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Sensitivity and stability analysis of a rigid canopy flow

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Abstract:

The present research project aims at examining the possibility of using walls covered by flexible filaments as a means of reducing drag; we first study a rigid canopy flow model that involves several initial parameters. A sensitivity study using adjoints allows to pinpoint the parameters which influence the most both the shape of the profile and integral quantities associated to it. The intrinsic stability properties of the flow, which result from the model, are analysed with a linear local stability approach in the temporal framework. A second approach, based on adjoint stability equations, provides the sensitivity of the eigenvalues of the stability problem to changes in the flow profile, changes which can arise from modifications of the parameters of the model.

1 Introduction

Our research interest covers the modelling of flow over complex surfaces (compliant, porous, not smooth ...) to investigate possible drag reduction. A surface covered by an array of rigid cylinders seems to be a good starting problem.
to highlight the parameters which might have the larger effects on the flow over such a rigid, porous and anisotropic medium.

With the purpose of the coupled flow through and over permeable layers the mathematical model proposed by Ghisalberti and Nepf [1] has been analysed. From a set of experiments over a rigid canopy [1] a parametric complex model has been built using a drag function, a turbulence mixing length model and the momentum fluid flow equations.

In [4] a simpler model is proposed but the comparison with experimental data appears to be of lower quality than the initial model. Some more complex approaches can be found in [5, 6] but we investigate here the first parametric original canopy model [1] because it is build from theoretical equations and experimental data fitting and because the profile can be rapidly computed with some quite good agreement with experimental data.

Since the model is largely based on empirical functions extrapolated from experimental data a sensitivity analysis has to be performed. The main idea is to define a quantity which qualifies the profile. As in boundary layers, an integral thickness, such as the displacement thickness $\delta_1$ may be appropriate and it is thus set as objective functional. The sensitivity analysis allows to determine the gradient of this functional with respect to any variable of the model to shed light on the dependencies and the importance of the choices proposed in [1].

After this study we also analyze the linear stability of the profile based on a modal approach, and we initiate the study of the sensitivity of the eigenvalues due to modification of the base flow.

2 Canopy parametric model

The flow configuration and notations are defined on figure 1 (right). The problem is assumed to be one-dimensional and parallel, the incompressible boundary layer flow is set in motion by the slope of the ground surface $(S)$. The contribution of the canopy to the base flow is modelled by a drag coefficient as a function of normal coordinate $z$ with an equilibrium equations written as:

$$gS = \frac{\partial \bar{u}'w'}{\partial z} + \frac{1}{2} C_D(z) a U(z)^2 \quad (1)$$

$g$ is the gravity acceleration. $\bar{u}'w'$ is the turbulent stress and $U(z)$ is the mean streamwise velocity. From this equation a complete mathematical model is derived and detailed in [1]. Here we simply report the main equations to solve which constitute the constraints of the sensitivity analysis:
In the present work we specifically introduce an additional equation to impose the continuity of the derivative with respect to \( z \) (prime exponent) at the top of the canopy \( (z = h) \) \( u'(h^-) = u'(h^+) \) with a \( \chi \) function:

\[
\chi(z) = 1 - e^{-\frac{z-h}{\xi}}
\]

(14)

The canopy is made of small cylinders of height \( h \), of diameter \( d \) and \( a \) frontal area per reference volume (density of the cylinder). The problem is divided into two regions (see fig. 1 right), the first one (index 1) inside the canopy and with the streamwise velocity \( u_1(z) \) and the second one above it (index 2) with the velocity \( u_2(z) \). The drag coefficient \( C_D \) is set to zero above \( z = h \) and is modelled with several functions inside the canopy. The canopy model begins at \( z = z_1 \), under this height the velocity does not change. The turbulence model is associated with three length \( t_{ml}, l_c \) and \( l_{ac} \). In addition, the equation 1 can be integrated analytically above the canopy to yield \( u_2(z) \). Finally, a nonlinear optimization problem with 3 nested Newton iterations is required to solve the full set of equations.

Figure 1 shows the profile obtained for case H of the reference paper and it is compared to the experimental data (with error bars). Some small discrepancies are observed due to the correction made through the \( \chi \) function to set continuity
of slope at the canopy interface. Without the $\chi$ function our results are in a good agreement with the integrated profile of the paper by Ghisalberti and Nepf [1] (not shown here here).

![Figure 1: Case H from experimental reference [1].](image)

### 3 Sensitivity analysis

We have to define all the parameters that are taken into account in the canopy model. The $n_1 = 20$ interdependent parameters are written as components of the vector $q_i$ while the $n_2 - n_1 = 3$ functions parameters are written as $Q_i(z)$:

$$ q_i = [a, d, h, \alpha, \xi, z_1, z_2, U_h, U_1, U_2, l_c, l_{ac}, \bar{q}_1, \beta_1, l_{ml}, C_{DC}, C_{DA}, U_h', \sigma, \beta] $$

$$ i = 1..n_1 $$

$$ Q_i = [\chi, C_D, u_1] \quad i = 1 + n_1..n_2 $$

In order to study the sensitivity of the model we define a lagrangian functional:

$$ \mathcal{L} = \mathcal{E} + L $$

where $\mathcal{E}$ is the reference quantity to analyse. A generic objective function has the form:

$$ \mathcal{E} = \epsilon_0 E + \sum_{i=1}^{n_1} \epsilon_i \frac{Q_i^2}{2} + \sum_{i=1+n_1}^{n_2} \epsilon_i \int_0^h \frac{Q_i^2}{2} dz $$

where $E$ has been chosen to be an integral parameter of the profile. Similarly to boundary layers, a displacement thickness is proposed as:
\[ E = \delta_1 = \int_0^{z_2} [U_2 - u(z)] \, dz = \int_0^h [U_2 - u_1(z)] \, dz + \int_0^{z_2} [U_2 - u_2(z)] \, dz \]

The weights \( \epsilon_i \) are user-dependent and take into account the dimension of the parameters. Many other choices for \( E \) are possible naturally, and corresponding mathematical expressions can become quite long and complex.

\( L \) describes the constraints of the optimization problem. In \( L \) are included the constraint with constant parameters referred as \( c_i \) and constraints with functions referred as \( C_i(z) \). Respectively, the Lagrange multipliers associated to constant and functions are referred as \( p_i \) or \( P_i(z) \):

\[ L = \sum_{i=1}^{n_1} p_i c_i + \int_0^h \sum_{i=1+n_1}^{n_2} p_i(z) C_i(z) \, dz \]  
\[ \text{(18)} \]

The sensitivity \( s_q \) or \( S_{Q_i} \) are determined from the derivatives of the Lagrangian functional with respect to the chosen parameters as

\[ \delta \mathcal{L} = \delta E = \sum_{i=1}^{n_1} s_q \delta q_i + \int_0^h \sum_{i=1+n_1}^{n_2} S_{Q_i}(z) \delta Q_i(z) \, dz \]

The variation of the Lagrangian functional with respect to all the parameters leads to a linear system for the unknown Lagrange multipliers. The full system is made by 23 rows (number of remaining parameters) and 17 columns (number of remaining constraints) which can be solved by reduction. The \( z \)-dependent sensitivity functions can be first solved analytically and finally a linear system is left, i.e.:

\[ \frac{\partial \mathcal{L}}{\partial q_i} = A_{i,j} p_j + \frac{\partial \mathcal{E}}{\partial q_i} \]

where, for example, the matrix on the right hand side, for \( i = j = 1 \), reads:

\[ A_{1,1} = \frac{\partial L}{\partial a} = \sum_{i=1}^{n_1} 2 \frac{p_{12} g S}{a^2 C_D \bar{\eta}_1} - p_1 d 16 U_h \]
\[ - p_8 C_D C_s \left( 3 a^2 d^2 (59.8) + 2 a d^2 (38.6) + d (-9.3) \right) + \]
\[ + p_{10} \left( \left( 1 - \frac{U_1^2}{U_h^2} \right) (h - z_t) \Omega C_D \bar{\eta}_1 - 2 (a d 16 + 1) d 16 \right) + \]
\[ + \int_0^h \frac{p_{17}(z) C_D(z) u_1^2(z)}{k_c z} dz \]  
\[ \text{(19)} \]

In the previous equation the function \( p_{17}(z) \) is the adjoint of the velocity profile, and it is the solution of the adjoint equation with homogeneous boundary condition:
\[-2 \frac{d}{dz} \left( \frac{du_1(z)}{dz} \frac{dp_{17}(z)}{dz} \right) = \frac{p_{17}(z)C_D(z)au_1(z)}{l_c^2} = h \] (21)

To get a well posed problem with a square invertible matrix, some parameters have to be assumed constant, and the corresponding sensitivities vanish.

In the following we focus on the sensitivities associated to the following four parameters: \(a\) the frontal area of the canopy per unit volume, \(d\) the cylinder diameter, \(h\) the height of the canopy and \(\alpha\) the fraction of the shear layer that lies within the canopy. A second analysis was done assuming \(\alpha\) constant and by adding the sensitivity to the parameters \(\xi\). It allows to verify how the sensitivity of \(a\), \(d\) and \(h\) can change when we select different parameters to solve the system. To compare sensitivity which are defined with different unit, we define the relative error as:

\[
\frac{\Delta E}{E} = s_{q_i} \frac{q_i \Delta q_i}{q_i} 
\]

We arbitrarily set \(\Delta q_i/q_i = 1 = \epsilon_{q_i}\).

<table>
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<th>(\epsilon_{r_a})</th>
<th>(\Delta E)</th>
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<th>(\epsilon_{r_d})</th>
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Table 1: Relative error table from two analysis.

As first conclusion, the results show that the parametric model is sensitive to the inputs \(a\) and \(d\) which are geometrical parameters of the canopy. The model appears to not be very sensitive to \(\alpha\) and \(\xi\) which are parameters related at the mixing length theory. This analysis can be improved by other tests and by performing further sensitivity analyses.

### 4 Linear Stability Analysis

The temporal linear stability analysis is based on the classical formulation in terms of normal velocity, and a decomposition in normal modes that reads:

\[w(x, y, z, t) = \tilde{w}(z)e^{i(k_x x - \omega t)}\]

with \(k_x\) the real streamwise wave number and \(\omega\) a complex number where the real part is a frequency and the imaginary part a growth rate,. The decomposition of the flow field with a mean flow (the previous profile \(U(z)\)) and the perturbation \(w(z)\) leads to the well know Orr-Sommerfeld equation, where
either $D^2$ or the double $\dot{\cdot}$ indicate the second derivative with respect to the normal direction $z$:

\[
\left( -i\omega + ik_x U \right) \left( D^2 - k_z^2 \right) - ik_x U'' - \frac{1}{Re} \left( D^2 - k_z^2 \right)^2 \tilde{w} = 0, \quad Re = \frac{U_b d}{\nu}
\]

$\omega$ is an eigenvalue for temporal stability and $\tilde{w}$ is an eigenfunction. The eigenvalue problem is solved using Matlab where the OS equation is discretized using a spectral scheme with Chebyshev polynomials (size $N$) and a mapping between the semi-infinite and the spectral domain [7].

The spectrum is shown in figure 2. The more unstable mode is shown with a red dot in the figure and the corresponding eigenfunctions are plotted in figure 3.

![Figure 2: The spectrum from case H profile, $Re = 360$, $k_x = 0.05$, $N = 60$.](image.png)
Both the vertical and the horizontal velocity have a maximum in correspondence of the inflection point of the canopy flow, typical of an instability of inflectional nature.

5 Sensitivity to variations in the mean flow

Since we have seen that some variations or errors of the input parameters can change the shape of the mean flow, it is interesting to compute the variation of the spectrum due to changes of the mean flow.

Following the work presented in [3] we determine the function named $G_u$ which measures the sensitivity of the eigenvalues to changes in the mean flow profile.

$$\begin{align*}
\epsilon &= \frac{\omega}{\alpha}, \\
\delta c &= \int_0^\infty \frac{G_u}{K} \delta U \, dz \\
G_u &= \overline{w^* (D^2 - k_x^2)} \hat{w} - \overline{(w^* \hat{w})''}
\end{align*}$$

The quantity $w^*$ is the adjoint eigenfunction related to $\hat{w}$, and the overline means complex conjugate. The constant $K$ can be computed from integration of the eigenfunction $\hat{w}$ (see [3]). The adjoint function is the solution of the adjoint Orr-Sommerfeld equation, see [2] and [3] for details.

The real and imaginary part of the function $G_U$ are plotted in figure 4 and it can be seen that the variation of $G_U$ are correlated to the derivative of the mean flow, and therefore with its stability properties. Let us further note that the second derivative is filtered by a gaussian function before its use in the OS equation.
5 Conclusion

A sensitivity model of a canopy flow has been derived and some preliminary conclusions have been drawn. The variation of geometrical parameters can lead to large variations of the mean flow, as compared to variations of some constants in the turbulence model which appear to have a minor influence.

The linear stability analysis suggests that the canopy profile is highly unstable due to the inflexion point at the boundary of the canopy. This statement must however be tempered by the fact that in the present analysis the drag force within the canopy has been neglected. The sensitivity of the eigenvalues with respect to mean flow variations has been computed and supports the significance of inflexion points in defining the instability. The next step will be to couple the sensitivity of the model parameters to that of the the eigenvalue (perturbation growth rate and wave number) with respect to the mean flow changes.

\[ \delta q_i \rightarrow \delta U \rightarrow \delta c \]

6 Acknowledgements

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References


