the 2-D gravity wave Simulation 6.2.1. Using a 2-D ground forcing that depends on \( x \) and \( y \) in order to validate the propagation in the \( y \) direction. We will compare results to the analytical solution based on dispersion relation (B4). Thus, in contrast to the 2-D validation case we will perform a three dimensional Fourier Transform to take into account propagation in the \( (x, y) \). We take a similar ground forcing as in Simulation 6.2.1 but convolved with a Gaussian that depends on \( x \) and \( y \). We thus consider the following 3-D forcing, \( \forall \mathbf{x} \in \Gamma_{PS} \):

\[
f(\mathbf{x}, t) = \left( e^{-\frac{(x-x_0-P/4)^2}{\sigma^2}} - e^{-\frac{(x-x_0+P/4)^2}{\sigma^2}} \right) \times \left( e^{-\frac{(y-y_0-S/4)^2}{\sigma^2}} - e^{-\frac{(y-y_0+S/4)^2}{\sigma^2}} \right) e^{-\frac{(d(x))^2}{\sigma^2}} ;
\]

where \( P = 1600 \) s is the dominant time period of the forcing signal, \( S = 80 \) km is the dominant spatial period along \( x \) of the forcing signal, \( t_0 = 1400 \) s is the starting forcing time, \( x_0 = y_0 = 500 \) km is the position of the bottom forcing in the \( (x, y) \) plane and \( d \) is the distance from the point \( (x_0, y_0) \) such that \( d(\mathbf{x}) = \sqrt{(x-x_0)^2 + (y-y_0)^2} \).

The atmosphere is considered isothermal and described in Table 14.

In Fig. 17, one notes a good fit between the 3-D numerical and 3-D analytical signals in terms of phase and vertical displacement amplitude, with a maximum amplitude error of less than 5 per cent over time. Geometrical spreading is visible since amplitudes in this case are smaller than in the previous validation case 6.2.1, that is, the case of Bottom 2-D ‘low-frequency’ forcing in a 3-D windy atmosphere with exponentially decaying density. The Doppler effect also impacts gravity wave propagation: upwind waves have a larger period and larger amplitude than downwind ones.

9 CONCLUSIONS AND FUTURE WORK

We have considered the linearized Navier–Stokes system of equations for acoustic and gravity wave propagation in a stratified and viscous moving medium. We have implemented a high-order
Table 13. Simulation parameters corresponding to the isothermal model 3.1 subject to wind for Simulation 8.2, that is, the case of a bottom 1-D ‘low-frequency’ forcing in a 3-D windy atmosphere with exponentially decaying density. In this table, we express parameters with the following dimensions: \(\rho\) (kg m\(^{-3}\)), \(c\) (m s\(^{-1}\)), \(\eta_V\) (kg m\(^{-1}\) s\(^{-1}\)), \(\mu\) (kg m\(^{-1}\) s\(^{-1}\)), \(g_z\) (m s\(^{-2}\)) and \(w_x\) (m s\(^{-1}\)).

<table>
<thead>
<tr>
<th>(L_x \times L_y \times L_z) (km)</th>
<th>(\Delta x) (m)</th>
<th>(\Delta t) (s)</th>
<th>(\rho)</th>
<th>(c)</th>
<th>(\eta_V)</th>
<th>(\mu)</th>
<th>(g_z)</th>
<th>(w_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1600 (\times) 1600 (\times) 400</td>
<td>5000</td>
<td>varying</td>
<td>652.82</td>
<td>0</td>
<td>0</td>
<td>-9.831</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Figure 16. Vertical displacement for the 3-D finite-difference numerical solution (‘Numerical 3-D’), and the 2-D analytical solution (‘Analytical 2-D’) as well as the difference between the 3-D analytical signal and the 2-D analytical one (‘N-A’). Signals are shown through time for Simulation 8.2, that is, the case of a bottom 1-D ‘low-frequency’ forcing in a 3-D windy atmosphere with exponentially decaying density, with uniform forcing along \(y\) (52) at two recording stations located at the same altitude \(z = 52.5\) km and whose positions along \(x\) are, from top to bottom, \(x = 850\) and 750 km, and position along \(y\) is \(y = 500\) km. The solid line in the bottom frame is the arrival time of both 2-D and 3-D gravity waves. The atmosphere is considered isothermal (Table 13).

Table 14. Simulation parameters corresponding to the isothermal model 3.1 subject to wind for Simulation 8.3, that is, the case of a bottom 2-D ‘low-frequency’ forcing in a 3-D windy atmosphere with exponentially decaying density. In this table, we express parameters with the following dimensions: \(\rho\) (kg m\(^{-3}\)), \(c\) (m s\(^{-1}\)), \(\eta_V\) (kg m\(^{-1}\) s\(^{-1}\)), \(\mu\) (kg m\(^{-1}\) s\(^{-1}\)), \(g_z\) (m s\(^{-2}\)) and \(w_x\) (m s\(^{-1}\)).

<table>
<thead>
<tr>
<th>(L_x \times L_y \times L_z) (km)</th>
<th>(\Delta x) (m)</th>
<th>(\Delta t) (s)</th>
<th>(\rho)</th>
<th>(c)</th>
<th>(\eta_V)</th>
<th>(\mu)</th>
<th>(g_z)</th>
<th>(w_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000 (\times) 1000 (\times) 400</td>
<td>2500</td>
<td>5.10(^{-2})</td>
<td>varying</td>
<td>652.82</td>
<td>0</td>
<td>0</td>
<td>-9.831</td>
<td>10</td>
</tr>
</tbody>
</table>

Figure 17. Vertical displacement for the 3-D finite-difference numerical solution (‘Numerical 3-D’) and the analytical 3-D solution (‘Analytical 3-D’) as well as the difference between the 3-D numerical signal and the 3-D analytical one (‘N-A\(_{3D}\)’). Signals are shown through time for Simulation 8.3, that is, the case of a bottom 2-D ‘low-frequency’ forcing in a 3-D windy atmosphere with exponentially decaying density, at two recording stations located at the same altitude \(z = 73.75\) km and whose positions along \(x\) are, from top to bottom, \(x = 552.5\) and 447.5 km, and position along \(y\) is \(y = 500\) km. The atmosphere is considered isothermal (Table 14).

finite-difference scheme that handles both acoustic and gravity waves simultaneously, in 2-D or 3-D media. We have also taken into account complex atmosphere models with strongly varying wind and adiabatic sound velocities.

We validated the simulations by comparison to analytical solutions in several benchmark cases involving acoustic and gravity waves in a stratified windy and viscous atmosphere. We obtained very good agreement in terms of vertical displacement and
pressure. The simulation results for validation cases exhibit interesting gravity wave characteristics and show the expected features: wave amplitude increases in vertical displacement with decrease of atmospheric density with altitude, and conversely wave amplitude tends to decrease with altitude due to viscosity, which mainly impacts high frequencies.

We also presented simulation results for an atmosphere model based on MSISE-00 and for the cases of tsunami and seismic waves, and finally for an atmospheric explosion in the lower thermosphere. This showed that simulations are stable for complex media and exhibit interesting physical features such as change in wave front curvature with gradients in wind and sound velocities. Both acoustic and gravity waves propagate up to the upper-atmosphere. However, with strong gradients in sound and wind velocities, one also observes wave refraction and wave concentration in the thermosphere. Finally, one notices that the Doppler shift of the wave frequency spectrum has a significant impact on wave shape and arrival times.

This new numerical modelling tool can thus give insights on gravity wave dynamics in the atmosphere and enable one to study real signals such as those recorded by the GOCE satellite. It can also provide benchmark solutions in complex cases (such as the MSISE – 00 empirical atmosphere model) for future numerical developments.

Future developments should include absorbing boundary conditions instead of non-realistic horizontal periodic conditions in order to properly model wave propagation in the upper atmosphere. The technique should also take into account topography because it has an impact on the generation and propagation of gravity waves. Finally, coupling with a solid Earth and an ocean should be implemented to better model the whole process from seismic underground perturbation to atmospheric wave propagation.

ACKNOWLEDGEMENTS

The authors thank Bruno Voisin for discussion about his work (Voisin 1994) and for giving them more details on the pressure response of an explosion in a moving medium. They are also grateful to Bernard Valette for mathematical discussion about the Euler equations. We acknowledge two anonymous reviewers for their constructive reviews improving this study. Computer resources were provided by granted projects no. p1138 at CALMIP computing centre (Toulouse France), nos. t2014046351 and t2015046351 at CEA centre (Bryères, France). This work was also granted access to the French HPC resources of TGCC under allocations 2015-gen6351 and 2015-gen7165 made by GENCI. They also thank the ‘Région Midi-Pyrénées’ (France) and ‘Université de Toulouse’ for funding the PhD grant of Quentin Brissaud. This study was also supported by CNES/TOSCA through space research scientific projects.

REFERENCES


Gravitoacoustic wave propagation in the atmosphere

\[ \partial_t \rho_p = -(w_i \partial_t p + w_j \partial_j p) - \rho \varepsilon^2 (\partial_i v_i + \partial_j v_j + \partial_k v_k) + \varepsilon g_z, \]
\[ \rho \partial_t v_i = -\rho (v_j \partial_j w_i + w_i \partial_j v_i + w_j \partial_i v_j - \partial_i p) + \left( \eta \frac{2}{3} \mu \right) \partial_j [\partial_i v_j + \partial_j v_i + \partial_k v_k] + \mu [2 \partial_i^2 [v_i + w_i] + \partial_j [\partial_i v_j + \partial_j v_i] + \partial_k [\partial_i v_k + \partial_j v_k] + \partial_j [\partial_i v_j + \partial_j v_i] + 2 \partial_i^2 v_i] + g_z \rho p \] (A1)

**APPENDIX B: DISPERSION RELATIONS FOR THE VALIDATION CASES**

The three validation cases presented above involve the following analytical formulation of the dispersion equations.

**B1 Acoustic wave forcing in a 2-D heterogeneous windless atmosphere**

The dispersion equation, without any source inside the domain and when one considers a windless atmosphere with varying density, sound velocity and viscosity (not considering shear viscosity), reads

\[ k_z^2 \left( 1 - \frac{i}{D} \right) - \frac{D}{H^2} k + \frac{1}{4 H^2} \left( 1 + \frac{i}{D} \right) - \left( \frac{0}{c} \right)^2 = 0, \] (B1)

where \( D = \frac{\rho^2}{\rho v^2} \).

**B2 Atmospheric explosion in 2-D windy atmosphere**

We consider a monochromatic point source \( Q \) that reads

\[ Q = \frac{2 i A}{\rho \omega} e^{-i \omega t} \delta (x) \delta (z) \]

\[ \partial_t p = -\mathbf{w} \cdot \nabla p - \rho \varepsilon^2 (\nabla \cdot \mathbf{v} + Q), \] (B2)

where \( A \) is the amplitude of the source pulse and \( \delta \) the Kronecker symbol, from Ostashhev et al. (2005), in the far-field approximation \( k, R \gg 1 \), we get

\[ \hat{p} = \frac{A (\sqrt{1 - M^2 \sin \beta^2} - M \cos \beta)}{2 \pi k R (1 - M^2)(\sqrt{1 - M^2 \sin \beta^2} - M \cos \beta) k R + \frac{i}{c}}, \] (B3)
where \( k = \frac{\omega}{c} \), \( M \) is the Mach number \( (M = \frac{v}{c}) \), \( \beta \) the angle between the \( x \) axis and the receiver and \( R = \sqrt{(x - x_S)^2 + (y - y_S)^2 + (z - z_S)^2} \) the source–receiver distance, with \((x_S, z_S)\) the Cartesian coordinates of the source.

The theoretical pressure response for gravity waves for an explosion in a stratified windy atmosphere is more difficult to implement and can be found for instance in Voisin (1994) and Godin & Fuks (2012).

**B3 Gravity-wave forcing in a 3-D stratified windy isothermal atmosphere**

The dispersion equation, without any source inside the domain and when one considers a windy and inviscid atmosphere with varying density and sound velocity, reads

\[
k^2_N = \frac{(k_x^2 + k_y^2)N^2}{\Omega^2} - \frac{1}{4H^2} - k_x^2 - k_y^2,
\]

where \( k_x, k_y \) are the wavenumbers respectively along \( x \) and \( y \), such that \( k_x = \frac{2\pi}{\lambda_x} \) and \( k_y = \frac{2\pi}{\lambda_y} \), \( \lambda_x, \lambda_y \) the wavelengths respectively along \( x \) and \( y \), \( \Omega \) is the intrinsic frequency such that \( \Omega = \omega - w_xk_x - w_yk_y \), and \( H \) the scale height.

**B4 Atmospheric explosion in a 3-D atmosphere**

For a monochromatic point source \( \hat{Q} \) in a system following eq. (B2) from Goldstein (1976) one has

\[
\hat{p} = \frac{\hat{Q}}{4\pi R} e^{ikR}
\]

where \( k = \frac{\omega}{c} \) and \( R = \sqrt{(x - x_S)^2 + (y - y_S)^2 + (z - z_S)^2} \) is the source–receiver distance, with \((x_S, z_S)\) the Cartesian coordinates of the source. In this case, no far-field assumption needs to be made because the full analytical solution is known in the whole domain.

**APPENDIX C: REALISTIC ATMOSPHERE MODEL**

In Fig. C1, we present all the physical parameters plotted against altitude extracted from the MSISE-00 atmosphere model (Section 3.2).

**APPENDIX D: VARIABLES**

Table D1 summarizes all the variables used in the article. By ‘Total’ in Table D1 we refer to the sum of the mean and fluctuating parts, see Hypothesis (iii).

**Figure C1.** Vertical profiles extracted from the MSISE-00 atmosphere model and used for construction of the realistic atmosphere models in Section 3.2.
### Table D1. The main variables used in our article.

<table>
<thead>
<tr>
<th>Name</th>
<th>Meaning</th>
<th>Name</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>Displacement perturbation</td>
<td>$c$</td>
<td>Sound speed</td>
</tr>
<tr>
<td>$v$</td>
<td>Velocity perturbation</td>
<td>$\rho$</td>
<td>Atmosphere mean density</td>
</tr>
<tr>
<td>$w = (w_x, 0, 0)$</td>
<td>Wind background velocity</td>
<td>$\rho_p$</td>
<td>Density perturbation</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure perturbation</td>
<td>$\eta_V$</td>
<td>Volume viscosity</td>
</tr>
<tr>
<td>$I_d$</td>
<td>Identity tensor in $\mathbb{R}^3$</td>
<td>$\mu$</td>
<td>Dynamic viscosity</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>Eulerian stress tensor</td>
<td>$g$</td>
<td>Gravitational acceleration</td>
</tr>
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<td>$I$</td>
<td>Internal energy</td>
<td>$C_i$</td>
<td>Heat capacity at constant volume</td>
</tr>
<tr>
<td>$Q$</td>
<td>Heat input</td>
<td>$C_p$</td>
<td>Heat capacity at constant pressure</td>
</tr>
<tr>
<td>$T$</td>
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<td>$U$</td>
<td>Total displacement</td>
<td>$P$</td>
<td>Total adiabatic atmospheric pressure</td>
</tr>
<tr>
<td>$V$</td>
<td>Total velocity</td>
<td>$G$</td>
<td>Total gravitational acceleration</td>
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<td>$F_{ext}$</td>
<td>External volumic forces</td>
<td>$R$</td>
<td>Gas constant</td>
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<tr>
<td>$\gamma$</td>
<td>Ratio of specific heat</td>
<td>$L$</td>
<td>Mean free path</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Thermal conductivity</td>
<td>$\alpha$</td>
<td>Absorption coefficient</td>
</tr>
<tr>
<td>$f, w$</td>
<td>Frequency, Pulsation</td>
<td>$Q$</td>
<td>Quality factor</td>
</tr>
<tr>
<td>$k$</td>
<td>Wavenumber</td>
<td>$N$</td>
<td>Brunt–Väisälä frequency</td>
</tr>
<tr>
<td>$\Delta x, \Delta y, \Delta z$</td>
<td>Spatial step along $x, y$ and $z$</td>
<td>$\Delta t$</td>
<td>Time step</td>
</tr>
<tr>
<td>$L_x, L_y, L_z$</td>
<td>Mesh dimension along $x, y, z$</td>
<td>$\beta$</td>
<td>Angle between the $x$ axis and the receiver position</td>
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<td>Forcing boundary</td>
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<td>Dirichlet boundary</td>
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