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Recovering the initial state of a Well-Posed Linear System with skew-adjoint generator

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ISAE – Supported by IDEX-"Nouveaux Entrants"

Workshop New trends in modeling, control and inverse problems
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Session “Time optimal control and observers”
1 Introduction

2 The reconstruction algorithm

3 Main result
   • With bounded observation operator
   • With unbounded observation operator

4 Application

5 Conclusion
1 Introduction

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4 Application

5 Conclusion
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- $A : \mathcal{D}(A) \subset X \to X$ be a skew-adjoint operator,
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**Considered systems**

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\begin{aligned}
\dot{z}(t) &= A z(t), \quad \forall \ t \in [0, \infty), \\
z(0) &= z_0 \in \mathcal{D}(A).
\end{aligned}
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\]

For instance:

\[
A = \begin{bmatrix}
0 & I \\
\triangle & 0
\end{bmatrix}
\] (+ Dirichlet boundary conditions) on \( \Omega \subset \mathbb{R}^n \) and

\[
X = H^1_0(\Omega) \times L^2(\Omega).
\]

\( \quad \downarrow \)

the classical wave equation.
Let

- $Y$ be another Hilbert space
- $C \in \mathcal{L}(X, Y)$
- $\tau > 0$
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The classical wave equation, with $C = \begin{bmatrix} 0 & \chi O \end{bmatrix}$:

\[
\begin{align*}
y(t) &= \begin{bmatrix} 0 & \chi O \end{bmatrix} \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, \quad \forall t \in [0, \tau], \\
&= \chi O \dot{w}(t), \quad \forall t \in [0, \tau].
\end{align*}
\]
Let
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- $C \in \mathcal{L}(X,Y)$
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We observe $z$ via $y(t) = Cz(t)$ for all $t \in [0, \tau]$.

The classical wave equation, with $C = \begin{bmatrix} 0 & \chi_\Omega \end{bmatrix}$:

$$y(t) = \begin{bmatrix} 0 & \chi_\Omega \end{bmatrix} \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, \quad \forall t \in [0, \tau],$$

$$= \chi_\Omega \dot{w}(t), \quad \forall t \in [0, \tau].$$

Our problem

Reconstruct the unknown $z_0$ from the measurement $y(t)$. 
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K. Ramdani, M. Tucsnak, and G. Weiss

Recovering the initial state of an infinite-dimensional system using observers (Automatica, 2010)

Intuitive representation

2 iterations, observation on $[0, \tau]$. 
Some bibliography


- **2008**: Phung and Zhang (*SIAM J. Appl. Math.*) introduced the Time Reversal Focusing (TRF), for the Kirchhoff plate equation.

- **2010**: Ramdani, Tucsnak and Weiss (*Automatica*) generalized the TRF, based on the generalization of Luenberger’s observers.
• **2005:** Auroux and Blum (*C. R. Math. Acad. Sci. Paris*) introduced the Back and Forth Nudging (BFN), based on the generalization of Kalman’s filters

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We construct the **forward observer**

\[
\begin{aligned}
\dot{z}^+(t) &= A z^+(t) - C^* C z^+(t) + C^* y(t), \quad \forall \ t \in [0, \tau], \\
z^+(0) &= z_0^+ \in \mathcal{D}(A).
\end{aligned}
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\end{align*}
\]

We subtract the observed system

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\begin{align*}
\dot{z}(t) &= Az(t), & \forall \ t \in [0, \tau], \\
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to obtain *(remember that \(y(t) = C z(t)\)),* denoting

\[
e = z^+ - z,
\]

the estimation error,

\[
\begin{align*}
\dot{e}(t) &= (A - C^* C) e(t), \quad \forall \ t \in [0, \tau], \\
e(0) &= z_0^+ - z_0,
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e(0) &= z^+_0 - z_0,
\end{aligned}
\]

which is known to be exponentially stable if and only if \((A, C')\) is exactly observable, i.e.

\[
\exists \tau > 0, \exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 \, dt \geq k_\tau^2 \|z_0\|^2, \quad \forall \ z_0 \in \mathcal{D}(A).
\]
Exponential stability $\Rightarrow \exists M > 0, \beta > 0$ such that

$$\|z^+(\tau) - z(\tau)\| \leq Me^{-\beta \tau}\|z_0^+ - z_0\|.$$
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We construct a similar system: the **backward observer**, 

\[
\begin{aligned}
\dot{z}^-(t) &= Az^-(t) + C^*Cz^-(t) - C^*y(t), & \forall t \in [0, \tau], \\
z^-(\tau) &= z^+(\tau).
\end{aligned}
\]
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\dot{z}^-(t) = Az^-(t) + C^*Cz^-(t) - C^*y(t), & \forall \ t \in [0, \tau], \\
\dot{z}^-(\tau) = z^+(\tau). 
\end{cases}$$

After a time reversal $Z^-(t) = \mathcal{R}_\tau z^-(t) := z^- (\tau - t)$, we get 

$$\begin{cases} 
\dot{Z}^-(t) = -AZ^-(t) - C^*CZ^-(t) + C^*y(\tau - t), & \forall \ t \in [0, \tau], \\
Z^-(0) = z^+(\tau). 
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Exponential stability $\Rightarrow \exists M > 0, \beta > 0$ such that
\[ \| z^+(\tau) - z(\tau) \| \leq Me^{-\beta \tau} \| z_0^+ - z_0 \|. \]

We construct a similar system: the **backward observer**,
\[
\begin{cases}
\dot{z}^- (t) = Az^- (t) + C^* C z^- (t) - C^* y(t), & \forall t \in [0, \tau], \\
\tau^{-} (\tau) = z^+ (\tau).
\end{cases}
\]

After a time reversal $Z^- (t) = \Psi_\tau z^- (t) := z^- (\tau - t)$, we get
\[
\begin{cases}
\dot{Z}^- (t) = -AZ^- (t) - C^* CZ^- (t) + C^* y(\tau - t), & \forall t \in [0, \tau], \\
Z^- (0) = z^+ (\tau).
\end{cases}
\]

And from similar computations for $A^- := -A - C^* C$ as those for $A^+ := A - C^* C$:
\[ t \| z^- (0) - z_0 \| \leq Me^{-\beta \tau} \| z^+(\tau) - z(\tau) \| \leq M^2 e^{-2\beta \tau} \| z_0^+ - z_0 \|. \]
If the system is exactly observable in time $\tau > 0$, that is if:

$$\exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \quad \forall z_0 \in \mathcal{D}(A),$$


$$\alpha := M^2 e^{-2\beta \tau} < 1.$$
If the system is exactly observable in time $\tau > 0$, that is if:

$$\exists k_{\tau} > 0, \int_0^{\tau} \|y(t)\|^2 dt \geq k_{\tau}^2 \|z_0\|^2, \quad \forall z_0 \in D(A),$$


$$\alpha := M^2 e^{-2\beta \tau} < 1.$$

Iterating $n$-times the forward–backward observers with $z_n^+(0) = z_{n-1}^-(0)$ leads to

$$\|z_n^-(0) - z_0\| \leq \alpha^n \|z_0^+ - z_0\|.$$

This is the iterative algorithm of Ramdani, Tucsnak and Weiss to reconstruct $z_0$ from $y(t)$. 
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2 The reconstruction algorithm

3 Main result
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4 Application

5 Conclusion
Outline

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2 The reconstruction algorithm

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In this work, the exact observability assumption in time $\tau$

$$\exists k_{\tau} > 0, \int_0^\tau \| y(t) \|^2 dt \geq k_{\tau}^2 \| z_0 \|^2, \quad \forall z_0 \in \mathcal{D}(A),$$

is not supposed to be satisfied!
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$$\exists k_{\tau} > 0, \int_0^\tau \|y(t)\|^2dt \geq k_{\tau}^2\|z_0\|^2, \quad \forall z_0 \in \mathcal{D}(A),$$

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However, the observers don’t need this assumption to make sense.
In this work, the exact observability assumption in time $\tau$

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is not supposed to be satisfied!

However, the observers don’t need this assumption to make sense.

**Questions**

- Given arbitrary $C$ and $\tau > 0$, does the algorithm converge?
- If it does, what is the limit of $z_\infty(0)$ and how is it related to $z_0$?
Decomposition of $X$:

- Let us denote $\Psi_\tau$ the following continuous linear operator

\[
\Psi_\tau : X \rightarrow L^2([0, \tau], Y),
\]

$z_0 \mapsto y(t)$.
Decomposition of $X$:

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Intuitively, if $z_0$ is in $\text{Ker} \ \Psi_\tau$, then $y(t) \equiv 0$, and we have no information on $z_0$!
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- We decompose $X = \text{Ker} \, \Psi_\tau \oplus (\text{Ker} \, \Psi_\tau)^\perp$ and define

\[
V_{\text{Unobs}} = \text{Ker} \, \Psi_\tau, \quad V_{\text{Obs}} = (\text{Ker} \, \Psi_\tau)^\perp = \overline{\text{Ran} \, \Psi^*_\tau}.
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V_{\text{Unobs}} = \text{Ker} \; \Psi_\tau, \quad V_{\text{Obs}} = (\text{Ker} \; \Psi_\tau)^\perp = \overline{\text{Ran} \; \Psi^*_\tau}.
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Note that the exact observability assumption is equivalent to $\Psi_\tau$ is bounded from below and then $\Rightarrow X = \text{Ran} \; \Psi^*_\tau$. 

G. Haine
Recovering the initial state of a WPLS 15/31
Stability of the decomposition under the algorithm:

- Forward–backward observers cycle $\Rightarrow$ operator $T^- T^+$, i.e.

$$z^-(0) - z_0 = T^- T^+ (z^+_0 - z_0).$$
Stability of the decomposition under the algorithm:
Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $A^+ := A - C^* C$ (resp. $A^- := -A - C^* C$) on $X$.

- Forward–backward observers cycle $\Rightarrow$ operator $\mathbb{T}_\tau^- \mathbb{T}_\tau^+$, i.e.
  \[
  z^-(0) - z_0 = \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \left( z^+_0 - z_0 \right).
  \]

- Denote $\mathbb{S}$ the group generated by $A$, then (since $A = A^+ + C^* C$)
  \[
  \mathbb{S}_\tau z_0 = \mathbb{T}_\tau^+ z_0 + \int_0^{\tau} \mathbb{T}_{\tau-t}^+ C^* C \mathbb{S}_t z_0 \, dt, \quad \forall \ z_0 \in X.
  \]
Stability of the decomposition under the algorithm:

- Forward–backward observers cycle $\Rightarrow$ operator $T^- T^+$, i.e.
  \[ z^-(0) - z_0 = T^- T^+ (z_0^+ - z_0). \]

- Denote $S$ the group generated by $A$, then (since $A = A^+ + C^*C$)
  \[ S^{-}\tau z_0 = T^+_{\tau} z_0 + \int_0^\tau T^+_{\tau-t} C^* C S_t z_0 \Psi_{\tau} z_0 dt, \quad \forall \ z_0 \in X. \]

- Using this (type of) Duhamel formula(s), we obtain
  \[ T^- T^+ V_{\text{Unobs}} \subset V_{\text{Unobs}}, \quad T^- T^+ V_{\text{Obs}} \subset V_{\text{Obs}}. \]
Stability of the decomposition under the algorithm:

- Forward–backward observers cycle $\Rightarrow$ operator $T^- T^+$, i.e.

$$z^-(0) - z_0 = T^- T^+(z_0^+ - z_0).$$

- Denote $S$ the group generated by $A$, then (since $A = A^+ + C^*C$)

$$S_T z_0 = T^+_T z_0 + \int_0^T T^+_{T-t} C^* C S_t z_0 dt, \quad \forall \ z_0 \in X.$$

- Using this (type of) Duhamel formula(s), we obtain

$$T^- T^+ V_{Unobs} \subset V_{Unobs}, \quad T^- T^+ V_{Obs} \subset V_{Obs}.$$

The algorithm preserves the decomposition of $X$!
Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $V_{\text{Unobs}}$. 

Let us denote $L = T - \tau T + \tau |V_{\text{Obs}}|$, we have:

$$\|L_n z\| = o\left(\frac{1}{n}\right), \quad \forall z \in X$$

$$\|L\|_{L(V_{\text{Obs}})} < 1 \iff \text{Ran} \Psi^* \tau \text{is closed in } X$$

Sketch of proof

1. $L$ is positive self-adjoint.
2. Duhamel formulas

$$\Rightarrow \|L\|_{L(V_{\text{Obs}})} \text{ in term of } \inf \|z\| = 1, z \in V_{\text{Obs}} \|\Psi^* \tau z\|.$$

$$\text{Ran} \Psi^* \tau \text{closed in } X \iff \Psi^* \tau \text{bounded from below on } V_{\text{Obs}}.$$

Furthermore, it is easy to prove that:

$$z + \varepsilon \in V_{\text{Obs}} \Rightarrow z - n(0) \in V_{\text{Obs}}, \quad \forall n \geq 1.$$
Convergence of the algorithm:
- It is obvious that the algorithm has no influence on $V_{Unobs}$.
- Let us denote $L = T^\tau - T^\tau_+ |_{V_{Obs}}$, we have:

$$
\|L^n z\| = o\left(\frac{1}{n}\right), \quad \forall z \in X
$$
Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $V_{\text{Unobs}}$.
- Let us denote $L = T^- T^+ |_{V_{\text{Obs}}}$, we have:

\[ \|L^n z\| = o \left( \frac{1}{n} \right), \quad \forall z \in X \]

\[ \|L\|_{\mathcal{L}(V_{\text{Obs}})} < 1 \iff \text{Ran } \Psi^*_r \text{ is closed in } X \]
Convergence of the algorithm:
- It is obvious that the algorithm has no influence on $V_{\text{Unobs}}$.
- Let us denote $L = \mathbb{T}_-^\tau \mathbb{T}_+^\tau |_{\mathbb{V}_{\text{Obs}}}$, we have:

\[ \|L^n z\| = o \left( \frac{1}{n} \right), \quad \forall z \in X \]

\[ \|L\|_{\mathcal{L}(\mathbb{V}_{\text{Obs}})} < 1 \iff \text{Ran } \Psi^*_\tau \text{ is closed in } X \]

Sketch of proof

1. $L$ is positive self-adjoint.
Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $V_{Unobs}$.
- Let us denote $L = T_\tau^- T_\tau^+ |V_{Obs}$, we have:

1. $\|L^n z\| = o\left(\frac{1}{n}\right)$, $\forall z \in X$

2. $\|L\|_{\mathcal{L}(V_{Obs})} < 1 \iff \text{Ran } \Psi_\tau^* \text{ is closed in } X$

Sketch of proof

1. $L$ is positive self-adjoint.
2. $L^{n+1} < L^n$ from which we get $\lim_{n \to \infty} L^n = L_\infty \in \mathcal{L}(V_{Obs})$. 

G. Haine

Recovering the initial state of a WPLS
Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $V_{Unobs}$.
- Let us denote $L = T^{-}_\tau T^+_\tau |_{V_{Obs}}$, we have:
  
  1. $\|L^n z\| = o\left(\frac{1}{n}\right), \quad \forall z \in X$

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Sketch of proof

1. $L$ is positive self-adjoint.
2. $L^{n+1} < L^n$ from which we get $\lim_{n \to \infty} L^n = L_\infty \in \mathcal{L}(V_{Obs})$.
3. $\forall z \in X$, $\sum_{n \in \mathbb{N}} L^n z$ converges absolutely in $X$. 
Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $V_{\text{Unobs}}$.
- Let us denote $L = \mathbb{T}_\tau^{-} \mathbb{T}_\tau^{+} |_{V_{\text{Obs}}}$, we have:

  1. $\|L^n z\| = o\left(\frac{1}{n}\right)$, \quad $\forall z \in X$

  2. $\|L\|_{\mathcal{L}(V_{\text{Obs}})} < 1 \iff \text{Ran } \Psi^*_\tau \text{ is closed in } X$

Sketch of proof

1. $L$ is positive self-adjoint.
2. $L^{n+1} < L^n$ from which we get $\lim_{n \to \infty} L^n = L_\infty \in \mathcal{L}(V_{\text{Obs}})$.
   - $\forall z \in X$, $\sum_{n \in \mathbb{N}} L^n z$ converges absolutely in $X$.
3. Duhamel formulas $\implies \|L\|_{\mathcal{L}(V_{\text{Obs}})}$ in term of
   $$\inf_{\|z\|=1, z \in V_{\text{Obs}}} \|\Psi^*_\tau z\|.$$
Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $V_{\text{Unobs}}$.
- Let us denote $L = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$|$_{V_{\text{Obs}}}$, we have:
  
  $\|L^n z\| = o\left(\frac{1}{n}\right), \quad \forall z \in X$

$\|L\|_{\mathcal{L}(V_{\text{Obs}})} < 1 \iff \text{Ran } \Psi_\tau^* \text{ is closed in } X$

Sketch of proof

1. $L$ is positive self-adjoint.
2. $L^{n+1} < L^n$ from which we get $\lim_{n \to \infty} L^n = L_\infty \in \mathcal{L}(V_{\text{Obs}})$.

- $\forall z \in X$, $\sum_{n \in \mathbb{N}} L^n z$ converges absolutely in $X$.

2. Duhamel formulas $\Rightarrow \|L\|_{\mathcal{L}(V_{\text{Obs}})}$ in term of

$$\inf_{\|z\| = 1, z \in V_{\text{Obs}}} \|\Psi_\tau z\|.$$ 

- $\text{Ran } \Psi_\tau^*$ closed in $X$ $\iff$ $\Psi_\tau$ bounded from below on $V_{\text{Obs}}$. 

Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $V_{Unobs}$.
- Let us denote $L = \mathbb{T}_\tau^- \mathbb{T}_\tau^+ |_{V_{Obs}}$, we have:
  \[ |L^n z| = o\left(\frac{1}{n}\right), \quad \forall z \in X \]

\[ \|L\|_{\mathcal{L}(V_{Obs})} < 1 \iff \text{Ran } \Psi_\tau^* \text{ is closed in } X \]

Sketch of proof

1. **L** is positive self-adjoint.
2. $L^{n+1} < L^n$ from which we get $\lim_{n \to \infty} L^n = L_\infty \in \mathcal{L}(V_{Obs})$.
3. $\forall z \in X$, $\sum_{n \in \mathbb{N}} L^n z$ converges absolutely in $X$.
4. Duhamel formulas $\Rightarrow \|L\|_{\mathcal{L}(V_{Obs})}$ in term of
   \[ \inf_{\|z\|=1, z \in V_{Obs}} \|\Psi_\tau z\| . \]
5. Ran $\Psi_\tau^*$ closed in $X \iff \Psi_\tau$ bounded from below on $V_{Obs}$.

Furthermore, it is easy to prove that:

\[ z_0^+ \in V_{Obs} \implies z_n^-(0) \in V_{Obs}, \forall n \geq 1. \]
Theorem

Denote by \( \Pi \) the orthogonal projection from \( X \) onto \( V_{\text{Obs}} \). Then the following statements hold true for all \( z_0 \in X \) and \( z_0^+ \in V_{\text{Obs}} \):

1. For all \( n \geq 1 \),
   \[
   \|(I - \Pi) (z_n^- (0) - z_0)\| = \|(I - \Pi) z_0\|.
   \]

2. The sequence \( (\|\Pi (z_n^- (0) - z_0)\|)_{n \geq 1} \) is strictly decreasing and
   \[
   \|\Pi (z_n^- (0) - z_0)\| = \|z_n^- (0) - \Pi z_0\| \xrightarrow{n \to \infty} 0.
   \]

3. There exists a constant \( \alpha \in (0, 1) \), independent of \( z_0 \) and \( z_0^+ \), such that for all \( n \geq 1 \),
   \[
   \|\Pi (z_n^- (0) - z_0)\| \leq \alpha^n \|z_0^+ - \Pi z_0\|,
   \]
   if and only if \( \text{Ran} \Psi^*_\tau \) is closed in \( X \).
Outline

1 Introduction

2 The reconstruction algorithm

3 Main result
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4 Application

5 Conclusion
What happens if $C$ is unbounded?

- Main issue $\Rightarrow A - C^*C$ has no more meaning (as a generator).

How to close the system?
What happens if $C$ is unbounded?

- Main issue $\implies A - C^*C$ has no more meaning (as a generator).
  How to close the system?

- Main tool $\implies$ Stabilization by colocated feedback law for well-posed linear system (Curtain and Weiss 2006) allowing admissible $C$. 

Well-posed linear system:

$$z(t) | [0,t] = \Sigma t \left[ z(0) u | [0,t] \right], \forall t \geq 0,$$

where $u \in U := L_2([0,\infty), U)$ and $y \in Y := L_2([0,\infty), Y)$ are the control and the observation (with $U$ and $Y$ two Hilbert spaces).

Well-posedness means that for all $t \geq 0$:

$$\Sigma t = [T_t \Phi_t \Psi_t F_t] \in L(X \times U, X \times Y).$$
What happens if $C$ is unbounded?

- Main issue $\implies A - C^* C$ has no more meaning (as a generator). How to close the system?

- Main tool $\implies$ Stabilization by colocated feedback law for well-posed linear system (Curtain and Weiss 2006) allowing admissible $C$.

- Well-posed linear system

$$\begin{bmatrix} z(t) \\ y|_{[0,t]} \end{bmatrix} = \Sigma_t \begin{bmatrix} z_0 \\ u|_{[0,t]} \end{bmatrix}, \quad \forall \, t \geq 0,$$

where $u \in U := L^2([0, \infty), U)$ and $y \in Y := L^2([0, \infty), Y)$ are the control and the observation (with $U$ and $Y$ two Hilbert spaces).
What happens if $C$ is unbounded?

- Main issue $\implies A - C^*C$ has no more meaning (as a generator). How to close the system?

- Main tool $\implies$ Stabilization by colocated feedback law for well-posed linear system (Curtain and Weiss 2006) allowing admissible $C$.

- Well-posed linear system

$$\begin{bmatrix} z(t) \\ y|_{[0,t]} \end{bmatrix} = \sum_t \begin{bmatrix} z_0 \\ u|_{[0,t]} \end{bmatrix}, \quad \forall t \geq 0,$$

where $u \in U := L^2([0, \infty), U)$ and $y \in Y := L^2([0, \infty), Y)$ are the control and the observation (with $U$ and $Y$ two Hilbert spaces).

- **Well-posedness** means that for all $t \geq 0$:

$$\Sigma_t = \begin{bmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{bmatrix} \in \mathcal{L}(X \times U, X \times Y).$$
Let $A \in L(D(A),X)$ be the infinitesimal generator of $T$. We denote $X_1$ the Hilbert space $D(A)$ (with the graph norm) and $X^{-1}$ its dual with respect to the pivot space $X$.

Associated triple $(A,B,C)$:

There exist a control operator $B \in L(U,X^{-1})$ and a observation operator $C \in L(X_1,Y)$ such that

$$\Phi_t u = \int_0^t T_{t-s} Bu(s) \, ds, \quad \forall u \in U,$$

and

$$\Psi_t z_0(s) = \begin{cases} C^T s z_0, & \forall s \in [0,t] \\ 0, & \forall s > t \forall z_0 \in X_1. \end{cases}$$
M. Tucsnak and G. Weiss

*Well-posed systems – The LTI case and beyond* *(Automatica, 2014)*

Let $A \in \mathcal{L}(\mathcal{D}(A), X)$ be the infinitesimal generator of $\mathbb{T}$.
We denote $X_1$ the Hilbert space $\mathcal{D}(A)$ (with the graph norm) and $X_{-1}$ its dual with respect to the pivot space $X$. 

Let $A \in \mathcal{L}(\mathcal{D}(A), X)$ be the infinitesimal generator of $\mathbb{T}$. We denote $X_1$ the Hilbert space $\mathcal{D}(A)$ (with the graph norm) and $X_{-1}$ its dual with respect to the pivot space $X$.

**Associated triple** $(A, B, C)$: There exist a control operator $B \in \mathcal{L}(U, X_{-1})$ and an observation operator $C \in \mathcal{L}(X_1, Y)$ such that

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds, \quad \forall \ u \in U,$$

and

$$\Psi_t z_0(s) = \begin{cases} C \mathbb{T}_s z_0, & \forall \ s \in [0, t] \\ 0, & \forall \ s > t \end{cases} \quad \forall \ z_0 \in X_1.$$
Let $\Sigma$ be associated with $(A, C^*, C)$, with $A$ skew-adjoint.

**Theorem (Curtain and Weiss 2006)**

There exists $\kappa \in (0, \infty]$ such that for all $\gamma \in (0, \kappa)$, the feedback law $u = -\gamma y + v$ ($v$ is the new control) leads to a closed-loop system $\Sigma \gamma$ which is well-posed. Furthermore:

$$
\Sigma \gamma - \Sigma = \Sigma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma \gamma = \Sigma \gamma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma.
$$
Let $\Sigma$ be associated with $(A, C^*, C)$, with $A$ skew-adjoint.

**Theorem (Curtain and Weiss 2006)**

There exists $\kappa \in (0, \infty]$ such that for all $\gamma \in (0, \kappa)$, the feedback law $u = -\gamma y + v$ ($v$ is the new control) leads to a closed-loop system $\Sigma^\gamma$ which is well-posed. Furthermore:

$$\Sigma^\gamma - \Sigma = \Sigma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma^\gamma = \Sigma^\gamma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma.$$  

Applying this theorem to $\Sigma$ associated with $(A, C^*, C)$, we obtain a closed-loop system $\Sigma^+$. 

$z(t) = z(0) + T\left(t, 0\right)(z(t) - z(0)),$ $\forall t \geq 0,$ $z(t) \in X,$ where $T\left(t, 0\right)$ is the semigroup of $\Sigma$. Under some additional assumptions (namely optimizability and estimatability), the closed-loop system is exponentially stable. In other words, the associated semigroup is: $z^+$ is a forward observer of $z$. 
Let $\Sigma$ be associated with $(A, C^*, C)$, with $A$ skew-adjoint.

**Theorem (Curtain and Weiss 2006)**

There exists $\kappa \in (0, \infty]$ such that for all $\gamma \in (0, \kappa)$, the feedback law $u = -\gamma y + v$ ($v$ is the new control) leads to a closed-loop system $\Sigma\gamma$ which is well-posed. Furthermore:

$$
\Sigma\gamma - \Sigma = \Sigma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma = \Sigma \gamma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma.
$$

Applying this theorem to $\Sigma$ associated with $(A, C^*, C)$, we obtain a closed-loop system $\Sigma^+$. Let $z^+$ be the trajectory of $\Sigma^+$ with control $v = \gamma y$ (for simplicity we suppose $u \equiv 0$), then we have

$$
z^+(t) - z(t) = \mathbb{T}_t^+ (z_0^+ - z_0), \quad \forall t \geq 0, z_0^+ \in X,
$$

where $\mathbb{T}^+$ is the semigroup of $\Sigma^+$. Under some additional assumptions (namely optimizability and estimatability), the closed-loop system is exponentially stable. In other words, the associated semigroup is:

$z^+$ is a forward observer of $z$. 

G. Haine

Recovering the initial state of a WPLS

22/ 31
Theorem (Curtain and Weiss 2006)

There exists $\kappa \in (0, \infty]$ such that for all $\gamma \in (0, \kappa)$, the feedback law $u = -\gamma y + v$ ($v$ is the new control) leads to a closed-loop system $\Sigma^\gamma$ which is well-posed. Furthermore:

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$$z^+(t) - z(t) = \mathbb{T}^+_t \left( z_0^+ - z_0 \right), \quad \forall \ t \geq 0, \ z_0^+ \in X,$$

where $\mathbb{T}^+$ is the semigroup of $\Sigma^+$. Under some additional assumptions (namely optimizability and estimatability), the closed-loop system is exponentially stable. In other words, the associated semigroup is: $z^+$ is a **forward observer** of $z$. 

Let $\Sigma$ be associated with $(A, C^*, C)$, with $A$ skew-adjoint.
The idea is now to construct the backward observer. There is mainly two ways to do that using the dual of a well-posed linear system.

\[ \Sigma_d = \begin{bmatrix} T_d & \Phi_d & \Psi_d & F_d \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & R_t \end{bmatrix} \begin{bmatrix} T^* & \Phi^* & \Psi^* & F^* \end{bmatrix} \begin{bmatrix} I & 0 & 0 & R_t \end{bmatrix}. \]

Then \( \Sigma_d \) is a well-posed linear system with input space \( Y \), state space \( X \) and output space \( U \), associated with \((A^*,C^*,B^*)\).

1. We can construct the closed-loop system \( \Sigma^\sim \) of \( \Sigma_d \).
2. Or, equivalently, define \( \Sigma^\sim \) as the dual of \( \Sigma^+ \).

We then obtain the same theorem as for bounded \( C \), using \( z^+ \) and \( z^- \), the respective trajectories of \( \Sigma^+ \) and \( \Sigma^- \), as forward and backward observers.
The idea is now to construct the backward observer. There is mainly two ways to do that using the dual of a well-posed linear system.

**Dual system**

Define $\Sigma^d$ by

$$
\Sigma^d_t = \begin{bmatrix} T^d_t & \Phi^d_t \\ \Psi^d_t & F^d_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J_t \end{bmatrix} \begin{bmatrix} T^*_t & \Psi^*_t \\ \Phi^*_t & F^*_t \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J_t \end{bmatrix}.
$$

Then $\Sigma^d$ is a well-posed linear system with input space $Y$, state space $X$ and output space $U$, associated with $(A^*, C^*, B^*)$.

Where $J_t u(s) := u(t - s)$ is the time reversal operator.
The idea is now to construct the backward observer. There is mainly two ways to do that using the dual of a well-posed linear system.

**Dual system**

Define $\Sigma^d$ by

$$
\Sigma_t^d = \begin{bmatrix}
T_t^d & \Phi_t^d \\
\Psi_t^d & F_t^d
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & \mathcal{H}_t
\end{bmatrix}
\begin{bmatrix}
T_t^* & \Psi_t^* \\
\Phi_t^* & F_t^*
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & \mathcal{H}_t
\end{bmatrix}.
$$

Then $\Sigma^d$ is a well-posed linear system with input space $Y$, state space $X$ and output space $U$, associated with $(A^*, C^*, B^*)$.

Where $\mathcal{H}_t u(s) := u(t - s)$ is the time reversal operator.

- We can construct the closed-loop system $\Sigma^-$ of $\Sigma^d$. 

The idea is now to construct the backward observer. There is mainly two ways to do that using the dual of a well-posed linear system.

**Dual system**

Define $\Sigma^d$ by

$$
\Sigma^d_t = \begin{bmatrix}
T^d_t & \Phi^d_t \\
\Psi^d_t & F^d_t
\end{bmatrix} = \begin{bmatrix} I & 0 \\
0 & \mathcal{R}_t \end{bmatrix} \begin{bmatrix} T^*_t & \Psi^*_t \\
\Phi^*_t & F^*_t \end{bmatrix} \begin{bmatrix} I & 0 \\
0 & \mathcal{R}_t \end{bmatrix}.
$$

Then $\Sigma^d$ is a well-posed linear system with input space $Y$, state space $X$ and output space $U$, associated with $(A^*, C^*, B^*)$.

Where $\mathcal{R}_t u(s) := u(t - s)$ is the time reversal operator.

1. We can construct the closed-loop system $\Sigma^-$ of $\Sigma^d$.
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The idea is now to construct the backward observer. There is mainly two ways to do that using the dual of a well-posed linear system.

**Dual system**

Define \( \Sigma^d \) by

\[
\Sigma^d_t = \begin{bmatrix} T^d_t & \Phi^d_t \\ \Psi^d_t & F^d_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathcal{R}_t \end{bmatrix} \begin{bmatrix} T^*_t & \Phi^*_t \\ \Psi^*_t & F^*_t \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathcal{R}_t \end{bmatrix}.
\]

Then \( \Sigma^d \) is a well-posed linear system with input space \( Y \), state space \( X \) and output space \( U \), associated with \( (A^*, C^*, B^*) \).

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We then obtain the same theorem as for bounded \( C \), using \( z^+ \) and \( z^- \), the respective trajectories of \( \Sigma^+ \) and \( \Sigma^- \), as forward and backward observers.
Introduction

The reconstruction algorithm

Main result
- With bounded observation operator
- With unbounded observation operator

Application

Conclusion
Example

Let

- \( \Omega \subset \mathbb{R}^N, \, N \geq 2, \) with smooth boundary \( \partial \Omega \)
Example

Let

- $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial \Omega$
- $\partial \Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$
Example

Let

- \( \Omega \subset \mathbb{R}^N, \, N \geq 2 \), with smooth boundary \( \partial \Omega \)
- \( \partial \Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}, \, \Gamma_0 \cap \Gamma_1 = \emptyset \)

Consider the following wave system

\[
\begin{cases}
\ddot{w}(x,t) - \Delta w(x,t) = 0, & \forall x \in \Omega, \, t > 0, \\
w(x,t) = 0, & \forall x \in \Gamma_0, \, t > 0, \\
w(x,t) = u(x,t), & \forall x \in \Gamma_1, \, t > 0, \\
w(x,0) = w_0(x), \, \dot{w}(x,0) = w_1(x), & \forall x \in \Omega,
\end{cases}
\]

with \( u \) the control, and \((w_0, w_1)\) the initial state.
Observation

Let $\nu$ be the unit normal vector of $\Gamma_1$ pointing towards the exterior of $\Omega$, we observe the system via

$$y(x, t) = -\frac{\partial(-\Delta)^{-1}w(x, t)}{\partial \nu}, \quad \forall x \in \Gamma_1, t > 0.$$
Observation

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Observation

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$$y(x, t) = -\frac{\partial(-\Delta)^{-1}w(x, t)}{\partial \nu}, \quad \forall x \in \Gamma_1, t > 0.$$ 

- Curtain and Weiss (SIAM J. Control Optim., 2006) ⇒ construction of forward and backward observers (formally $A^\pm = \pm A - C^*C$).
Let $\nu$ be the unit normal vector of $\Gamma_1$ pointing towards the exterior of $\Omega$, we observe the system via

$$y(x, t) = -\frac{\partial (\Delta)^{-1} w(x, t)}{\partial \nu}, \quad \forall x \in \Gamma_1, t > 0.$$ 

- Guo and Zhang (SIAM J. Control Optim., 2005) $\Rightarrow$ well-posed linear system.
- Curtain and Weiss (SIAM J. Control Optim., 2006) $\Rightarrow$ construction of forward and backward observers (formally $A^\pm = \pm A - C^*C$).
- So we can use the algorithm.
For instance, let us consider the following configuration

\[ \Gamma_0 \]

\[ \Omega \]

\[ \Gamma_1 \]
For instance, let us consider the following configuration
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Choosing a suitable initial data

- \( \text{Supp}(w_0) \) has three components \( W_1, W_2 \) and \( W_3 \), such that
  - \( W_1 \subset V_{\text{Obs}} \)
  - \( W_2 \subset V_{\text{Unobs}} \)
  - \( W_3 \cap V_{\text{Obs}} \neq \emptyset \) and \( W_3 \cap V_{\text{Unobs}} \neq \emptyset \)

- \( w_1 \equiv 0 \)
Choosing a suitable initial data

- \text{Supp}(w_0) \) has three components \( W_1, W_2 \) and \( W_3 \), such that
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  - \( W_2 \subset V_{\text{Unobs}} \)
  - \( W_3 \cap V_{\text{Obs}} \neq \emptyset \) and \( W_3 \cap V_{\text{Unobs}} \neq \emptyset \)

- \( w_1 \equiv 0 \)

To perform the test, we use

- Gmsh: a 3D finite element grid generator
- GetDP: a general finite element solver

\begin{quote}
G. Haine and K. Ramdani

Reconstructing initial data using observers: error analysis of the semi-discrete and fully discrete approximations
(Numerische Mathematik (Numer. Math.), 2012)
\end{quote}
The initial position (Left) and its reconstruction (Right) after 3 iterations

⇒ 6% of relative error in $L^2(\Omega)$ on the “observable part”.
1 Introduction

2 The reconstruction algorithm

3 Main result
   • With bounded observation operator
   • With unbounded observation operator

4 Application

5 Conclusion
Conclusion

More?

G. Haine

*Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator*  
(Mathematics of Control, Signals, and Systems (MCSS), January 2014)
More?

G. Haine
*Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator* (Mathematics of Control, Signals, and Systems (MCSS), January 2014)

**Application to thermo-acoustic tomography:**

G. Haine
*An observer-based approach for thermoacoustic tomography* (Mathematical Theory of Networks and Systems (MTNS – Gröningen), July 2014)
Conclusion

More ?

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*Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator*

(Mathematics of Control, Signals, and Systems (MCSS), *January 2014*)

**Application to thermo-acoustic tomography:**

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*An observer-based approach for thermoacoustic tomography*

(Mathematical Theory of Networks and Systems (MTNS – Gröningen), *July 2014*)

**Still to be done:**

- Stability of $V_{\text{Obs}}$ and $V_{\text{Unobs}}$ with noisy observation $y$
- Generalization ($A^* \neq -A$)
- Optimization of $\gamma$